Algebraic Topology
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I've been through immigration services this morning. I was standing in a line down there at 10:00 just to go in, and some miracle happened.

The Soviet Union must have been worse.

Okay, so, uh, I've decided to continue the class next semester. The first class is the first Tuesday in February? Okay, great. February first? That's the first week of classes, right?

[Classes begin on Thursday.]

Oh, okay. Jonathan went down to get some papers.

Well, so, the agenda,

1. cohomology as obstructions to constructing mappings cross-sections to bundles, well, deformations of algebraic structures, although that's not usually considered topology. We constructed cohomology briefly by dualizing our chains and taking homology. We've considered homology geometrically as generalized manifolds, here cohomology is forced on you when you ask certain natural questions.
2. More on algebraic topology of manifolds.
3. With homology and cohomology one is naturally led into homotopy theory; there are groups and fibrations. There is kind of a DNA picture of a space in terms of fibrations. The twisting is determined by cohomology. The picture is quite useful. It's only in the homotopy category. There's also rational homotopy.
4. This only makes sense on a manifold, but we have Poincaré duality and a ring structure on homology. Cohomology also has a ring structure. Maybe also a little string topology, which includes an extension of the homology product.

So I thought I would start with, finish this semester with discussing the Hurewicz homomorphism.

[I have a question. Do you think that people with a background in algebraic topology who didn't do the course this semester could join in?]

Yes, officially I haven't done much. Unofficially I gave a picture of this.

[I will also have a second semester differential geometry course.]

Will you be proving things? I don't like proving things. It sounds like a good match.

[Will there be any mention of gauge theory and characteristic classes?]

Yes, thank you. One of the main easy applications is the definition of Chern classes, Stieffel-Witten classes, actually, that's how it's done in Steenrod's book.

In some sense there hasn't been a proper, well, this is too sophisticated a statement for this seminar because it's vague, but I don't think there's been a proper treatment of gauge theory in terms of algebraic topology. Think of bundles as being some kind of nonabelian cohomology. There is actually a formulation of bundles in terms of Cech cocycles. There's not a really good clear picture of the analog of cycles and homologies. In homology you have the idea of chains, cycles and homologies. There's no boundary operator in gauge theory. It's some kind of, someone's thesis was in that direction, but it was more speculative.

That's kind of an interesting, I'll write this as a general vague question:

Problem: Give a boundary operator for gauge theory. Gauge theory means the theory of G-bundles where G is a Lie group. For example, bundles and connections.

I don't know what I mean by this question precisely, I just feel it in my bones. Let me just say a word about it. You have a space, and over it a continuous family of homeomorphic spaces, possibly with extra structure; for example they could be vector spaces. For every path you have a structure above called a connection.

Given a vector above the starting point, you can move this along continuously using a connection. These maps compose, and you can move linearly. It's a functor from (points, paths) to (vector spaces, maps). With this group theory property, you see that small paths make this like a differential equation; then you can differentiate and think of it as integrating and differentiating. Hurewicz invented this concept, this fibration, in the 40s. They were figuring out what connections were at MIT in the 50s. He died, fell off the pyramid in Mexico in '56, backing up to take a photo.

He was working more generally; the fiber could be anything, with some appropriate isomorphism. You can do the circle and diffeomorphisms, or even homotopy type and homotopy equivalence.

Two of these things are isomorphic if there is another family and something that makes all of this commute. The space of isomorphism classes is an infinite dimensional space, a moduli space, which contains Riemannian metrics, Einstein, gravity, so on.

What do I mean, a boundary operator for gauge theory. Suppose I take a path and move it a little bit to another path. Then the isomorphism will change a little bit by something having to do with the loop, the infinitessimal square, then the discrepancy around this little square is in the limit what's called the curvature of the connection. It's a two-form with values in endomorphisms of the fiber. You write it as one plus something where the something is small, the something is the curvature.

It's a little complicated, because you have a couple of big pieces, and two little pieces between.

$$\begin{tikzcd} & L \arrow[dr] & \\ B \arrow[ur] \arrow[dr] & & B' \\ & L' \arrow[ur] & \end{tikzcd}$$

This sort of looks like algebraic topology, but it's nonabelian. Everybody gets this far, and then stops. If this were abelian, this square would just fall off.

