Algebraic Topology
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Let me talk a little bit about integrands again. The more I think about it, the more clear it is that it's not just the way topology was found, but also a good way to think about it.

We think of k-integrands as the objects which can be integrated over an oriented k-domain inside an open subset of Euclidean space. We think of it as the limit of a sum, and so on.

So when you have a surface, an integrand or 2-form should assign a number to each small parallelogram, which should be proportional to the area, up to a sign, and additive. You arrive at this idea that at every point you should assign to each pair of tangent vectors a number with the property that $\omega(v_1, v_2)$ should be linear in each variable and antisymmetric. So imagine this varying smoothly, and you break it up into a grid. Evaluate your form on the vectors and that gives you a number. Now add those up. It also shows you how you transform things. If you have a mapping between two k-dimensional things, and you have a map F then $F^*\omega$ is defined by taking $F^*\omega(v_1, v_2) = \omega(F_*(v_1), F_*(v_2))$. So an integrand can be pulled back.

That's somehow strange, you have these things and then the dual notion, where you have tangent vectors and you can just concatenate them together to get the exterior algebra. If you have fields of these things, there is no F_* or F^*. These things cannot be transformed. If it were invertible then the thing could be transformed.

It's kind of strange that only one of these is functorial.

Anyway, so these integrands are objects which have this interesting property, that things can be pulled back. That's how we integrate things, with a standard chart.

The top form, up to a constant, is just a constant multiple of the determinant of the Jacobian, since there's only one k-multilinear skew-symmetric form up to a constant.

So these are the important things. Now let's look at this one-form in the plane minus the origin. η, which you would secretly call $d\theta$, is $\frac{xdy - ydx}{x^2+y^2}$. If you plug in $x = r\cos\theta, y = r\sin\theta$, you can see this. It's not defined at the origin. So integrating η just gives you the angle that you've moved through.

Why is that true? The integral $\int_\gamma d\psi$ is the difference of the endpoints; then if γ is a closed curve, this is zero. Locally, let $\tilde{\theta} = \theta\rho$, where ρ is an antibump function. Now θ is multivalued so $d\theta$ doesn't have any meaning; this is kind of a cartoon picture. But take $\eta - d\tilde{\theta}$ and you're deforming η, and the integral of a perfectly round circle is 2π, so that the integral of $\tilde{\theta}$ is also 2π.

Since you can do this for any ray, you get this result.

$$\int_\gamma \eta = \int_\gamma \tilde{\theta} = \#signed\ intersections$$

It's a very general fact that you can always deform a representative down to something so that the integral is obtained by going around this something.

You can now define ω off the origin as $\frac{xdydz+ydzdx+zdxdy}{(x^2+y^2+z^2)^{3/2}}$. This measures the solid angle; if you integrate around a solid sphere you get the total signed solid angle. So now suppose we had a surface map in here that surrounded the origin; then the integral of ω over a closed surface is equal to an integer times 4π. In some sense it's the number of time the surface wraps around the origin.

We can use the same method to prove this fact. The statement about the solid angle means that ω can be thought of, on the standard sphere, under the stereographic projection onto the plane, well,

[long pause]

Is there an area preserving map from the sphere to the disk? It's clear there is.

[disagreement]

If you look at ω under this, it just looks like $dxdy$ so this $d(xdy)$, which we'll call $d\eta$. Take a bump function ψ, and consider $\psi\eta$. We can pull this form back to the sphere and consider $\omega - d(\widetilde{\psi\eta})$.

So you're concentrating your thing on the neighborhood of a ray. Then you extend to $\mathbb{R}^3$ by radial projection; this will give another representative. Count the number of intersections; this works just like the other case.

Here's something that's a little like deforming a surface. This is the sort of thing that Betti was playing around with. You can talk about equivalence classes where $\omega_1 \cong \omega_2$ if and only if their difference is $d\eta$.

You can come up with a picture for a form, based on a foliation.

As an example, we can pull back a point missing from a plane to a line missing from 3-space. Then integrating the pullback of the $d\theta$ form gives the same thing, the number of pages of the book we go around. Given that picture we can generalize quite extensively.

This makes sense for knots as well. There's something called the Seifert surface of a knot.

If you take the trefoil knot, you can consider a bounding 3-sphere approaching the edge of the knot. Now you can come up with a bounding surface which is orientable, in this case T^2 minus a disk. It has one boundary component so you always get something minus a disk.

Exercise 1 *A knot bounds an orientable surface in* S^3*.*

You can think of this as a sheet of a $d\theta$ and think of a form concentrated on one of these surfaces. When you integrate across it you pick up a constant. More generally, this surface is guiding a 1-form which is defined near the surface. Then $\int_\gamma \eta_K$ is equal to the signed number of intersections with S. You can also call this the linking number with the knot.

So anyway, you see that there are interesting forms on sets in Euclidean space obtained by removing certain subsets, like points or compact subsets or one dimensional subsets. Now we're ready to define homology. We're forced to one definition by these considerations, just the way Poincaré was. We're psychologically ready now.

The idea of a definition is to take linear combinations of "domains of integration" mapping to the space for imaginary integrands that might live there.

This is like sending radio waves to aliens who might be out there. We can make these definitions even when the space doesn't have these integrands. We'll use Stokes' theorem as a template. So what is a domain of integration? It's something defined by equations which define a smooth surface. So you write equations with algebra. Now suppose you have a mapping into your space. We're going to take connected and oriented regions. If I were mapping into Euclidean space I could pull back and integrate here.

1. So we take linear combinations of these domains with the relation that $(R, o) = -(R, -o)$.
2. If we have R then we can define the boundary of R, which inherits an orientation. This boundary is made of reasonable looking domains; I want them to be smooth on the interior. So δR is actually a sum of connected pieces.
3. We define a cycle as a linear combination with boundary 0.
4. $z_1 \sim z_2$ if and only if their difference is a boundary.

Then homology is actually an Abelian group.

We've taken the calculus and erased the calculus part. This is exactly how Poincaré defined it. There are a number of details we need to talk about, and I'm sure Scott has objections.

[Yes and no]

I don't have time to talk today; I have time on Wednesday and on Friday.