EAST ASIAN SYMPLECTIC CONFERENCE

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1. Nov. 2: Huai-Liang Chang: Mixed-Spin-P field moduli and its virtual fundamental class

Thanks to the organizers for inviting me here. I will try to introduce this newly developed technique to approach the calculation of [unintelligible]in higher genus. Let me start from some history about this problem. In mirror symmetry, physics starts from this 2d sigma model. Witten tried to but supr symmetry and [unintelligible]with geometry. They perform the A and B twists. The classical version of mirror symmetry is to compare counts of holomorphic curves in a compact Calabi–Yau. On the B side it's something related to the period but something more mysterious in higher genus. Around 99, in genus zero, the A side, well Givental and LLY. This was done in 90 by Candelas et al. on the B side. In genus one, around 2009, Zieger–Li calculated this for the quintic. On the B-side, Vafa and other people did this around 91. BCOV did genus 2, and HQK extended to genus less than 51. The things on the B-side are not defined. On the A side, the path integral is known to be the Euler class of the following setup.

The talk today, I want to highlight this point. The true virtual phenomena appear in positive genus. I will try to define this. The way this appears is via the so-called ghost map, which is captured by the mixed Spin P moduli spaces, using FJRW theory. This is some framework I'll try to explain.

Let me start with a short definition of Gromov–Witten. You take the count of all curves of genus g and degree d to your quintic. We have

$$\begin{array}{c} \mathcal{E}^\infty = \sqcup_f \Omega_R^{0,1}(f^* T_Q) \\ \downarrow \\ \mathcal{M}^\infty = M_g^{sin}(Q,d) \end{array}$$

[some fast discussion] In algebraic geometry you can recover all your information by intersecting this cone with the zero section. This doesn't help us calculate the Gromov–Witten invariant at all.

Kontsevich's approach to calculating this is, look at the moduli of stable maps. This will be a subset of the stable maps to $\mathbb{P}^4$. From this point of view, this moduli is obtained by cutting out those things where $f_1^5 + \cdots + f_5^5 = 0$. In genus 0, this V is a bundle. If I write this as quintic sections, then we write $[\overline{\mathcal{M}}_g(Q,d)]^{vir}$ is recaptured as the Euler class of this model. This is only true in genus 0. This is the genus zero approach. Every part of this fails when the genus is 1. The main reason this fails is because of the following new sheaf. You pull back $\mathcal{O}(5)$ and get another sheaf over your moduli space, which is 0 for $g = 0$. There exists a stable map [picture]. Over a curve (that contracts a genus one component to a point) V'

is nonzero. Because it's not zero, V is not a bundle. So approximating a virtual fundamental class as an Euler class fails.

The observation is that if you have $f : C \to Q$, then you can write down the following sequence

$$0 \to H^0(C, T^Q) \to H^0(C, T_{\mathbb{P}^4}) \to H^0(C, \mathcal{O}(5)) \to H^1(C, T_Q) \to H^1(C, T_{\mathbb{P}^4}) \to H^1(C, \mathcal{O}(5)) \to 0$$

So the first entry is just the deformations of the curve in the quintic and in $\mathbb{P}^4$ (the fourth and fifth nonzero entries are obstructions in this moduli problem). Then $H^0(C, \mathcal{O}(5))$ is the obstructions of extension between Q and $\mathbb{P}^4$. So the final thing is a higher obstruction that isn't clear from the point of view of geometry. There's no longer a bundle. The f, we call it a ghost elliptic curve. Over the ghost you have a higher obstruction. We spent a long time trying to throw away the higher obstruction. The counting of curves in the quintic is governed by obsruction theory in X. Similarly for $\mathbb{P}^4$. In the higher genus case, the existence of these terms tells us that we sholud enlarge our moduli space, taking duals and adding to the moduli.

$$Y^p := \overline{\mathcal{M}}_g(\mathbb{P}^4, d)^P = \{f : C \to \mathbb{P}^4, p \in P(C, \omega_c \otimes f^*\mathcal{O}(-5))\}$$

This will be a moduli space over $\mathcal{D}_g = \{(C, L)\}$ which is notoriously big and hard but at least smooth. Then $Ob_{Y^p}/\mathcal{D}_g$ is $H^1(C, L)^{\oplus 5} \oplus H^1(\omega_c L^{-5})$.

Now talking about this obstruction, a bad part of this Y_P is that you get a non-compact moduli space.

[missed a bit]

The theorem is that you can use σ to do a perturbation and you get $[Y^p]^{vir}$ which is equal to $[M_g(Q, d)]^{vir}$ up to a sign. At that time we were very surprised that this kind of thing can happen. In genus zero, this is the Euler class of your vector bundle. Near the boundary, you have a preferred perturbation of your zero section. This is actually a typical phenomenon in so-called Landau–Ginzburg theory after you quantize on the A-side. That finite dimensional model admits a Hamiltonian Floer homology description, where the Hamiltonian is almost like that σ.

Now, we cannot calculate both, still, we cannot do many things. But if you restrict to the case where you have only maps of degree 0, then that P can be understood as, I should say $\overline{M}_g(\mathbb{P}^4, d)^P$ is "$\overline{M}_g(K_{\mathbb{P}^4}, d)$." Now using this point of view and the form of σ as, well, look at the superpotential [missed] then the critical locus is the quintic.

This is nothing but counting curves in this Landau–Ginzburg space. So $\overline{M}_g(Q, d)^{vir}$, up to sign is $[\overline{M}_g(\mathbb{P}^5, d)^P]^{vir}$. You can look in $\mathbb{C}^6/\mathbb{C}^*$ with weights $(1, 1, 1, 1, 1, -5)$. You can get an orbifold if $p \neq 0$, this is $\mathbb{C}^5/\mathbb{Z}_5$ with superpotential $W = \sum^5 x_i^5$. This is good because it's affine.

If you quantize everything here, then roughly speaking you'll get a curve $(C, L) \in \mathcal{D}_g$ along with sections $f_1, \ldots, f_5$ in $\Gamma(C, L)$ and $p \in \Gamma(c, \omega_c \otimes L^{-5})$. When $x \neq 0$ then the f_i have no common zeros and the p has no constraint. This is the moduli space we had on the left. If instead, p is nonzero and f_1 through f_5 have no constraint, then $L^{\otimes}5$ is isomorphic to ω_C, and then this isomorphism tells you you're taking the fifth roots of the canonical line bundle, and take five sections. This moduli is nothing but the moduli giving the FJRW theory. To make this possible you need your degree to be divisible by 5. To make freedom to make this possible, you let Cbe an orbi-Riemann surface.

Roughly speaking, if you allow orbi-Riemann surfaces with a lot of orbipoints and one additional numerical condition—because ω has [missed].

Now you have freedom. So this moduli, given the numerical condition, is written as $\overline{\mathcal{M}}^{\frac{1}{5}}_{g,m_1,\dots m_\ell}$ and you can come up with a virtual cycle, which is exactly the Floer homology with [unintelligible]. Since this is about singularities, you can calculate that the virtual dimension is $\sum(2-m_i)$ and if $g>0$ the only nonzero virtual cycle happens only in the case $\overline{M}^{\frac{1}{5}}_{g,2^a,1^b}$.

It's funny, this comes up in Witten's work in a different context.

This has virtual dimension, and the virtual dimension is zero for $[\overline{\mathcal{M}}^{\frac{1}{5}}_{g,2^a}]$. This is primitive FJRW for the quintic.

The problem is that there's no reason for these two numbers to be related, they are just stimulating each other to exist.

Here comes the great idea of [unintelligible]. He says let's quantize the Kähler parameter. He had this great idea for two years, and eventually I became convinced that this may help. What do I mean by quantize? Before you take all your maps to your target, [missed]

A mixed spin P field is a collection of objects $(C, L, N, \phi, \rho, \nu)$ where C, L are orbifold, N a line bundle, ϕ a section of $(C, L^{\oplus 5})$, ρ a section of $(c, \omega_c L^{-5})$ and $\nu = (\nu_1, \nu_2)$ a pair of sections of $(C, L \oplus N \oplus N)$ where (ϕ, ν_1) have no common zero and (ϕ, ν_2) have no common zero.

Now you impose one more serious condition, which is that this collection of data has no automorphism, $\operatorname{Aut}\xi < \infty$. This is a mapping space of curves to something called "variations of GIT." [too fast]

[pictures]

Somehow W is $[W_{g,\ell,d_0,d_\infty}]$ where d_0 is the degree of $L \otimes N$ and d_∞ the degree of N, and then you can take the virtual fundamental class, once you prove that $Ob_W \xrightarrow{\sigma} \mathcal{O}_W$ is proper, this is a very difficult, serious thing, due to W.-P. Li and J. Li.