In a group, you have somehting like $B'M_1B''$ and $B'M_2B''$, so you get $B'M_1M_2^{-1}B'^{-1}$. So the difference is somehow conjugated. If you add another piece the B changes. You have to integrate and put these together. So you don't really have a chain complex for this.

Well, the way this sort of gives you a hint. One thing you can do is keep going until your path comes back to the starting point. Then for each such path γ you push this around and you get a linear automorphism of the fiber at the starting point.

So from a closed path at p you get a linear automorphism of the vector space at p. Then if you go around two of these the auotmorphisms compose. This is called *hol* and it respects composition. Then *hol* sends a composition of paths to a compostion of automorphisms. So this kind of a group of closed paths and then you have the Lie group, I'm going back to the Lie group case, and you have a homomorphism from this "group" to the Lie group. You have to do something clever with the parameterization due to Mohr and then do something with equivalence classes that Milnor worked out to get this to actually be a topological group, the group of closed paths.

A special case is, suppose the curvature of the connection, the lifting of paths that moves the fibers over the paths, is zero, which means the lifting of paths doesn't change if you deform, then you get a group of endpoint fixing homotopy classes of closed paths mapping to the Lie group. This is what's called the fundamental group, π_1, the group of closed paths up to homotopies fixing the endpoint. If you vary the bundle by an isomorphism, this will change by an inner automorphism of the Lie group. So it's only defined up to conjugation.

So this gauge theory, in some sense, the analog of homology is this, is π_1. Being a cocycle is like the curvature being zero and then you get a homology class. So this structure is somehow just the beginning of a structure. This is hard to appreciate until we get more algebraic topology under our belt. This is all we've got.

So why do we care? This is all we've got for fifty years, but now the physicists have talked about interesting functions on this space of all connections. They used it to define the standard model, and it fits with experiments. What the heck are they doing? There's some mathematics missing that underlies what they're doing. It already started in the 80s when Donaldson came up with new invariants of four manifolds, taking the Lie group to be $SU(2)$. They have been later reinterpreted as part of this picture. In dimension three, a few years later, you get the Chern Simons operator, the boundary theory of the one in dimension four. The functional was introduced in the seventies.

Incidentally, Chern just died, a week ago. He and, well, other things happened, Vaughan Jones got the Fields medal for finding these new knot invariants using algebraic methods. Witten reinterpreted them heuristically but mathematicians don't really know how to define them using gauge theory.

Donaldson used the Riemannian structure as a helping lemma; one of the definitions of characteristic classes uses a metric. Physicists can do this with a Lorentzian manifold; the physical theories are much bigger than these Donaldson invariants, it depends on the metric, but the topological part, which doesn't depend on the metric, is called the sector part. This is old hat, they're using 6-manifolds and 10-manifolds and string theory and gauge theory, they just assume that the quantizations of gauge theory exist. They can actually generate a computation and when they make them, they get something significant.

Compare to us, we can make definitions but we can't compute anything significant; they can compute but they can't make definitions. I like our way better.

The topologist looking at this has to think of it as a crude group, (s)he can't actually use the extra structure from it being a Lie group.

Let me say a little more, whatever gauge theory is, it's completely determined by, you have this sort of functor from paths to a category of vector spaces, I'm doing the vector space case. Suppress the fact that these vary continuously. For each point you have a vector space. The functor is from the category whose objects are points and whose morphisms are paths to the category of vector spaces. Up to equivalence, you can take one object in each category, and up to equivalence it's determined by this part of it. This is an equivalent subcategory. In dynamical language this is, like, the dynamical part. It's a little more elegant to thing with the whole picture, but up to equivalence you can think with the one-object picture.

