Algebraic Topology
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Guest lecture by Scott

In my mind there is some confusion about which sort of complexes we're talking about, then I'll state and prove the correct version of the simplicial approximation theorem. Then I'll show a neat geometric interpretation of the Lefschetz number for a simplicial map.

Dennis defined a cell complex as you start with points, glue on line, then discs, et cetera. We know what a cell complex is. What I want to talk about today are simplicial complexes and Δ-complexes. Now Δ-complexes will be certain kinds of cell complexes.

The standard k-simplex=$\{x \in \mathbb{R}^k + 1 | \sum x_i = 1, x_i \geq 0.\}$. So the 0-simplex is a point in $\mathbb{R}^1$, a 1-complex is a line, a 2-complex is a triangle. A Δ-complex will be, someone hands you a bag of pieces and you fit them together nicely. Suppose I hand you 254 triangles and you glue them together, it's the disjoint union of a bunch of simplices, quotiented by some identification map, linearly.

A simplicial complex just has one more condition, and that is that a simplex is uniquely determined by its vertices. It is a Δ complex such that each simplex is uniquely determined by its vertices.

So I can make a cell complex that is not a Δ-complex.

By construction every point in a simplex, I can write p in a simplex σ as a linear combination of the vertices $\sum t_i v_i$ such that $\sum t_i = 1$.

A point is in the first barycentric subdivision has new cells wherever two coordinates are equal. It's not really a simplicial complex but I don't want to get into categorical definitions.

A circle with two vertices is not a simplicial complex because the two vertices do not determine a unique edge. It's a Δ-complex and a cell complex but not a simplicial complex.

So I just think this was necessary because it wasn't clear what kind of complexes or maps we were working with. You heard me babble about how simplicial blah blah blah is categorical. Maps are categorical too. So a map between simplicial complexes should map simplices to simplices linearly. Once you know where the vertices go, you know where everything goes. $f(p) = f(\sum t_i v_i) = \sum t_i f(v_i) = \sum t_i w_i$.

I was confused last time because we were talking about simplicial maps that stretch, and that's not right. Simplicial maps don't stretch.

Any neighborhood of any point only intersects finitely many simplices. You have to be careful if you want to allow infinitely many simplices. Whitehead worried about this. I think that's where you get W and CW-complexes.

I want to give a counterexample to the theorem. I'll write what the theorem was last time and give a counterexample, and then I'll write the right theorem.

Theorem 1 *$f : K \to L$ is a continuous map of simplicial complexes. Then there exists a simplicial map $g : K \to L$ such that $g \cong f$.*

Maybe the statement had to do with subdivisions, but let me give you an example to show that you need subdivision.

Imagine that K and L are spheres. If they have fixed simplicial structure, a simplicial map is determined by its map on the vertices. We know that there are countably many homotopy classes of maps from the n-sphere to itself. That would cause a contradiction because you'd get that this countable collection was homotopy equivalent to a finite collection.

Subdividing the range doesn't even help because if the number of vertices in the range exceeds the number in the domain then your map won't be surjective and will be of degree zero.

The confusion maybe last time was it wasn't clear if we were working with simplicial maps or not.

Theorem 2 *$f : K \to L$ is a continuous map of finite simplicial complexes. Then there exists a simplicial map $g : K' \to L$ such that $g \cong f$, where K' is a subdivision of K.*

Give K a metric from Euclidean space.

Let ϵ be the Lebesgue number of the collection $\{f^{-1}(st(w)) : w \text{ is a vertex of } L\}$. Subdivide K so that the diameter of the star of each vertex of K' is at most $\epsilon/2$.

So $f(st(\alpha)) \subset st(w)$ for some vertex w. Then there's a map g such that $g(v) = w$ for some such w. Extend g linearly. We want to show that it is homotopic to f.

Let's do this by induction. Choose $\sigma = [v_0, \cdots, v_n]$ a simplex of K', and let $x \in \sigma$. Then $x \in st(v_i)$ for $0 \leq i \leq n$. Then $f(x) \in f(st(v_i)) \subset st(g(v_i))$ for all i.

We claim that $\cap st(g(v_i)) = st([g(v_0), \cdots, g(v_n)])$.

Lemma 1 *Suppose P is a simplicial complex with vertices labeled by v. Say I compute $st(v_1) \cap$*

$\cdots \cap st(v_n)$ to be either empty or $st(\sigma)$ where $sigma = [v_1, \cdots, v_n]$. The star of σ is the union of the interior of all simplices containing σ.

