Algebraic Topology
December 10, 2004

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December 13, 2004

The final exam is scheduled for Friday the 17, from 11:00 to 1:30. This is the last officially scheduled class, but there will be an additional class on Monday, December 13. We can do questions and review.

In some sense what I've done is defined homology very geometrically, without systematically developing its abstract properties. Next semester, this course continues. It should be 540, it should be a first year course.

[It's 600-something.]

It shouldn't be. They're lazy. So we're forced into homotopy theory by homology. There are various things like groups, fibrations, rational homotopy theory, and this semester homology, then cohomology appears as obstructions to doing various things in homotopy theory. I'll present it that way. All this together might be called algebraic topology, but then you can discuss algebraic topology of manifolds. You have characteristic classes, string topology, which only makes sense on manifolds, and general geometric topology.

Okay. So I haven't defined officially the fundamental group. I've defined what it means for a path-connected space to be simply connected; any path between two points can be deformed to any other or a loop can be filled with a disc. Any multi-valued function is actually well-defined.

There's a discussion you find in all the books and so on, which is called the fundamental group of a space X. Most of you know this, but I'll just say the words. You consider all maps of a circle into the space, where a fixed point in the circle goes to a fixed point in the space. Then there's a composition $\alpha \cdot \beta$, and up to homotopy this is associative and has inverses. So the fundamental group is the homotopy classes respecting a base point. So $(S^1, *) \to (X, *)$. That would be how it was presented in any book.

There are some things wrong with this definition. How do you choose a base point? There's really a more elaborate structure. You could say that for every point you have a group. Whenever two points are close together, in a locally contractible space, there is a canonical isomorphism. That's the same thing as saying they have a natural flat connection. If you move around a hole you get an inner automorphism. So there is a monodromy given by inner isomorphism. That structure is closer to the fundamental group.

Grothiendieck thinks the real idea is the fundamental groupoid, where you consider homotopy classes of paths between any two points.

Say you're looking at the Hopf fibration over the two-sphere, or say, the circle of unit tangent vectors, which is sort of the dual of the Hopf fibration. You might say, "let's construct the universal covering space of the unit circle, the real line," and have a real line over every point. Then you project a cross section down and get a nonzero section of the sphere. So the construction of the universal cover really depends on the base point.

So if you want a basepoint-free idea of the fundamental group you have to look at it in one of these ways. So the fundamental group is okay, but it's not great.

So often you get representations $\pi_1(X) \to G$ where G is maybe a Lie group. You only care up to conjugation, so that's a basepoint free notion where what you have is a flat connection. This is how the whole business started with Riemann and Poincaré. Anyway, representations of the fundamental group up to conjugation have geometric realization over the space without mentioning any basepoint.

There's another point, which is that group theory is an undecidable branch of mathematics, but representation theory is computable and knowable. From some points of view you can pursue it.

When you consider algebraic topology the way I am presenting it, there are two natural extensions. First you can use twisted coefficients, algebraic topology over the flat connection, or even in further generalization, sheaf coefficients. Over each piece a nice sheaf will give a flat connection. Next semester the courses by Kirillov and de Cataldo will be about representations and sheaves. They could assist with the geometric picture I'm making.

So what I wanted to do was mention, the geometric pictures we mentioned for H_1 and H_2. The first thing is that there's a map from π_1 to H_1. So π_1 was a much stricter equivalence relation but the arrow between them is a homomorphism, the Hurewicz homomorphism.

Consider the torus as the quotient of a square. If you cut a small hole out of the torus, and then the loop going around it is freely homotopic to $\bar{a}\bar{b}ab$. So what you see is if you have a loop mapping into a space X, and it bounds a torus, then the loop is a commutator. If a circle bounds a torus then it's a commutator. Similarly, if you take the surface of genus two, well, a commutator is zero if and only if a and b commute. So the circle in the center of the octagon to make the two-genus surface is equivalent to a product of two commutators. So a loop that bounds a surface of genus n then it is a product of n commutators.

Now suppose some loop α is in the kernel of the Hurewicz map. Then it's the boundary of a 2-chain. So if something bounds a 2-chain then it bounds a 2-manifold. It bounds a homology class if and only if it bounds a surface, so if and only if it is a product of commutators. So the kernel is the commutator subgroup and the map is just Abelianization. Riemann and Poincaré knew this, they used it, whether they stated it.

Okay, so that's, in some sense, homology of a space is some kind of general Abelianization. In fact, last Monday Gabriel showed, that's too fast. Better later. What about H_2 and H_3? There are corresponding things you can put here, let me not define them but allow them to be forced upon us. We showed that any element of H_2 can be represented by maps of a surface into a space, up to deformation and local moves.