So if you think this way there's kind of a third thing you can do. For every point you have the closed paths, and so, um, so, the set of all these things is called the free loop space, the set of all closed paths at every point. If you take your closed path and start moving the base point, keeping the path the same, so it's kind of a circle action, and if you mod out by it you get the equivariant loop space. And then for every one of these closed curves, you can go around and take the holonomy, and then there is a well-defined conjugacy class of automorphisms associated to every closed curve. Then you can look at the traces of these. There's a system of functions on the space of closed curves, and this is the data the physicists use, taking averages of these using Feynman path integrals.

Gauge theory is working with functions on these, they have a whole calculus, but the math doesn't work out. There's a whole group of physicists at Penn State who've been to the seminar, who try to carry through the quantization procedure for the Einstein equation.

Somehow it's a big generalization of the fundamental group, which is just a shadow of it, because it's nontrivial for simply connected things. The mathematicians have a definition, we know what a connection is, but we don't have the rest.

The boundary operator, well, do I have homework here? I didn't bring it. Well, uh, physicists have a way of, uh, um, taking very simple classical physical systems like a harmonic oscillator or something like that. If the variables or observables, like position or momentum, become operators, multiplication or differentiation, and then on Hilbert spaces you get the quantum version. What's crucial is that the variables are functions on an affine space, because you'll use the parallel translation of affine space to get differentiation.

In some sense that's all they know how to do, super elaborations of that. The space of all connections is nonlinear and has nontrivial topology. They have this procedure, where if they start with something linear and want to put in constraints, so you start out with linear variables, but then you want to divide by some gauge group, what you do is add more variables. If you want to add $x^2 = t$ you add a new variable z and write $Dz = x^2 - t$. I'm kind of simplifying. They perform the imposition of restraints and dividing by actions, and if you take cohomology you get what you want on the quotient. This is called BRST. These are called ghost variables, formal variables to put in the constraints. It's some kind of algebraic topology construction. Then they do a lot more stuff with it.

After they build up a whole thing, there are a lot of technical problems, because there are infinitely many linear variables, one for each point in space time. The formulas they write down then don't make sense. So they'll be talking about the equivariant loop space, and they pretend like every point on this thing is a variable.

Algebraic topology gives a way of treating these things by looking only at finite dimensional spaces. String topology is intersection operations on the loop space, it's kind of like the delta. I haven't spent much time on this per se, because I've been trying to figure out the background.

Apparently this guy Yang looked for, in electromagnetism, you can rotate at every point something that they're talking about, and Weyl called it the gauge group, and there's a fairly successful quantization of Maxwell, quantum electrodynamics. If you compute with thousands and thousands of graphs, you get things to 10 or 12 decimal places, it's the most precise theory in science. Think of gauge theory and one dimensional complex vector spaces. For ten years Yang was trying to write down nonabelian versions, at Stonybrook, physicists had a Wednesday lunch, and Simons pointed out to Yang that he was writing down something related to the way connections transform. He was working at Brookhaven National Lab and Mills was his officemate. That's the basic bosonic model of the standard model.

Anyway, from that discussion we have, let me remember this part. $\pi_1(X)$ is the group of homotopy classes of closed curves in X. The first Hurewicz homomorphism is the map h from

π_1 to H_1. You have a point and a closed loop and then you can just regard this as a cycle and take its homology class.

Let X be connected; then the first observation is that this is onto. You can deform it to the base point and make it a single curve.

What about injectivity? No. Our example will be the plane with two points removed. We make some local moves, and then deform this, which way was this going, and then bring this around like that. This is a special case, of if you look at the torus, and you cut a little hole out of the torus, it's homotopic to $aba^{-1}b^{-1}$. That's what I had here.

So in fact there's an actual map of the torus into the space, where the boundary goes to this thing here. So here's another picture of the homology. There are more complicated homologies. Take something in the kernel, and then I know that it bounds a 2-chain, since it's in the kernel. Being zero in π_1 means you can fill it with a disk; in H_1 that means it's a surface. It's in the kernel if it bounds a map of a surface into the space; that's why we made this geometric picture.

For genus two, this picture would be like this. Then this element, going all the way around, is an element α in the kernel of H that bounds a surface, so in this case a product of commutators.