A common technique is $\int_{[W]^{vir}} t^{rd\ W} = 0$. [pictures]

The consequence is that we can link the two invariants. Tracing the link you obtain the relation, and FJRW determines Gromov–Witten, in fixed genus. The consequence is that for genus g degree d Gromov–Witten, they are determined by the first [[unintelligible]] Gromov–Witten and $\theta_{g,k}$. So $[\mathcal{N}_{g,d}]_d$ for $g=2$ is determined only by $\theta_{g=2,2^2}$. There are no curves, this only uses $\mathcal{M}_{g,n}$ and the spin version. This number can be determined explicitly. It also shows that the FJRW invariant can be calculated in genus 1 no matter how many 2 you put.

Okay thanks.

2. Huijin Fan: Analytic theory of gauged linear sigma model

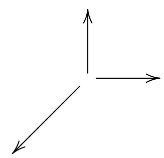
[I do not take notes at slide talks]

3. Hiroshi Iritani: Global mirror symmetry for toric stacks and its applications

We've been studying functoriality of quantum cohomology. The theme today, the content today, I've talk several times about. We have some small progress, gradually, and there are many things we haven't written. Today I'd like to concentrate on some small technical piece, a certain compactification.

When we are talking about mirror symmetry, classical cohomology is maybe already know. As it turns out, it is useful to have [unintelligible]in the picture, which gives a certain compactification of the mirror.

Let me start with an example. We consider $\mathbb{P}^2$, this is a fan of $\mathbb{P}^2$



Then the mirror is $W = c_1 x + c_2 y + \frac{c_3}{xy}$, which is a Laurent polynomial on $(\mathbb{C}^\times)^2$.

This is mirror in the following sense. If I rescale x and y then I get $x + y + \frac{q}{xy}$ where $q = c_1 c_2 c_3$. This is in $\mathbb{C}^\times$. They are mirror to each other in the following sense.

This q on this side for W, plays the role of the complex structure. The q corresponds to the Kähler parameter for $\mathbb{P}^2$, in $H^2(\mathbb{P}^2, \mathbb{C}^\times)$, the so-called Kähler parameter.

For instance, we have $QH^*(\mathbb{P}^2)$, some commutative algebra structure on $(H^*(\mathbb{P}^2), \star_q)$, which should be isomorphic to the Jacobian ring of W.

This Jacobian ring is $\mathbb{C}[x^\pm, y^\pm]/(\partial_x W, \partial_y W)$.

This is one instance of mirror symmetry. We also have a quantum connection of $\mathbb{P}^2$, also called the Dubrovin connection. I'm not going to go into the details, but in formulas, this is a connection on a one parameter family of the form $q\frac{\partial}{\partial q} + \frac{1}{z}(p \star_q)$, this is a trivila bundle over $\mathbb{C}^\times$ equipped with this connection. This quantum connection should be isomorphic to, let me call it Saito theory, some singularity theory with connection associated to W. Here I can write this as $H^2(\Omega_{(\mathbb{C}^\times)^2}[z], zd + dW\wedge)$ where z is the same variable and this is algebraic differential forms and we take cohomology. This symmetry is somehow known, an almost easy computation.

I should also say that the corresponding connection is the Gauss–Manin connection.

In this case it is very natural how to compactify this picture. What I want to add is [unintelligible]. I just want to add some compactification, I want a partial compactification. I just write $w_1 = x$ and $w_2 = y$ and $w_3 = \frac{q}{xy}$. Then the Landau–Ginzburg model should be compactified in the following way. I have

$$\begin{tikzcd} \mathbb{C}^3 = \{(w_1, w_2, w_3)\} \arrow[r, "W"] \arrow[d] & \mathbb{C} \\ \mathbb{C}_q & \end{tikzcd}$$

and let me look at the fiber at $q = 0$. Then in this fiber, we see the singular variety, $W = w_1 + w_2 + w_3$ on the variety $w_1 w_2 w_3 = 0$. Then we want to study the Jacobian ring on this variety. We want to give this a log structure. I want to consider the logarithmic vector field tangent to this variety. We consider the following. Here the logarithmic tangent sheaf is generated by $\langle w_i \frac{\partial}{\partial w_i} - w_j \frac{\partial}{\partial w_j} \rangle$ and what we get, in this case by an elementary calculation, is

$$Jac(W) = \mathbb{C}[w_1, w_2, w_3]/(w_1 w_2 w_3, w_i - w_j)$$

which is $\mathbb{C}[p]/p^3$ which is the classical comohology of $\mathbb{P}^2$.

This is true in the quantum level but I want to get some nice compactification.

This example has been generalized to weighted projective space by de Gregorio–Mann. They consider weighted projective space. This example is very simple, but if you consider weighted projective space, for instance $\mathbb{P}(1,1,2)$, which is $\mathbb{C}^3\backslash 0/(\lambda,\lambda,\lambda^2)$. Then we consider

$$\begin{tikzcd} \mathbb{C}^3 = \{(w_1,w_2,w_3)\} \arrow[r, "W=w_1+w_2+w_3"] \arrow[d, "q=w_1w_2w_3^2"'] & \mathbb{C} \\ \mathbb{C}_q & \end{tikzcd}$$

If you do the same computation you don't get the correct answer.

In this case the log tangent sheaf is generated by $w_1\frac{\partial}{\partial w_1} - w_2\frac{\partial}{\partial w_2}$ and $2w_1\frac{\partial}{\partial w_1} - w_3\frac{\partial}{\partial w_3}$ Then

$$Jac(W) = \mathbb{C}[w_1,w_2,w_3]/w_1w_2w_3^2, w_1-w_2, 2w_1-w_3$$

and this is $\mathbb{C}[p]/p^4 \ncong H^*_{CR}(\mathbb{P}(1,1,2))$.

The solution to this discrepancy due to de Gregorio and Mann is the following. The correct mirror, we should compactify it as a stack. This is just a variety compactification. We should pull back the mirror family by $q = t^2$, and then I get $\{w_1,w_2,w_3,t)|w_1w_2w_3^2 = t^2\}$ and then take a normalization, so that I add a rational function t/w_3, called u, nad what I get is

$$\{w_1w_2, w_3, u|w_1w_2 = u^2\}$$

This is equipped with a similar projection to the t plane and by W. If you consider this, maybe I omit the calculation, you get the correct cohomology. Indeed, you should think of this as an orbifold. There is a $\mathbb{Z}/2\mathbb{Z}$ action on this model because I take a square root of q, where $t\mapsto -t$ and $u\mapsto -u$. If I take the quotient in the coarse moduli space I recover the picture that gave the wrong answer. We should think of this thing as an orbifold, as a stack.

Somehow we want to generalize to something more general than toric stacks and then you get the following. I'm not going into the details of the construction. Let $N=\mathbb{Z}^n$. In general one can allow torsion, but for simplicity let me have this be the fan lattice. I fix a finite set S inside N, arbitrarily big. In the $\mathbb{P}^2$ case I start with $\mathbb{Z}^2$ and $(1,0)$, $(0,1)$, and $(-1,-1)$. But I could add irrelevant ghost vectors. Then I can cook up the following abstract sequence

$$0\to\mathbb{L}\to\mathbb{Z}^S\xrightarrow{\beta}N\to 0$$

The map $\mathbb{Z}^S\to N$ is the tautological map, and $\mathbb{L}$ is defined to be the kernel of β.

We consider the dual sequence, which is

$$0\to N^*\to(\mathbb{Z}^S)^*\to\mathbb{L}^*\to 0$$

and then we construct a Landau–Ginzburg model by tensoring $\mathbb{C}^\times$ by the second and third term. So what I get is $(\mathbb{C}^\times)^S$ mapping to $\mathbb{L}^*\otimes\mathbb{C}^\times$ and also equipped with a map to W by summing variables over S.

This model should be partially compactified. How can I do this? At the level of varieties this is a standard procedure. The partial compactification, the so-called "secondary toric variety" first considered by Gelfand–Kapranov–Zelevinsky and appears in recent work of Demer–Katzarkov–Kerr.

They work in a slightly special situation where the finite set S lies in some hyperplane of distance 1 from the origin. There's also some stacky version considered here. It seems slightly different from theirs but I'm not completely sure.