So if we take any cycle, suppose that X is now simply connected so that a loop into the space can be deformed to a point fixing the basepoint fixed (which is equivalent to any loop being filled in). So we can deform the map to make a local move by deforming it to something small, and so any element can be moved to be represented by S^2 into the space. So consider a map of the two sphere into the space, with a fixed point of the sphere as a reference point. Then take homotopy equivalence. If you have two of these you can map a sphere to a one-point union of spheres, which turns out to be homotopy commutative, like islands floating in the sea. You can't do this for π_1 because it's like two boats in a narrow channel, like a one-lane highway. The inverse, with the opposite orientation, can be pulled back to a point.

So there's a group π_2 with strict homotopy. This $\pi_2 \to H_2$ map is called the Hurewicz homomorphism, so assuming it's simply connected, this Hurewicz map is onto. It's also into. You need simply connected. For instance, in a high genus surface, π_2 is zero but H_2 is not.

If the degree of a map is nonzero, that if, if $f_*(H_2\Sigma_1 \to H_2\Sigma_2) \neq 0$ then $g\Sigma_1 \geq \Sigma_2$.

What is a dominant map in algebraic geometry? I think it's just a map of nonzero degree. Anyway, so you can't map a two-sphere into a higher genus surface with nonzero degree.

You can do the same thing for π_3. There is an infinite sequence of groups π_n with a Hurewicz homomorphism to H_n, whose image is the spherical homology, those homology elements which can be represented by spheres.

There's a nice fact. If the space is simply connected, the third Hurewicz map is also onto. There's a corresponding statement where if these are zero up to a certain point, the first time they're nonzero, it's an isomorphism, and the next one is onto.

So how do we show it's onto in dimension three? I started proving this and the proof was harder than I thought. But maybe there's an easier proof.

Suppose you have a 3-cycle in some simply connected space. First I want to show theat this is homologous to a 3-sphere. We can take this cycle and pull simplices apart and so on to make this a 3-manifold. We could do it easily up to this dimension. Doing it the next time required the theorem that every 3-manifold is the boundary of a 4-manifold (Stiefel). We shouldn't need it but I'll use it. We have the 3-manifold mapping to the space, so I remove a ball from the four-manifold and get the preimage of a possible homology from my 3-manifold to the ball. Take a Morse function going from the three manifold to S^3. There are 1-handles, 2-handles, and 3-handles. A two handle looks like a thickened igloo. I can't draw a 3-handle; it's like a 1-handle in reverse. I can add an arc, so I can extend the map on a 1-handle. I'm just adding a solid torus 1-handle. Just draw an arc between them and squeeze this down. You can also extend over a two-handle for the same reason. You extend to a map of the disc, and extend that to an igloo.

Now we say, what is this? It's attaching 1-handles to the sphere. If you attach a 1-handle to the sphere you get $S^1 \times S^2$; a bunch of these gives you $\#_n S^1 \times S^2$. You can then extend this by a map of a specific kind of manifold in. We can do the same thing, pull the circle down to a point and eliminate it. So these circles in $S^1 \times S^2$ can be surgered away so you get the connect sum of a bunch of copies of S^3.

A general position statement says that you can, by a global perturbation, put all of the handles of the same dimension with one another, and order them by dimension, which I'm using. That's like lemma 0 of Morse theory.

Now let's go back and prove it's into. Onto is like an existence statement. Existence in $k+1$ implies uniqueness in dimension k, which is a universal thing that you see all the time. You apply existence to the $k+1$ dimensional equivalence between two k things. So the general yoga is existence in $k+1$ implies uniqueness in k.

You say, suppose I have two spheres mapping into the space that are homologous, so I find a 3-chain connecting them. Then I make this a manifold, apply the procedure of smoothing to make it a manifold, and then abstractly cap the spheres off; then I can get this as the boundary of a 4-manifold; In the cobordism I can run an arc betwen the caps I glued on, and cut out a neighborhood of an arc, and I do the same proof of the onto-ness staying away from the points on arcs. You do the previous proof, this looks like an old fashioned jukebox, right, these are like the lights, you have to stay away from the lights, you do the previous proof but stay away from these reasons.

So the into statement is more subtle than the onto statement at the same dimension and essentially the same as the onto statement at dimension one higher.

Now there's a nice construct that has enjoyed a lot of attention in the past thirty years. What am I going to do now? Right.