Theorem 1 *Probably Hurewicz, maybe Hopf, Riemann, Poincaré, I don't know who stated it.*

$h : \pi_1 \to H_1$ *is onto and its kernel is the commutator subgroup, that is, the subgroup generated by commutators. It's also the normal subgroup generated by commutators. If you conjugate a commutator, you get a commutator. So this is equal to abelianization.*

In some deep sense this is true beyond just dimension one. You can think that the passage from all of homotopy theory to homology theory is some kind of big Abelianization. If I don't get to it remind me at the end; it's easy to say, called the Dold-Thom theorem. We used the proof of it two times, the relationship between roots and coefficients of the characteristic polynomial.

The second is the map from $\pi_2 \to H_2$, and so on, $\pi_n \to H_n$. There's a little bit to say, a nice result about these. I haven't officially said what this is. I'm going to define these groups and define these maps. The image of h in H_n are the spherical homology classes. These are homology classes that can be represented by maps of spheres into the space. The sum of two connects and is closed under sum so forms a subgroup, called spherical homology. That's the image. The result is, suppose π_1 is zero. If you don't, we can't say anything about the higher ones. $H_2(T^2) = \mathbb{Z}$ but since the sphere is simply connected, you get a map of the sphere into $\mathbb{R}^2$ which makes the diagram commute, so it's nulhomotopic, so you can't repressent any homology class by a map of the sphere. The same argument works for any surface, a map of a sphere into a surface can be deformed to a point.

Theorem 2 *But if you suppose again that the space is connected and simply connected, then* *all homology classes in dimension two and three are spherical, but not four in general.*

Why doesn't this work in dimension four? Take $\mathbb{CP}^3$, with $\mathbb{CP}^2$ sitting in it as a hyperplane. If you move this and intersect it with itself, it, uh, take $\mathbb{CP}^2$, and it has two-spheres sitting in it as hyperpanes. Two of these intersect in a point. $\pi_1(\mathbb{CP}^2)$ is zero, and the homology groups are $\mathbb{Z}$ in dimensions two and four, 0 in three and one. It has the element in dimension four because it's a manifold. Take the preimage of the two two-spheres in a supposed four-sphere representative. The intersections under a simplicial approximation would be the number of times that the sphere covers that simplex, and thus the cycle. On the other hand one of the homology classes bounds, and if you intersect the thing it bounds with the other one you get intersection points cancelling in pairs.

It's a little elaborate; this is what it will be like next semester, using the ring structure. To prove that things don't exist you have to work a little. There's no map of the sphere that maps around with positive degree.

Why is it true in two? More in the details next semester? This is very easy; we know it already, because if you take any cycle in the space, if you take an element in H_2, and represent it by a cycle, I can squeeze it down to a point, get a tube that's small, and do a surgery, and do the same thing for one of the others, and that's a sphere. It kind of looks like those things that swim in the Amazon, what are they called, Manitobas? Manatees? They're like big sea-walruses, there's another one that has arms, another strange creature.

Why is it onto in dimension three? A three dimensional cycle, let me think.

Think of a cell complex that's simply connected. The first idea, we've kind of done this before, is choose a maximal tree, and then squeeze that to a point. Then it has one cells, two cells attached to them, and three cells attached to them. So I can collapse the loops to points, so now I have two cells attached at a point with three-cells attached by maps of the two-sphere.

Now, this is supposed to be a two sphere and this a 3-cell. If you have a map of a 2-sphere into this space, wrapping around this bunch of balloons, call points in these balloons a and b. You can take a simplicial approximation and take a in the middle of a simplex, and see some number of preimages, a plus a and a $-a$ and another $+a$, and then some b and cs. You consider little islands around these points and get a picture of the map. Up to homotopy, every map of S^2 into a bouquet of spheres has a little clump of as, a little clump of bs, and a little clump of cs, and little islands around each one individually. There's a sea around the islands. The map is to take the beach to the point at the center of the bouquet, and all of the sea, and then the islands to the balloons.

Look at the preimages in a simplicial approximation. You'll get some minus signs. A plus and a minus sign is folded over and then mapped in. So you can push this across, and thence cancel all the minus signs and assume these are all plus signs. So everything else maps to the complement. If you put holes in each balloon, the air goes out and they homotope down to the base point. So you get this picture of the map. Any map of the two sphere into the bouquet of two spheres is like this. You make a rigorous proof with the simplicial approximation theorem. You're just looking at pictures.