We can take a stacky compactification $(\mathbb{C}^\times)^S \subset Y$ and $\mathbb{L}^* \otimes \mathbb{C}^\times \subset M$ with $Y \to M$, both of these singular toric DM stacks, some singular orbifolds.

So if you have a simplicial fan Σ such that $\Sigma(1)$ is in $\mathbb{R}_{\geq 0} b$ for $b \in S$, then we can consider every simplicial fan such that the set of one dimensional cones [unintelligible]. This corresponds to some toric chart $U_\Sigma \subset \mathcal{M}$ such that 0corresponds to, this is $q = 0$, the large radius limit. This is an affine chart. Let me give the following example.

Let $N = \mathbb{Z}^2$.

(1) Let S be three points $(-1,1)$, $(0,1)$, and $(1,1)$. I have two charts. I have two cones and the other chart we have is this one [pictures].

In terms of geometry this corresponds to $\mathcal{O}(-2)$ over $\mathbb{P}^1$ and the other one to $\mathbb{C}^2/(\mathbb{Z}/2)$.

This is the first example.

Let me write the corresponding compactified Landau–Ginzburg model. I consider these points as 1, 3, and 2, then I have

$$\{(w_1, w_2, w_3) | q w_3^2 = w_1 w_2\}$$

over $M = \mathbb{P}(1,2)$. The other chart is around $q = \infty$, the first is around $q = 0$. The $q = \infty$ part is the orbifold point.

This example I have a crepant resolution.

(2) The other example is, I have three vectors $(0,1)$, $(1,1)$, and $(1,0)$. I can consider these two fans [pictures].

I have (sorry if it's similar)

$$\{(w_1, w_2, w_3) | q w_3 = w_1 w_2\}$$

over $\mathbb{P}^1$, non-stacky $\mathbb{P}^1$, with coordinate q. Again $q = 0$ is the first chart and $q = \infty$ the second one. This is a discrepant resolution from $\mathbb{C}^2$ blown up at the origin to $\mathbb{C}^2$.

In general it's much more complicated.

What we can show in this setting is that

Theorem 3.1. *Coates–Corti–I–Tseng (in preparation) The big quantum connection of $\mathcal{X}_\Sigma$ is isomorphic to the Saito theory associated to W restricted to the formal neighborhood of 0 inside U_Σ.*

We can consider a formal version of Saito theory for this.

We also have a correspondence between the Poincaré pairing and the higher k-residue pairing.

This gives you a way to compare quantum cohomology for toric stacks related by these transformations.

I need some convergence to say things about these crepant [unintelligible]. The new thing here is the big quantum connection, first of all, and the big quantum cohomology, I take redundant vectors which correspond to big quantum cohomology.

Maybe I'll state this as a conjecture but it's probably true. When we work over $\mathbb{C}[[z]]$, then the mirror isomorphism extends to an analytic neighborhood of $0 \in U_\Sigma$.

Then we have some chance to compare the two things. Maybe just look at the Jacobian ring before comparing the quantum connections themselves. For the Jacobian rings, this is similar to the work of González–Woodward. They consider the toric minimal model program and how quantum cohomology changes under toric mmp. Their result can be understood more easily in terms of this compactification.

If we consider the family of Jacobian rings $Jac(W_q)$ over $\mathcal{M}$, then the previous examples,

(1) (crepant case), we have the following. The Jacobian ring, this is over $\mathcal{M} = \mathbb{P}(1,2)$, the dimension of the cohomology is the same, they're two dimensional, and I have a double point. I'm doing the spectrum of the Jacobian ring. There is some singularity at $q = \frac{1}{4}$ where the Jacobian ring is infinite dimensional.
(2) discrepant case, you have over $\mathcal{M} = \mathbb{P}^1$, if I calculate the ring, we have a branch that just diverges at ∞. In some sense this is expected, we have a $\mathbb{C}^\times$ action, the Euler field acts trivially in the crepant case but non-trivially here. A non-constant section should then always be like this, depending on how the Euler field acts on $\mathbb{P}^1$, is it positive or negative?

We consider some toric curve $\mathcal{C}$ inside $\mathcal{M}$ in general, which connects two toric stacks $\mathcal{X}_1$ and $\mathcal{X}_2$, and you either see the crepant or discrepant picture.

In the crepant case, we have the following. In this result we just work, we don't know the answer to the conjecture, but we're almost close. First I'll state the theorem.

Theorem 3.2. *(Coates–I.–Jiang) In the crepant case, you can upgrade this to a quantum connection, $QConn(\mathcal{X}_1) \cong QConn(\mathcal{X}_2)$ under analytic continuation across some formal neighborhood of this curve $\widehat{\mathcal{C}}$ inside $\mathcal{M}$.*

Moreover, in this theorem, we didn't use mirror symmetry, this is compatible with derived equivalence. The vertical map is called the Γ-structure. In the crepant case, in the discrepant case, what we have, the corollary of the previous conjecture and the mirror theorem is the following. I assume some convergence, then we can say that

Corollary 3.3. $\overline{QConn(\mathcal{X}_1)}$ *is a direct summand of* $\overline{QConn(\mathcal{X}_2)}$ *on a neighborhood of* $\mathcal{C}$.

Here $\bar{\ }$ means I tensor over $\mathcal{C}[z]$ with $\mathcal{C}[[z]]$.

Maybe I finish my talk by considering a non-completed version of this. We make the following conjecture. There is some Stokes phenomenon, this quantum connection has irregular [unintelligible], but if you choose some sector, an angular sector I, then there exists an analytic lift of the following shape $QConn(\mathcal{X}_1)|_I \cong L \oplus QConn(\mathcal{X}_2)|_I \oplus R$ (where $arg\ z$ is in I). This is a lift of this direct summand relation which corresponds to a semiorthogonal decomposition of the derived category of coherent sheaves $D^b_{coh}(\mathcal{X}_1) = \langle L, D^b_{coh}(\mathcal{X}_2), R \rangle$.

We have some evidence for this, for example from the gamma conjecture. If you have a toric morphism $\mathcal{X}_1 \to \mathcal{X}_2$, not just a birational map, then there is some evidence, this should be identified with the [unintelligible]pullback. Maybe I'll stop here.

4. Miles Reid: Ice cream and orbifold Riemann–Roch

$$\sum_{i=0}^{\infty}\left[\frac{3n}{7}\right]t^i = \frac{t^3+t^5+t^7}{(1-t)(1-t^7)}$$

This is intended to be a very elementary talk. I've written down this very obvious formula. I'm taking the integral part, when $i=0$ I get 0, when $i=1$ I get 0, and then when I get to 3, I get an integer solution, $1t^3+1t^4+2t^5+2t^6+3t^7+\cdots$ and it continues in a periodic way. There are jumps at 3, 5, and 7 modulo 7. The $(1-t)$ bit tells you to look at the jumps. The $(1-t^7)$ tells you this is periodic with period seven. This quantity $t^3+t^5+t^7$ has some structure. This tells you a lot of things about orbifold Riemann–Roch.

You can take this $t^3+t^5+t^7$ and think about it as, if you are allowed to change, on day 1 we'd get $\frac{3}{7}$, on day 2 you'd get $\frac{6}{7}$, I'm losing $-\frac{3}{7}$, $-\frac{6}{7}$, $-\frac{2}{7}$, $-\frac{5}{7}$, $\frac{1}{7}$, $\frac{-4}{7}$, and 0.

The point I want to make, this list of numbers here, this is some kind of Dedekind thing. This is palindromic, and it's centered at 5, this $t^3+t^5+t^7$. What's 5 got to do with it? The number we get here, $\frac{15}{7}$, this is $\frac{1}{7}$, so 5 is InverseMod$(7,3)$. Why would you want this?

Let's say I have an elliptic curve and I want a polarizing divisor, I'll write $A=\frac{3}{7}P$, I'll allow a rational function to have $\frac{3}{7}$ of a pole. If I write the section ring $R(E,A)$, this is an easy bit, and I add $\frac{t^3+t^5+t^7}{(1-t)(1-t^7)}$, this corresponds to E being a curve of degree 15 in the weighted projective space $\mathbb{P}^2(1,5,7)$. I'm having slightly more complicated weighted projective spaces than in the last talk. You're supposed to count three things, since this is a plane, and I want to multiply to make it of degree 0, and that's -13. So $K_E=2A$ where $2=(15-13)$, the elliptic curve I'm talking about has a point on it p which is of type $\frac{1}{7}(5)$. This is the same 5. My orbifold points are polarized, $\frac{1}{7}(5)$ is the same as $\frac{1}{7}(1)$ or anything else, but these are polarized, so I've specified which one is which.