I should say open star. Why is this true. Well, by definition, every one of these guys is the union of the interior of all simplices containing v_i. So a guy in the intersection is the interior of a simplex containing all v_i. We know a simplex is uniquely determined by its vertices. So if $[v_1, \cdots, v_n]$ is a simplex we just need to check that the intersection is equal to the star.

So the star of σ contains things which contain all the v_i, and something in the star of every one of the v_i.

In our case it's not empty because it contains $f(x)$. So that means that there is a point, so I showed there was a linear extension of g. I have to show that it's homotopic to f.

So how do I show that. The idea will be that each cell is contained in the star of something. Simplices have linear coordinates. Then I can use a straight-line homotopy. It's the picture that Dennis drew last class. It's key to subdivide the domain so that the image of the star of each vertex is contained in the star of some vertex of L. Then I can use $tf + (1-t)g$; then you have to argue that these fit together and everything is continuous. So this will be continuous everywhere.

So I think third time's the charm or something.

All right. So that's the simplicial approximation theorem. I have like thirty minutes, so let's say a few cool things. The Lefschetz theorem goes through the same way. There are zeros along the diagonal, the trace is zero.

In case it wasn't obvious:

Corollary 1 *Brouwer fixed point theorem. A map from the disc to itself has a fixed point.*

Compute the homology. H_0 is $\mathbb{Q}$, H_1 is 0, H_2 is 0. For every f from the disk to itself the induced maps are $1, 0, 0$. So the Lefschetz number of f is 1, which is nonzero, so it has a fixed point. The Lefschetz number is defined $Lf = \sum (-1)^i tr\ (f_i)_* H_i \to H_i$.

If $f : X \to X$ then $Fix(f)$ is $\{x : f(x) = x\}$. If $f : K \to K$ is a simplicial map then $L(f) = \chi(Fix(f))$.

If you try to read about simplicial things, it's masochistic. There are hardcore category guys who make this very intimidating at first reading.

Andrew mentioned it's a subcomplex, the fixed point set. I have to show that I can make it a subcomplex. Let me state the claims.

The first claim is that $Fix(f)$ is a subcomplex of the first barycentric subdivision of K.

The second claim is that $\chi(Fix(f))$ is the Lefschetz number of f.

So I can regard my map as a map on the first subdivision. If claim one is true I can compute the Lefschetz number of f, the alternating sum of the traces, I can do this over the chain. I have an entry 0 if it doesn't map the cell to itself, and 1 if it does map the cell to itself. The Lefschetz number tells me whether it takes the cell to itself. 1 if it's in the fixed point set, 0 if it's not. So adding the alternating sum of the trace is exactly the Euler characteristic of the fixed point set.

You have the two short exact sequences $0 \to Z_i \to C_i \to B_{i-1} \to 0$ and $0 \to B_i \to Z_i \to H_i \to 0$. If you have an exact sequence the dimension of the middle is the sum of the dimension of the two. Multiply everything by $(-1)^i$ and add them all up. Then the alternating sums will cancel; this is true over a field. You get a number if you map the cell to itself. It's nonzero if and only if it's in the fixed point set.

So why is claim one true? Well, so, I want to show the fixed point set is a subcomplex of the first barycentric subdivision. If f maps a simplex to some other simplex, there's no fixed point in the interior. So $x \in \sigma$ is a fixed point implies $f(\sigma) = \sigma$. So we can consider nondegenerate maps of the k-simplex to itself. These correspond to $s \in \Sigma_{k+1}$. I can map two vertices on a 1-simplex to themselves or to each other.

So to prove that the fixed point set is a subcomplex, I've argued that you're determined by your image on the vertices. Say $p = \sum t_i v_i$. Then $f(p) = \sum t_i s(v_i)$. Suppose that $f(p) = p$. Then $\sum t_i s(v_i) = \sum t_i v_i$. If $p \in Fix(f)$ then the following holds. If $s_f(v_i) = v_j$ then $t_i = t_j$. It's the collection of coordinates where they're equal. Then $t_i = t_j$ when $s_f(v_i) = v_j$. So if s takes v_i to v_j then t_i, t_j are equal.

If you have a cycle $(123)(45)$ It maps v_4 to v_5, interchanges them. So then t_1, t_2, t_3 are equal. Also, t_4, t_5 are equal. So that's it.