Apparently this proof, there's a slight generalization, where you want to move the map off a given cell. Stallings has a paper where he proved this and then used it to prove the Poincaré conjecture, but then he had to admit that there was something wrong with the proof. Things being simply connected make things easy. I forget the statement.

Every map looks like that, so, um, so now what I have is a kind of chain complex. The 2-spheres are like $\mathbb{Z}$s; each one is like a cycle. When I attach a map, for each cell I have another bunch of $\mathbb{Z}$s. The boundary of one cell is attached to some sequence. I collapse the 1-skeleton to a point, and this looks like a chain complex for the homology. When you have a map between free abelian groups, you can change the bases to make it diagonal. Doing these operations enough times you can change it to being diagonal. You can think of $A + B$ as a seperate sphere, so think of this to make it diagonal, and then you have nonzero coefficients and some zeros. The ones that map to 0 will be the cycles.

It's a pretty hard proof. The general claim is that I can rearrange these cells in the space to look like this diagonal boundary map. I've got to do more to prove this, and I won't have time. You can get a very simple picture of the space, the simply connected space, the three cells are attached onto these, maybe with more than one attached to one, and some won't be hit at all. So then some arre attached by degree zero so they attach at the base point and don't cover anything. It will look just like a chain complex with homology. You have spheres in dimension two. These are the cycles and so anyway, they're spherical. This is harder.

[What was wrong with the way you did it at Stonybrook?]

Nothing, that one doesn't generalize to higher dimensions, this one does. I was challenging myself. I need more preparation.

[What is the general statement?]

Let me finish here. If it's onto in dimension three, you take your homology and improve it to a homotopy. Onto in dimension three becomes more precise to give you into in dimension two. The general statement is, and this is what's called

Theorem 3 *The Hurewicz theorem*
If $\pi_i = 0$ for $i < k$, then $\pi_j \to H_j$ is bijective for $1 \leq j \leq k$. and $\pi_{k+1} \to H_{k+1}$ is onto.

π_i is the group of maps of S^i into a space up to homotopy; these are abelian because of the island idea.

I gave a different argument with manifolds at Stonybrook that only works in this low dimension. When I took topology at Princeton with Steenrod, this theorem was the point of the whole year course. If you really understand it, it's like the only thing you need to know. It's a good building block for homotopy theory.

There are some additional statements for rational homotopy theory. That's a good place to stop. You have the statement of the Hurewicz theorem and some glimmer of what you have to do, to get into homology from the point of view of cells.

One more anecdote, I was working on this when I was doing my thesis; I had a big space B that I was trying to classify. It was periodic of period four with groups $0, \mathbb{Z}_2, 0, \mathbb{Z}$. The DNA picture, you have nucleotides, what are they called?

[base pairs!]

Base pairs, yes, these are $K(A, n)$ which has homotopy group A in dimension n. Then homotopy will agree up to $n + 1$. Every other space up to homotopy is from one of these spaces, take these building blocks, take the next one and glue it in as a twisted product. The simplest case is when it's just a product. Maybe $B = \prod blocks$. It's a theorem that the homotopy groups of a product are the direct sums of the homotopy groups of the factors.

So I went into the common room and there was this guy George Cook sitting there. I asked "When is a space the product of the building blocks?"

He just answered right away that it is if and only if $\pi_i \to H_i B$ is a map to a direct summand. So I did it, and bingo. It was one of the most beautiful conversations I've ever had in my life. If you can find the homotopy with this invariant and split the map then you know the space. This space, at the prime two, anyway, this is possible. That's jumping ahead and you have to have this DNA picture of homotopy. So this gives you a building block picture that is algebraic. The cell picture gives you blocks that are geometric. This is called the Postnikov system.

I was talking to a Russian the other week in Disney World. This was actually done by Cartan and Serre, but he wrote a thesis about it. It was kind of the golden age of algebraic topology in some sense.

Okay.