There's another example, where $E'=E(9,10)\subset\mathbb{P}(2,3,5,7)$. If you get bored, you can calculate a similar thing for this example.

When I have a variety X and a divisor D. Normally [unintelligible]. There's something called $R(X,D)$, the "section ring" which is $\bigoplus_{m=0}^{\infty}H^0(X,\mathcal{O}_X(mD))$. You want to calculate this ring by generators and relations. This has a Hilbert series. Each vector space is finite dimensional. You write $P_n(X)=dimH^0(mD)$, and $P_{X,D}(t)=\sum P_m(t)t^m$. If D is Cartier and ample then P is polynomial divided by $(1-t)^{n+1}$. You can get this formula if you know Riemann–Roch but this comes much earlier.

My case is that X is an orbifold and I'm only interested in cyclic quotients, isolated cyclic quotient points.

I'm going to use this notation $\frac{1}{r}(a_1\cdots a_n)$, where this is [unintelligible].

In this case there's a wonderful theorem. I need to say one or two more words. I told you what happens if D is Cartier and ample. I want that $K_X=k_xA$, for A the polarizing divisor.

So then $\chi\mathcal{O}_x(mA)=\chi\mathcal{O}_Xk_xmA$ up to a sign $(-1)^{\dim X}$. If you know Riemann–Roch or Serre duality, this is Serre duality.

The ring $R(X, D)$ is a Gorenstein ring. I won't give the definition of this, but the way to think about this is, a hypersurface, codimension 2, any complete intersection is okay. It's like [pictures].

And this condition is sort of like homology sphere. To put it another way, lots and lots of cohomology groups [unintelligible]. Anyway, what I want to say is that $R(X, D)$ is Gorenstein of weight k_X, the same k_X. This is an internal property, then $D(X, K_X)$ has this symmetry,

$$t^k P_x([unintelligible]) = (-1)^{[unintelligible]} P_X(t)$$

So the result is that if X is an orbifold and $R(X, D)$ is Gorenstein, then the Hilbert series $P_X(t)$ is the sum of terms, $A_t/(1-t)^{n+1}$, exactly what happens with an ample Cartier divisor, plus a sum over the orbifold points,

$$P_{orb}(\frac{1}{r}(a_1, \dots, a_n), k_A)$$

Let me say what's happening. So $A(t)$ is the integral of a palindromic polynomial of degree c which is $k_X + n + 1$. This c is called the coindex, a term introduced by Mukai, for instance, projective space has coindex 0. A nonsingular quadrant has coindex 1.

If $c < 0$ then $A(t)$ is 0. Otherwise this is $1 + a_1 t + \cdots + a_1 t^{c-1} + t^c$. This is then determined by the first $[c/2]$ coefficients. Now $\mathcal{B}$, the orbifold points, I'm writing a whole basket of them, not giving them special names. Now P_{orb} has as its denominator $(1-t)^n(1-t^r)$ and numerator $B(t)$, with $X(t)$ a palindromic integral polynomial in the interval $[c/2]+1, \dots [c/2+n-1]$. It's InverseMod$(\frac{1-t^r}{1-t}, \prod \frac{1-t^{a_i}}{1-t})$.

If I only have one of these terms, I'm doing ice cream. This is something you can do by hand, it's written out explicitly so I can do it by computer.

Let me give you a kind of family of examples. So for $S = \mathbb{P}(1, 1, 3)$, then $A = \mathcal{O}(5)$. The coordinates here are u_1, u_2, and v. The u_1 and u_2 are in [unintelligible]. I'll get degree 5 symmetric monomials in u_1 and u_2 and degree 2 symmetric monomials in these times v. I'm writing down its generators in degree 1, these are them, there are nine of them in $H^0(S, A)$. In degree 10 for $H^0(S, 2A)$, I've got $S^{10}(u_1, u_2)$, and at the end I get $u_1 u_2 v^3$. In degree $H^0(S, 3A)$. I'll call these guys y_1 and y_2 and call v^5 by z. Then P is the cone and $z(P)$ is nonzero so I can set it to 1 to normalize. Then y_1 and y_2 are orbinates at P. If I want to see the point P close up, I need to take $y_1/z^{[unintelligible]}$, have I got that right? Something like that.

So this means that this point here is something like $\mathbb{C}^2/\frac{1}{3}(1, 1)$, and y_1 and y_2 get an ϵ^2. According to my formula, there is an expression, which is $\frac{1+6t+t^2}{(1-t)^3} + \frac{t^2}{(1-t)^2(1-t^3)}$. I want to blow S up at one or two or more poitns, at d general points, where $d = 1, 2, 3, \dots 8$, so I replace 6 with $6 - d$. When $d = 0$, the first thing I want to know with a projective variety, what's the dimension? It's three, the number of factors in the bottom. I sum the coefficients in the numerator to get the degree, it's $8 + \frac{1}{3}$, so it's $\frac{25}{3}$. This variety blown up d times is a log del Pezzo surface, and the case $d = 8$, the most complicated one, this is $S(10)$ in $\mathbb{P}(1, 2, 3, 5)$ if I've got it right. The Hilbert series is $\frac{1-t^{10}}{(1-t)(1-t^2)(1-t^3)(1-t^5)}$. This is counting the number of homogeneous polynomials in each degree. The other part is [unintelligible].

So [unintelligible]and Liana [unintelligible]have classified all log del Pezzo surfaces with [unintelligible]. I've told you nine families, the others mostly belong to similar cascades.

What time did I start? Half past? So if I want to talk about the proof of this, when we make databases of Fano three-folds, they were using this technology. [missed a little]

Let me try to get the flavor of the proof. This is reduction to global quotient case. If I have this surface $\mathbb{P}(1,1,3)$, then locally at this point I can take a third root, I get a cyclic cover. It'll be ramified at the divisor $z=0$, and then there's a local analysis to say that the function P_{orb} exists and in the local quotient case, this depends only on $\frac{1}{r}(a_1,\ldots a_n), k_X$. In the global quotient case, we use this thing called Lefschetz, holomorphic Lefschetz fixed points. Probably the last one of these is sort of the easiest to explain. If I've got a variety Y, and I take a group, divide out by $\mathbb{Z}/r$, I write $X/\mathbb{Z}/r$. If I do this map π, if I take $\pi_*(\mathcal{O}_Y)$, then the invariants, this has the action of μ_r, this breaks into eigenspaces, $\bigoplus \mathcal{L}_i$. The fixed point formula says if I have an equivariant sheaf $\mathcal{F}$ on Y, then I can ask for the group g^* acting on the cohomology of $\mathcal{F}$, $H^i(Y,\mathcal{F})$. That's the same as a certain sum of the element g acting in $\bigoplus \mathcal{L}_i$. If I take the alternating sum of these, the trace of these, this is called the Lefschetz number $L(g,\mathcal{F})$. Then any formula of this kind lies in $ch.Td$. In our case $\mathcal{F}=\mathcal{O}$. We imagine $\frac{\epsilon^i}{\det(1-g^*|_{T_p})}$.

Let me try to say this in a kind of simpleminded way. The proof of this is in our papers. The way we refer to this, we are referring to Atiyah Singer and Segal.

Maybe I'll explain something more basic. The thing here is a gorup acting on [unintelligible], the fixed point theorem means that it localizes to poreperties of $\mathcal{F}$ at the fixed points of the group action. This group here, I made the kind of restriction at the very beginning, the only fixed points are isolated fixed points. If you look at the group, the trivial element fixes everything. Then this formula is acting for $\chi(\mathcal{O}_Y)$ if $g=\mathrm{id}$ and it's a local function of $\frac{1}{r}(a_1,\ldots a_n)$ at $g\neq \mathrm{id}$. When the group is acting onthe tangent space, a basis is gien by $\frac{\partial}{\partial x_i}$ and you get $det(1-\epsilon^{-a_i})$.

I don't want to know $L(g,\mathcal{F})$, I want $\chi(\mathcal{L}_i)$. The way I'm thinking about this problem, there are r of these sheaves $\mathcal{L}_i$. These are r-points satisfying r linear equations. One of them is $\sum\chi(\mathcal{L}_i)$ is $\chi\mathcal{O}_X$. The other is, I have no idea what I'm supposed to be writing here, something like $\sum\epsilon\chi(\mathcal{L}_i)=$, let me not try to write that side.

