Algebraic Topology
December 13, 2004

Gabriel C. Drummond-Cole

December 13, 2004

I thought we'd just take questions and make a list of the ideas we talked about.

One thing we talked about was solutions of equations defining surfaces. We looked at $F(z, y) = 0$ and this gave us an abstract Riemann surface spread over the z-plane plus the point at ∞.

So what does this have to do with algebraic topology? That's the next question. Somebody else has to answer, instead of Scott. What idea in topology did I use this to suggest?

The topology is a methodology for doing geometry. Certain structures are very robust. If you shake the structure it doesn't change.

[Genus?] That's not the one I was thinking of, but Riemann associated an integer, the genus, with this, and then he showed something about the space of holomorphic 1-forms, that it was equal to the genus, and that if you vary the structure a little bit the genus doesn't change.

I was thinking about the idea of a generic point moving, and the points above it moving around and being perturbed, and the idea of a multivalued function or a function on paths. Of course, roots can be thought of as complex functions. If you go around in a closed loop you might get another root, so it's a multivalued function.

This whole idea of the multivalued function which eventually leads to the fundamental group; that's like the first topological idea that one extracts from this discussion. This idea of a covering space, you have it here. If you look at this example you're led to the covering space and the fundamental group.

[Was it a novel idea to study the space of all solutions?]

I think the novel idea, in fact, the stroke of genius, now it hardly seems an idea, for each point here you have a y. The root of this function is a multivalued function on the sphere, but a single-valued function when defined on this surface above.

So the general idea of building a surface on which multivalued functions become single-valued, that's a big idea. These are the first manifolds, and just about anyone in this department works on these things.

The thing I wanted you to say was the idea of a multivalued function or the fundamental group and the theory of covering spaces.

Then the next thing was there are these abelian multivalued functions. This is where when you move the path by a deformation it remains the same, but a path can also go through these local moves, what's the picture? There's actually some topology here. If you have to cross a handle it requires a local move. The ambiguity of the function here is the same as the ambiguity of the function here.

This leads to the idea of classifying closed curves up to deformation plus local moves, which is homology, H_1.

What I omitted was studying more nontrivial systems, nonabelian multivalued functions. I think I'll do that some next semester. The whole theory of connections and differential equations, that's a natural language for it.

There are two ways to go, nonabelian multivalued, which is representations of the fundamental group but has no higher level analogue, or abelian higher dimension, which give you the higher homology groups.

Last time I talked about higher homotopy groups, but these are Abelian so they are not the true analog to π. The full story here in some sense is what's called gauge theory, which is the theory of bundles and connections. If the curvature is zero, that corresponds to this one-dimensional thing; the curvature is a two-dimensional thing.

In some sense the proper topological interpretation of this has not been done. It's defined but we want to define it like homology, in terms of chain complexes.

So what did we say about homology. What did I say about it?

[First we did this thing you mentioned, then pursued the idea to higher dimension, which sent us into some Morse theory].

The idea of this is to give a picture of homology. We try to give, so, namely, a picture of a cycle and a picture of a homology.

Can anyone say what does this picture of a cycle look like?

[A closed surface sitting in a space is a cycle in the space.]

Is that general up to homology?

[I think so.]

It's a collection of cells, so there's a map of the surface onto the cycle. If you do this abstractly, and build several sheets, you can pull things apart?

How did the definition of homology come about as the sum of things with homology zero?

[If you look at homology as a time sequence with two end-paths, do the same thing with your n-surfaces as the end times and something happening between]

Why were the local moves allowed, though?

[Integration.]

That's right, we have Stokes' theorem. That $\int_{\delta R} \omega = \int_R d\omega$, so that even if has topology, if you integrate over the boundary of a closed thing you get zero, so if you integrate an exact thing over a closed thing you get zero too.

So we had examples of things, ω was like a solid angle or something like that. You have a surface around a hole in space. A form that assigns to each point a solid angle, you count the solid angle surrounding a hole.

In the nineteenth century they had general versions of this, and Poincaré said that the algebra of domains is exactly the definition of homology, so homology comes out of calculus.

These solid angles help calculate what happens around a little hole. On the other hand, these are natural objects, the idea of solid angles and so on. That's why topology is useful, because it has this geometric origin, that's why we teach it.

So I was trying to get a picture of a cycle and a homology; if you think of a homology capped off at the ends that gives you a cycle, in any dimension. So homology turned out to be the same as deformation and local surgeries on the space. You can locally cut and glue pieces with the same boundary. When you're doing actual manifolds, these can be described in precise ways. Ordinary homology theory requires singularities. In low dimensions we don't have to worry about it.

So Stokes theorem is what leads to higher dimensional homology. Just for completeness it's worth stating that there is this thing called de Rham's theorem. He proved it though it was known by Poincaré and used by Cartan. It wasn't written up until the 50s because he needed a theory of distributions to do it neatly.

So de Rham's theorym says $\int_C \omega$ yields an algebraic duality between ker $d/im\ d$ and ker $\delta/im\ \delta$ over an oriented manifold. This is called (de Rham) cohomology and turns out to be the algebraic dual to homology.

[This is for a manifold?]

Much more generally, this is true for any cell complex.

In Whitney's book it's discussed for general cell complexes. If you have something where the forms are defined on each piece, and the intersections are lower dimensional manifolds, then you can do this there.