There's the following construction. Take a space X; suppose it has a complicated π_1, like $GL(n, \mathbb{Z})$. You can make such spaces. Actually, for any space there is such a group. You just use loops for generators and 2-cells for relations. If you abelianize this, you probably just get Z_2 for $n >> 2$. So let $K \subset \pi_1(X)$ be the kernel of $\pi_1 \to H_1$. So let $\alpha_1, \alpha_2, \cdots$, be normal generators of K. Then α and all of its conjugates generate K. If you set these equal to zero you. If you set one transposition equal to zero you kill the symmetric group.

So each α is in the kernel, so bounds, I think there's some interesting discussion for that group about how much genus you need. So I attach a cell to X. I attach a cap to kill α. In this new space, $\pi_1 = H_1$. Anything can be deformed off of these new cells. But in X if it's in K it's zero, and you can see that you haven't changed the homology, because you have killed something that was already dead in homology, like you killed it again, like, [stamps on ground twice]. What you have done was change H_2.

I know, I need more hypotheses, I had the wrong hypotheses, I need K, when you abelianize K, you get 0. So K is equal to its own commutator subgroup. That's called a perfect group. Suppose K is perfect, e.g., if $H_1(X) = 0$. If you take $SL(n, \mathbb{Z})$, that's an example of a perfect group. So then the elements here are also in K. I could write somebody in K as a product of commutators in K, so the loops in my bounding surfaces can be squeezed down and I can move this whole cycle to a sphere. Okay? I can deform that, since these are elements of K. I'll have to drag these over. Now, these are spheres, now, these new homology classes I've created. I kill them by attaching 3-cells, say with f_i. What is the homology of the new space? You had the old chain complex for the original space, and you attached a bunch of new generators in dimension two.

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\begin{array}{l}
X \;\; 3 \longrightarrow 2 \longrightarrow 1 \longrightarrow 0 \\
\\
f_1 \longrightarrow e_1 \\
\\
f_2 \longrightarrow e_2 \\
\\
\vdots \longrightarrow \vdots
\end{array}
$$

So you've added six generators in dimension three and six in dimension two which interact like $f_i \to e_i$, so the homology is all the same. So given X and $P \subset \pi_1(X)$ a perfect subgroup we can attach 2 and 3-cells to X to form $X \hookrightarrow^{+} X^{+}$, and $+$ is an isomorphism on homology in all degrees.

This is kind of a universal construction. Any other map that kills P and is an isomporphism on homology factors through this map.

As a special case, if $H_1(X) = 0$ take $P = \pi_1(X)$; then P is perfect. Then X^+ contains X and the homology has $\pi_2(X^+) = H_2(X),\ \pi_3(X^+) \geq H_3(X)$, and you can say that the homotopy groups of X^+ are new invariants of X.

For example, take $SL(n, \mathbb{Z})$ and take a space with this as fundamental group. If it has π_2, attach three cells to kill it, and so on. So you get a space with fundamental group π_1 but whose higher homotopy is zero. This has some significance and classifies bundles. You call this X, with $\pi_1(X) = SL(n, \mathbb{Z})$ and $\pi_n(X) = 0$. Then $\pi_i(X^+)$ for $i < n$ are the algebraic K-groups of Z. Modulo torsion you get a $\mathbb{Z}$ in dimensions 5,9,13, but the torsion is hard. You can do this for any ring and get the algebraic K-groups of R. This was discovered around 1970.

So homotopy theory kind of burst forth, and you're kind of deeply penetrating into algebra and arithmetic. There's another interesting example. Hurewicz defined these and Hopf surprised the world by showing that $\pi_2(S^3) \neq 0$, that this is $\mathbb{Z}$. So $S^{n+i} \to S^n$ is interesting for $0 < i < n$. These are canonical finite groups $\pi_{n+i}(S^n)$. It's not complete in any way. There's a whole field of this. It's a big subject. It turns out with the plus construction, just take another theorem these groups assemble as the homotopy groups, well, take a space such that $\pi_1(X) = S_N$ for N large and all of the higher homotopy groups are zero. Then one example is to take a large dimensional Euclidean space and take n particles. This space is connected, simply connected, take a hundred points, the topology is trivial up until a million. If you unlabel the points, dividing by the symmetric group, you're introducing an action. So if you take configurations of a hundred points in high Euclidean space, you get $\pi_1 = S_n$ and higher groups zero. Now $\pi_i(X^+)$, here the alternating group is your perfect group, simple is better than perfect. So the construction kills the alternating subgroups and you get X^+. So $\pi_i(X^+_{Alt}) = \pi_1$.

So that's another example. People who work on this use a lot of algebraic K-theory. The symmetric group is like $GL(n)$ of the field with one element which doesn't quite exist. This example maps into algebraic K-theory. It all comes out of playing with curves mapping into a space.

That's all.