This is written out explicitly as a Vandermonde matrix, and the other side is a column vector. Then this gives a formula for $\chi(\mathcal{L}_i)-\frac{1}{r}\chi(\mathcal{O}_Y)$. These are Dedekind sums. They involve taking the index. This means that [unintelligible].

The local analysis, let me say reduction to the [unintelligible]is easy to say, I can make a proof on the spot in two minutes that [unintelligible], I could make it [unintelligible].

For example, suppose I have $\frac{1}{3}(2,2)$. Let me resolve the singularity, I'll see a -3 curve, and there's a discrepancy functor, so I have K_S is globally, well, in any case, if T is the minimal resolution then $K_T=K_S-\frac{1}{3}E$. The thing is f^*K_S, most algebraic geometers of my generation resolve singularities. We resolve the singularity, $f^*(K_S)=K_T+\frac{1}{3}E$. Then when I try to calculate $H^0(m-f^*K_S)$, I get $(-mK_T-\frac{2m}{3}E)$ which is $-(K_T+E)+\frac{m}{3}E$. This one here is nef and big so I can apply Riemann–Roch. What does it mean to apply Riemann–Roch to $\frac{m}{3}E$. If a rational function has poles, it has only integral value poles. You round down.

The moral is that by resolving the singularity you get an expression, evaluated by Riemann–Roch.

5. Nov. 3: Viktor Ginzburg: Non-contractible periodic orbits in Hamiltonian dynamics on closed symplectic manifolds

Well, thank you, it's a pleasure to be here, it's not my first time in Hong Kong but it's my first time on this campus and I'm enjoying it enormously.

This subject hasn't been studied much, for good reason, that I'm going to explain. I'll say that there is something there to talk about. Most of what I'm going to say is joint work with B. Gürel.

Of course non-contractible periodic orbits have been studied by many people [list of names] and the usual logic is that you take Floer theoretic invariant and calculate it for a certain Hamiltonian and find it's nonzero, so it's nonzero by invariance for any other Hamiltonian in the class you're interested in. The invariant does not have to be the full Floer homology. It could be the torsion or something else, but it's an invariant.

If you take a Hamiltonian diffeomorphism on a closed manifold there need not be any non-contractible periodic orbits so anything you choose will be zero.

I'll use Floer homology indiscrimately. It's a complex generated by the fixed points of one-periodic orbits. The homology of this complex is the Floer homology. When you look at, when you are interested in k-periodic orbits, they look like this. [picture]. These are the fixed points of φ^k where k happens to be 3 in this case. Again, k-periodic orbits generate a complex, and the homology of the complex is the Floer homology.

This is essentially an invariant of φ and the homotopy class of an orbit. If you have a Hamiltonian diffeomorphism without a periodic orbit in a homotopy class then these invariants are all zero. And you can always find a Hamiltonian diffeomorphism with *no* non-contractible periodic orbits.

Okay, so I'll start with a symplectic manifold (M, ω) of dimension $2n$. I denote by $\tilde{\pi}_1(M)$ the maps $[S^1, M] = \pi_1/\text{conjugation}$. I'll call something in this $[[x]]$ or f, and its homology class is $[x] \in H_1(M, \mathbb{Z})/\text{Tors}$.

Now I have a diffeomorphism $\varphi_H : M \to M$ which is given by a 1-periodic Hamiltonian $H : M \to \mathbb{R}$. I'll denote by $\mathcal{P}_k(\varphi, f)$, the k-periodic orbits in $f \in \tilde{\pi}_1$. The loop you traverse is in the class f. We'll say $P(\varphi, f)$ is just all periodic orbits in f.

In one hand we're talking about a map, and in the other about an isotopy. To say that an orbit is in f I need a continuous orbit. Given the end map φ, you can include it in many homotopies. But the free homotopy class is determined by φ, this is a consequence of [unintelligible][the Arnold conjecture?]

So $\mathcal{P}(\varphi, f)$ are the guys I'm interested in.

There's a general principle, more important than anything else I'm going to say.

If φ has more than necessary periodic orbits or orbits that don't have to be there, then the total number of periodic orbits of φ is infinite.

This general principle goes back to Hofer and Zander. In this more general form it's due to Gurel. This is based entirely on a low-dimensional result that is called Frank's Theorem. Let me recall what it says. It says, well, if you have a Hamiltonian diffeomorphisms φ on the 2-sphere, then the minimum necessary number of fixed points is 2. This was proved by [unintelligible]in the early 1970s.

If you have φ and the number of fixed points is greater than 2, then the number of periodic orbits is infinite. This was proved in the 1980s, but I think the only complete symplectic proof took a very long time to compute. So until recently this was the sole evidence for this principle. This calls for an idea of how to prove it. You should look at the complex and how it looks under iterations, somehow this hasn't taken anyone anywhere. If you take an orbit with some specific features, then the features of the orbit can be used to prove something.

Coming back to the problem in question, I'm talking about non-contractible periodic orbits. We don't need to have any, but we assume we have 1, then by the principle we have infinitely many.

Theorem 5.1. *(Gurel–G) Let M be atoroidal, this means that $\int_{T^2\to M}\omega = 0$. For instance any surface with genus at least two.*

Assume φ has $x \in P_1$ with $[x] \neq 0$. I'll let the free homotopy class of x be denoted f and it's not 1.

The assertion is that this principle is correct and indeed, for every large prime p there exists a simple periodic orbit in the class f^p. Its period is either p or p', the next prime.

When we talk about periodic orbits, there are two types I want to consider. They could be simple or iterated. If you go around a circle twice, that's iterated.

One extra remark, moreover, if π_1 is hyperbolic and torsion free, then the condition that $[x] \neq 0$ can be replaced by the condition $[[x]] \neq 1$.

Let me discuss a little bit the hyptheses of this.

What manifolds do I know that fit here? Examples. Well, surfaces of genus greater than or equal to 2, Kähler manifolds with sectional curvature negative, and some others that are neither of these classes. How likely am I to have a Hamiltonian diffeomorphism with one noncontractible periodic orbit? I'd say this. Given a free homotopy class, there is always a Hamiltonian diffeomorphism with a 1-periodic orbit in this class.

I really need x to be nondegenerate. I really want P_1 to be finite (otherwise your set is already infinite and there's nothing to talk about).

I want to say, given f there is always φ with $[[x]]$ in f, and such φ form an open set. So at least for any M, this class is nonempty.

Conjecture 5.2. *Assume that $\tilde{\pi}_1$ or H_1 is large enough. Then φ with noncontracible orbits form a C^∞ Baire second category set. Generically you have a noncontractible periodic orbit.*

The only manifold for which this is known is the 2-torus. This uses several nontrivial results in low dimensional topology.

I don't even know where to start to prove this for the n-dimensional torus.

The theorem and the other results I'm not going to talk about do not apply to the n-torus.

Conjecture 5.3. *The theorem holds for the $2n$-dimensional torus.*

It's currently known (and easy now) for T^2.

Let's talk a little bit about generic existence. For a free homotopy class f, look at the collection of Hamiltonian diffeomorphisms with at least one non-degenerate periodic orbit in the class f.

Inside it, I have a collection of Hamiltonian diffeomorphisms with infinitely many periodic orbits. I'll specify a collection of free homotopy classes, $f^{\mathbb{N}}$, that's the collection of powers of f.

Theorem 5.4. *I'll assume that no power of f is equal to 1. For example the homology class of f is nonzero. Then the set $\mathcal{F}_f^\infty = \{\langle\varphi|\#\mathbb{P}(\varphi, f^{\mathbb{N}}) = \infty\}$ is C^∞ second Baire in $\mathcal{F}_f$.*

Within this set, almost everything has infinitely many.

This is actually easy. The way the proof goes, Floer theory, to guess that—the Floer homology of $\varphi^k, f^{\mathbb{N}}$ is zero, which implies, basically, that there is a non-contractible non-hyperbolic orbit. It's known that when you have an orbit like this, you have infinitely many periodic orbits. This is called Birkhoff–Lewis–Moser.

One point here is that til recently, maybe five or ten years ago, people in symplectic topology tended to ignore C^∞ generic questions. It's useful to apply symplectic topology to these questions, you can often see quite a lot.

Maybe I want to say, state another theorem, a bit on the technical side, but, it slightly generalizes the first theorem, I want to go beyond the class of atoroidal manifolds. It's like going to monotone manifolds, I'll go to atoroidal monotone manifolds. These are manifolds where the integral of ω over any torus is λ times the first Chern class applied to that torus.