[Is there a geometric description of the algebraic duality over $\mathbb{Z}$?]

The dual constructions aren't the same with integr coefficients. But say everything is finite

dimensional. Then every such chain complex can be built out of direct sums and these
$0 \to \mathbb{Z} \to 0$
$0 \to \mathbb{Z} \to^{\lambda} \mathbb{Z} \to 0$
$0 \to \mathbb{Z} \to^{1} \mathbb{Z} \to 0.$

In cohomology you have the reverse. This is not true over a general ring, which is why we have algebraic K-theory and derived categories. There's a shift as to where the torsions are. The trick is to tensor with $\mathbb{Z}/n$. What's interesting is that what you get is $0 \to \mathbb{Z}/n \to^{\lambda} \to \mathbb{Z}/n \to 0$.

The homology and cohomology are then dual, and the rational coefficients are dual. At various points in my life I've replaced integer discussions with rational discussions.

There's a formalism for this involving universal coefficients and Tor,Ext, but I never had to brace that because I had these more primitive methods. This is something called Pontryagin duality.

You usually state it for the ring equal to a field.

So we define the homology of a space X, when it's a cell complex by the homology of the chain complex, and for Steenrod singular homology for maps into a space. You have to prove these two definitions are the same using the simplicial approximation theorem.

Then, now, we have the induced transformation, you take the singular definition that let's you see clearly that things are functorial, and then the other definition. You can say that H_* is a functor from spaces and continuous maps to the space of graded abelian groups. Clearly composition of maps is respected. I believe this is really the first example, this is where the language started developing.

Alexander, who was a really good topologist but also independently wealthy, became a rich banker and spent a lot of time making money like Jim Simons. This is J.W. Alexander. Last time, I, well, let's see, just having this functor, um, proves lots of theorems, just knowing this functor exists. For example, if I have a circle and I map it around itself twice, then this cannot be continuously extended over the disc. You have a functor and you want to know if you can make the diagram commute:

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d, "\subset"'] & S^1 \\ D \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\cdot 2"] \arrow[d, "\subset"'] & \mathbb{Z} \\ 0 \arrow[ur] & \end{tikzcd}$$

So since the filler doesn't exist here, we know that the map above is impossible.

Are there any questions? I'm just talking.

There's also this Poincaré duality, I just discussed this fast. Let's discuss this for closed manifolds. I'll prove this carefully (in my sense) next semester. This is when you look at closed manifold, the homology groups of dimension n are isomorphic to the groups of codimension n. What's really true is that the vector space in dimension n is isomorphic to the dual of the vector space in codimension n.

Poincaré just said that these ranks are the same. The duality is illustrated by the idea that for $S^k \times S^l$, there is an l-cycle intersecting a given k-cycle in a given point. If it's nonorientable you can use $\mathbb{Z}_2$ coefficients.

There's a differential form proof of this also, there is a cell complex proof but also one with differential forms using de Rham's theorem. You think of them in cohomology as generated by differential forms. There's another, this development has an important ramification when the manifold is defined inside complex projective space by polynomial equations (it is an algebraic variety). Then you have complex variables $dx_1 dx_2 dx_3$, you consider dz_i, and $d\bar{z}_i$, these being a set of generators. Then this picture of homology in dimension one, it breaks up into two pieces, it two into three pieces, in four into four pieces.

This is because you can talk about pieces that have some number of dz_i and some other number of $d\bar{z}_j$. The p, p groups are interesting. So this adds on some interesting complexity. So Scott proved something, the simplicial approximation and then that the Lefschetz number of a simplicial endomorphism is the Euler characteristic of the fixed point set. So suppose you have something like $\sum z_i^2 = 0$ in projective space. Then conjugation is a simplicial involution, so this is the Euler characteristic of the real locus of the equation, which is the Lefschetz number. So this is just a symmetry of the (p, q) lattice. So the Lefschetz number is zero for the (p, q) part, and it's $-1, +1$ for the $1, 1$ and $2, 2$ parts. So $\sum (-1)^p \beta_{p,p}(V_{\mathbb{C}}) = \chi(V_R)$. If we take two variables, $z_1^2 + z_2^2 = 0$, you get the right answer. If you take three variables you get $S^2 \times S^2$, and you get $1 - 2 + 1$, so that works.

This theorem gives a way of calculating things, since the way this works is like, except for the middle dimension, like projective space.

For any variety, cutting along a linear hyperplane, these have the same cohomology up to the middle dimension of the smaller one. Then the kernel has interesting structure. Without the middle part it's the weak Lefschetz theorem; with the middle part it's the hard Lefschetz theorem.

A relatively new idea is to consider all two-cycles and maybe consider those that satisfy a minimal area assumption; you can consider holomorphic maps. These form a moduli space or a family, and the pictures we've been talking about are what we've been talking about. So fix $\beta \in H_2(M)$, and look at holomorphic maps from a fixed genus into this. It's a finite dimensional algebraic variety and we say two are equivalent if there's a diagram to make things commute. It's noncompact precisely because of the finite moves that generate homology. If you add the degenerate pictures then you get a compact moduli space, and this is the essence of symplectic topology and Gromov Witten invariants.

It's remarkable to me how much mileage there is just in homology of surfaces. Since you didn't ask questions I've added some cultural remarks.

This starts again in January. I'll be here on Wednesday December 29 from 12:30 to whenever.