This is similar to monotonicity, except I used tori instead of spheres.

When this is true I can look at N_T, the minimum toroidal Chern class. This is simply the positive generator, this is determined by the condition that the group, the subgroup of $\mathbb{Z}$ generated by all such integrals, is $N_T\mathbb{Z}$.

Theorem 5.5. *Let M be toroidally monotone and assume that $N_T > \frac{4}{2} + 1$. Then everything else is as in the first theorem, excpt that φ has a hyerbolic orbit x with nonzero homology class.*

This is again along the lines of the general principle, instead of counting orbits, I'm specifying a type of orbit.

In some sense the mechanics of this proof is totally different from the generic existence proof. Here I'm using a hyperbolic orbit nad there I used a non-hyperbolic orbit.

Before I try to explain how the proofs work, let me say that the second theorem combined with the first conjecture should tell you that if you take a manifold with big enough π_1, you get infinitely many periodic orbit (generically).

Now a word about the proofs. I'll focus on the first theorem.

First, for open manifolds, how do people use Floer homology? If you can calculate some Floer theoretic invariant and show that it's nonzero, then it's nonzero for any choice of Hamiltonian. The idea here is different. The first guy here, x, is a seed that generates infinitely many periodic orbits. In contrast with [unintelligible]these don't have [unintelligible], you have to run your flow many times before x develops [unintelligible].

If I take the Floer homology filtered by the action, $HF^f(\varphi^k)$, for k large, this will look more or less like the Morse cohomology of $f(x) + \epsilon \sin(\omega x)$. So we think ϵ is fixed and ω goes to ∞. There are two parts of this, f is responsible for invariants and things, and the noise is responsible for infitely many periodic orbits, the general principle.

I'm not going to talk about the proof. So maybe I'll just finish early. In contrast to other approaches, the noise part is what we're trying to track down in the proofs.

This brings up sort of one connection I wanted to mention.

When you look at the proofs, which I have not talked about, they look awfully sort of similar to what people do in bar codes and persistent homology. Recently there have been a couple of papers in symplectic topology using these techniques, one by Polterovich–Scheluknin and the other by Usher–Zhang. It would be interesting to see if this could be improved. The bar codes also focus on the stable part, while what we need is the noise part. Maybe some sort of statistical machinery is what is needed.

The second part I want to bring up, of course for filtered Floer homlogy, you can take various filtered invariants, like the filtered Euler characteristic or filtered torsion, and see how that part changes under iteration. It's interesting to wonder if you can use these tools to prove the theorem or the conjecture for the torus.

There's one more point, about Floer homology of symplectomorphisms. I have a student who proved one theorem about noncontractible periodic orbits of symplectomorphisms. There are clearly parallels. That's probably pretty much what I wanted to tell you today.

6. Suguru Ishikawa: Spectral invariants of distance functions

Thank you for inviting me. I'm interested in the superheavy set in a closed symplectic manifold. This notion was introduced by Entov. This is defined using spectral invariants.

Let (M,ω) be a closed symplectic manifold of dimension $2n$. Let $\Omega_0 M$ be the space of contractible loops in M. Let $\widetilde{\Omega}_0$ be the standard covering space, which is (X,U) where x is a loop and u a map from $D\to M$ whose restriction to the boundary is x. The pairs (x,u) and (y,v) are equivalent if $x=y$ and $x-y$ is in the kernel of c_1.

Let H be a Hamiltonian and X_H its time-dependent vector field, $\iota_X\omega=-dH$. Let φ_t^H its flow. We say H is nondegenerate if and only if, for every contractible closed orbit of H, the linearization does not have eigenvalue 1.

For $\Phi:[0,1]\to Sp(2n)$, with $\Phi(0)=\mathrm{id}$, the Conley–Zehnder index is the intersection number with $\{A\in Sp(2n)|\ker(A-1)\neq 0\}$. This is a half-integer, since the intersection at the endpoints counts as a half. We say Φ is nondegenerate if and only if $\ker(\Phi(1)-1)=0$. In this case, then the Conley–Zehnder index is an integer.

The action A_H of the Hamiltonian H is a map $\widetilde{\Omega_0 M}\to\mathbb{R}$ defined by

$$A_H(x,u)=-\int_D u^*\omega+\int_0^1 H(x(t),t)dt.$$

If H is nondegenerate, we define $\operatorname{Spec}_k H$ as the critical values (x,u) of A_H such that the critical point, the Conley–Zehnder index is $-k$.

My convention is that the degree of Floer homology is the negative of the Conley–Zehnder index so that this coincides with the degree of Floer homology.

Floer homology is, roughly speaking, a chain complex C_k generated by the critical points of A_H and the differential is defined by counting. It is well known that this is actually a chain complex. For a precise definition we should use the Novikov ring, but this is a rough definition to explain the spectral invariant. This is actually a chain complex, and its homology group $HF(H, J) \cong \ker \partial / \mathrm{im} \partial$ is naturally isomorphic to $QH(M, \omega)$.

A spectral invariant is some kind of [unintelligible]. For a nondegenerate Hamiltonian H, consider the fundamental class on M is an element of quantum cohomology, so it can be represented by a chain in Floer homology, so take a representative in the chains and take

$$\max_{[x,u]} A_H(x,u)$$

that appears as a coefficient in a representative $\sum a_{(x,u)}(x,u)$ of $[M]$; then take the infimum over all representatives of $[M]$. This is the spectral invariant $c(H)$ of H.

What we'll use today from this invariant, which has very nice properties, is that, first, $c(H) - c(G)$ is bounded by $\int_0^1 \min_X (H_t - G_t) dt$ below and $\int_0^1 \max_M (H_t - G_t) dt$ above.

This property implies that the spectral invariant is Lipschitz continuous with respect to the C^0 norm of H. We can define for degenerate Hamiltonians the spectral invariant and then the spectral invariant of the function 0 is 0. Moreover, another important property is that if H is nondegenerate, then $c(H) \in \mathrm{Spec}_n H$.

[unintelligible]defined the homogenization of the spectral invariant, they called it the partial symplectic quasistate,

$$\zeta(H) = \lim_{k \to \infty} \frac{1}{k} c(kH)$$

for $H \in C^\infty(M)$

Definition 6.1. Let X be a closed subset of M. Then X is *heavy* if and only if, for any $H \in C^\infty(M)$ that vanishes on X, it has $\zeta(H) \geq 0$. It's called *superheavy* if for any $H \in C^\infty(M)$ that vanishes on X, it has $\zeta(H) \leq 0$.

Theorem 6.2. *(Entov, Polterovich 09) If X is heavy, then X cannot be displaced by Hamiltonian diffeomorphisms.*

If X is superheavy, then it cannot be displaced by symplectic diffeomorphisms.

Another important property is that heaviness and superheaviness are preserved by products. If X is (super)heavy and Y is (super)heavy then $X \times Y$ in the product ambient symplectic manifold is (super)heavy.

This implies that if you find a new example of a superheavy set, then you know the product of already known superheavy sets with the new one is also nondisplaceable by symplectic diffeomorphism.

An example of a superheavy set is the torus $\{(z_0, \cdots z_n) \in \mathbb{CP}^n : |z_0| = \cdots = |z_n|\}$.

My main theorem, I found a new example of a superheavy set. Assume $c_1 = \kappa\omega$ on $\pi_2(M)$ for any κ in $\mathbb{R}$. Then assume $U_j \subset \mathbb{R}^{2n}, \omega_i$ is a strictly convex open set, and $U_j \hookrightarrow (M, \omega)$ symplectically.

Theorem 6.3. *(I.) For any function in $C^\infty(M)$ which vanishes outside of $\amalg U_j$, then $\kappa \leq 0$ implies that the spectral invariant is bounded above by $\max c_0(U_j)$, and if $\kappa > 0$ and $\max c(u_j) \leq \frac{n}{\kappa}$, then the spectral invariant of F has a bound depending only on U_j, $c(F) \leq \max_j c(U_j)$.*

Here c_0 and c are constants I'll explain later.

Corollary 6.4. *If $\kappa \leq 0$ then $X = M \setminus \amalg U_j$ is superheavy. If $\kappa > 0$ and U_j is not so big, then X is superheavy.*

The definition of the constant c, for a strictly convex open subset $U \subset \mathbb{R}^{2n}$ containing 0, define $f : \mathbb{R}^{2n} \to \mathbb{R}$ by $t, y \mapsto t^2, y = \partial U$ and $t > 0$. [for whatever reason] $D^2 f > 0$, and $c_0(U)$ measures its positivity,

$$c_0(U) = \inf\{2\pi/a, a > 0 D^2 f > wh\}$$

For example, if U is an ellipsoid $E(r_1, \ldots, r_n)$, then $f = \sum_i \frac{|z_i|^2}{r_i^2}$. Then $c_0(U) = \pi r_1^2$ and $c(u) = \pi r_n^2$.

An example of the corollary is, consider Σ_g with $g \geq 1$. There is a CW decomposition $e^0 \cup e^1 \cup \cdots \cup e^1 \cup e^2$.

So π_2 vanishes and we can apply the corollary, which says that the complement of the 2-cell is superheavy.

This example was already known.

Remark 6.5. For the case that κ is positive, it is known that the largest ball whose complement is superheavy in $\mathbb{CP}^n$ is

$$\{[z_0 : \cdots : z_n] \in \mathbb{CP}^n : \frac{|z_0|}{\sum |z_i|^2} > \frac{1}{n+1}\}.$$

The constant $c(\beta)$ is n/κ, the largest ball such that the complement is superheavy.

Theorem 6.6. *(Seyfaddini) Assume further that U_j is a ball and each U_j is displaceable and $E(U_j) \leq \frac{1}{2|\kappa|}$. Then the same conclusion holds, for all $F : F|_{M \setminus \sqcup U_j} = 0$, we have*

$$c(F) \leq \max_j c(U_j)$$

This version uses displaceability but my version doesn't need that, howover big this thing is.

Let's talk about the proof of the main theorem. I'll talk about the version where $\kappa = 0$, the other case is similar.

Lemma 6.7. *If $\Psi : [0,1] \to Sp(2n)$ is C^0-close to Φ then the Conley–Zehnder index of Ψ is bounded by the Conley–Zehnder index of Φ plus $\frac{1}{2} \dim \ker(\Phi(j) - 1)$, and we'll call this sum the maximum Conley–Zehnder index of Φ.*

We also need that

Lemma 6.8. *For $H : \mathbb{R}^{2n} \to \mathbb{R}$ with $D^2 H < \begin{pmatrix} -C & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$, then the maximum Conley–Zehnder index of $\varphi^H_{[unintelligible]}$ is at most $-n - 2[\frac{c}{2\pi}]$.*

Lemma 6.9. *Let H be from $rR^{2n} \to \mathbb{R}$ and $\chi : \mathbb{R} \to \mathbb{R}$. Then the maximum Conley–Zehnder indices differ by at most 1.*

Let's start the proof for $\kappa = 0$. Let f_j be a quadratic function of U_j. Assume the inequality is satisfied for $D^2 f_j > \begin{pmatrix} a & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$. The goal of the proof is that for any function F that vanishes outside U_j, we have spectral invariant less than $\frac{2\pi}{a}$ We'll prove it for special H and then use monotonicity to prove it for general H.

Let ϵ be a small constant and χ_0, χ_1 be functions with a graph like this [picture] which is linear on $[0, 1-\epsilon]$ and its slope is less than $\frac{2\pi}{a}$ and vanishes near 1. Its maximum should be less that $\frac{2\pi}{a}$.

Then χ_1 is similar, but goes to $1-2\epsilon$. Let $\chi^s = \chi_0 + s\chi_1$. The special Hamiltonian H^s is $\sum_j \chi^s \circ f_j : M \to \mathbb{R}$. Then H has the following graph: [picture]

We claim that $c(H^s) = c(H^0)$ for all s in $(0, \infty)$.

The proof, let $x \in \coprod \{f_j \leq 1 - \epsilon\}$ then the slope of χ_0 is less than $\frac{2\pi}{a}$, so $\chi(f_j) D f_j(x) < \begin{pmatrix} -2\pi & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$. We can estimate the Conley–Zehnder index using the third lemma, the difference is bounded by 1. Then we can use the second lemma, since the first term is bounded in this way by 2π, so this is boundde by $-n-2+1 < -n$ and if x is in the complement of that reogion, then the function is independent of s. If G^s is a perturbation of H, then this implies that $\operatorname{Spec}_n G^s = \operatorname{Spec}_n G^0$. The orbit appear in this region does not have degree n, so the spectral set is independent of s. So $c(H^s) = c(H^0)$.

By making H^s large, arbitrarily, so any function F, there is some H^s bigger, so this implies that $c(F) \leq c(H^s) = c(H^0)$ which by monotonicity is at most $\max H^0$, but by construction, this is at most $\frac{2\pi}{a}$. This is the end of the proof, and the end of my talk. Thank you.

7. Ching-Hao Chang: The isotopy problems of nodal symplectic spheres and J-holomorphic spheres in rational manifolds

Thanks to the organizers for inviting me. This is my first time here since I was an elementary student. I'll say something about this isotopy problem. My spheres are two dimensional and my rational manifolds four dimensional. A symplectic submanfiold X of (M, ω) is first a submanifold and second of all $X, \iota_* \omega)$ is symplectic.

So $\mathbb{CP}^1$ embedded in $\mathbb{CP}^2$ appropriately is a symplectic submanifold.

So the question is about isotopy of symplectic submanifolds in a symplectic manifold. For $i_0 : S \to (M, \omega)$ and $i_i : S \to (M, \omega)$, the question is whether there exists an isotopy $H : S \times [0,1] \to M$ which is continuous, with $H_0 = i_0$ and $H_1 = i_1$ and H_t always a symplectic submanifold.

For a submanifold, you can talk about embedded and immersed submanifolds, so we could also ask the question about immersed submanifolds.

There are multiple ways to deal with this problem. One is to, well, say S is a submanifold satisfying some sectional curvature condition and a mean curvature condition. Then the flow will have a long time existence and will converge to something good like a holomorphic curve in a Kähler manifold. You have to deal with a second fundamental form, an almost complex structure, some other things, to ensure long time existence of the flow. I don't want to talk about this approach today.

I want to use the isotopy of J-holomorphic curves in a manifold to study this instead.

I'll define a J-holomorphic curve. A map between two almost-complex manifolds, at least C^1, is pseudoholomorphic if it satisfies the Cauchy–Riemann equation, which is, $du + JduJ_s = 0$ at every point. This is if and only if $Jdu = duJ_s$. In the holomorphic case this is the Cauchy–Riemann equation.

This definition has nothing to do with the symplectic manifold. This only relies on almost-complex structures. In order for J to be related to the symplectic manifold, I'll define what it means to be ω-tame. We say J is ω-tame if $\omega(u, Ju) > 0$ for all u. We can collect J_ω which is the set of ω tame almost complex structures for fixed ω.

To tame J we can associate the Riemannian metric $g_J = \frac{1}{2}(\omega(u, Jv) + \omega(v, Ju))$.

Sometimes you have a correspondence between symplectic submanifolds and J-holomorphic maps for some tame J. One direction is trivial and the other direction is also not so hard.

When we have this fact in some subcategory, you can consider the isotopy of J-holomorphic curves instead of symplectic submanifolds. The almost-complex structure can vary as t varies.

The J-holomorphic curves are *not* as good as holomorphic maps. For example, you have for the holomorphic case, you have f is holomorphic if and only if it's analytic and if $f^{(n)}(z_0) = g^{(n)}(z_0)$ then f is g on a neighborhood of z_0 and so for the whole connected component of z_0. For J-holomorphic curves you don't have this, you have something "vanishing to infinite order."

What you have is that $\int_{|z|<r} |\omega(z)|dz = O(r^k)$, this is vanishing to infinite order.

So if u and v, if their difference vanishes to infinite order, then you know [unintelligible].

If you have a J-holomorphic curve, smooth, defined outside a point, then you can extend it to the point to become J-holomorphic and smooth. So it's not as good as the holomorphic case but not quite a disaster.

The first theorem is given by Gromov.

Theorem 7.1. *Any symplectic sphere in $\mathbb{CP}^2$ of degree 1 is symplectically isotopic to an algebraic line.*

An algebraic line can be written $a_1x + a_2y = a_3$. So there's only one isotopy class in this homology class.

So the question is how many isotopy classes are there in a fixed homotopy class.

Theorem 7.2. *(Barrand) Any nodal symplectic sphere of degree d in $(\mathbb{CP}^2, \omega)$ is symplectically isotopic to an algebraic curve.*

This result was extended to arbitrary degree.

As a corollary, there are only finitely many isotopy classes in a homology class.

What does "nodal" mean? This was embedded in Gromov's theorem, but now it's immersed, with good self-intersection, it intersects itself transversally. For J-holomorphic curves, the intersection, this is either zero or two dimensional. The intersection of the tangent bundle is transversal, there is no $du|_Z = 0$, and no triple intersections. Also, you don't want anything reducible.

If an immersed curve of degree d has these properties then we call it a good nodal curve.

So you can have the following result.

Theorem 7.3. *(CH Chang) Let (M, ω) be a rational symplectic four-manifold. Then $A \in H_2(M, \mathbb{Z})$ with $c_1(M)A > 0$, then the space of nodal symlpecitc spheres in the class A, called S^0_A, has only finitely many isotopy classes.*

Rational means it's a finite blowup of $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$.

We prove this by studying the isotopies of J-holomorphic curves.

Theorem 7.4. *(CH Chang) For $A \in H_2(M, \mathbb{Z})$, and $c_1(M) \cdot A > 0$. Let J_0 and J_1 be ω-tame on (M, ω). Then let u_0 be a J_0-holomorphic sphere. Three is a u_t so that u_t is a J_t-holomorphic curve for all t.*

You have some kind of adjunction formula for J-holomorphic curves. In the same homology class A, you have 2
$delta(u) \leq A \cdot A - c_1(M) \cdot A + \chi(S)$, if the self-intersection is transversal, then this is an equality. The number of self-intersections is then fixed by fixing A, the Chern class, and χ. So we are interested in

$$\mathbb{P}_A = \{(u, J) | u = \mathbb{CP}^2 \to M, [u(\mathbb{CP}^1)] = A\}$$

We're only discussing $\mathcal{P}_{A,\{p_i\}_{i=1}^{d_A}}$ where $d_A = c_1(M) \cdot A - 1$. So we're talking about hitting this many points to lower the moduli space's dimension to what we want.

We have to deal with the bad curves. What are the bad curves? We have triple intersections, this is a bad curve. Or $\mathcal{P}_{A,\{p_i\}}$ with $du|_{z_a} = 0$. Or non-transversal intersection. Or irreducible curves could limit to reducible curves. When this happens, the number $c_1(M) \cdot A - 1$, then the sum

$$\sum_{j=1}^{N} c_1(M) \cdot A_j - 1$$

will be strictly less that d_A. So we can take that part, with enough marked points, this is a bad part.

The next thing, all these moduli spaces are in fact separable Banach manifolds. They are intersections of Banach manifolds. [picture] In fact this is a Banach bundle. One section is $\phi(u, J)$ is $du + [unintelligible]$. The other is the zero section. If they intersect transversally, their intersection is also a Banach manifold, in fact a J-holomorphic curve.

So if they intersect transversally, it is equivalent to considering the linearization of the section. The question is, or any α, can you always solve $d\Phi_{(u,J)}(v, \delta J) = \alpha$ and $Dv + (\delta J)du(J_s) = \alpha$.

If you have a marked point, can you always find $v(z_i) + du|_{z_i}([unintelligible])$ in $T_{p_i}M$.

If you look at this kind of thing, you have at a point (u, J), with z_i, that [couldn't understand]

[commutative diagram].

From this commutative diagram you have an induced map and then short exact sequence and then long exact sequence of cohomology. Since the Gromov operator is Fredholm, it's a bounded linear operator between Banach spaces with finite dimensional kernel and cokernel. By some exact sequences, this will eventually control the porjection [picture] which will become Fredholm.

Then you have

Theorem 7.5. *(Sard–Smale) For a Fredholm map from one Banach manifold to another, the regular values will be of second category in M provided that f is smooth enough.*

To be exact, if the index of $d\pi$, the dimension of the kernel minus the dimension of the cokernel, is k, then you should be C^{k+1}. A projection is always smooth enough. So the regular values of π are smooth in our moduli space. When the number is positive enough, we can know that this map [unintelligible]is always surjective. We can calculate that the kernel is of six real dimensions, the dimensions of the automorphisms of $\mathbb{CP}^1$.

We say that J is generic if it's a regular value of these things. It's a local submersion and it doesn't admit bad curves. For a generic J you have a curve connecting them. It has cokernel 1 or 0. But if it's a good J, then the cokernel is 0. It can't have cokernel 1. That's why the generic space of almost–complex structures is a subspace of the ω-tame structures.

That's why when you have (u_0, J_0) you can always find some path to (U_t, j_t).

[missed a little]

That's an example of how to use the isotopy of J-holomorphic curves to study the isotopy of symplectic submanifolds.

8. Byeongho Lee: Orbifolding Frobenius manifolds

Let's start with Frobenius manifolds, let's review. What is a Frobenius manifold? We'll say M is $\mathbb{C}^n$, in general we affine manifolds, but first let's look at $\mathbb{C}^n$, and fix a basis for M, $\{\partial_i\}$ and I'll be identifying M with its tangent space. Let x^i be the dual basis (flat coordinates) and $\mathcal{O}_M$ is analytic functions on M, and $\mathcal{T}M$ is holomorphic vector fields on M. Because M is a vector space, we can say g is a non-degenerate symmetric bilinear form and I want to view this as a constant metric on M. If you use coordinate vector fields, then g_{ij} will be constant.

Let's pick an analytic function $\Phi \in \mathcal{O}_M$ and define a multiplication $\circ_\Phi$ on vector fields as the following

$$\partial_i \circ_\Phi \partial_j = \Phi_{ijk} g^{k\ell} \partial_\ell$$

where Φ_{ijk} is $\partial_i\partial_j\partial_k\Phi$. So this is commutative but not necessarily associative.

We also should have associativity, the so-called WDVV equation $\phi_{abk}g^{k\ell}\Phi_{\ell cd} = \Phi_{adk}g^{k\ell}\Phi_{\ell bc}$.

Let me review semisimple Frobenius manifolds, since I'll be working with them.

So (M, g, Φ) is semisimple if the tangent spaces are isomorphic to $\mathbb{C}^n$ as $\mathbb{C}$-algebras. We have a multiplication structure on each tangent space. If we have one, we have canonical coordinates.

We have been using flat coordinates but we also have *canonical coordinates*. I'll write them as $\{u^i\}$ with vector fields $\{e_i\}$ such that $e_ie_j = \delta_{ij}e_j$.

If you look at this formula, it's automatically associative. If you calculate $g(e_i, e_j)$, it's no longer constant, we need another constraint, so it's flatness of g. Let's see the other piece of data is a metric potential $\eta \in \mathcal{O}_M$ so that $g(e_i, e_j) =$

$\delta_{ij}e_i\eta$. Then the flatness of g is a condition on the metric potential, the Darboux–[unintelligible]equation. The form of the equation is not that important so I won't state it, but it's simpler than WDVV.

There is an important theorem, one of the important theorems about semisimple Frobenius manifolds, which is that flatness of g guarantees the existence of Φ. What this means, suppose we don't know we have a Frobenius manifold. We want to define a semisimple multiplication. If g satisfies the flatness condition, then we have Φ. The proof is in Manin's book. The idea is that there's another formulation in terms of flat connections, and he uses that to prove the theorem.

So these are the ones I'm going to use.

What is orbifolding? Before talking about Frobenius manifolds, let's talk about Frobenius algebras, so there have been the work in the A-model of Fantechi–Göttsche and in the B-model the work of my advisor, Kaufmann. In their setting, they start with X a complex manifold and there's a finite group G that acts on it, they consider global quotients. Then they build something called $H^*(X, G)$, which is

$$H^*(X, G) \coloneqq \bigoplus_{g\in G} H^*(X^g, \mathbb{Q})$$

and they have a noncommutative product on this thing using geometric data of X and G. The details are not important to us.

The important properties are, they take the G-invariants of this larger ring, $H^*(X, G)^G$ is isomorphic to $H^*_{CR}([X, G])$.

In the B-model, Kaufmann's setting is a Jacobian Frobenius algebra of some polynomial f, this means that it's a Jacobian ring and then a Frobenius algebra on that, and G a finite group acting on the domain of f such that f is invariant under this action. Similar to that case, he writes down a larger ring, the algebra is a direct summand of the larger ring, a G-Frobenius algebra from fixed points and group cohomological data of G, again non-commutative, called A, and then A^G is a new Frobenius algebra.

Let's look at the common features. You start from a Frobenius algebra and a G-action, the sources are different, and then we build a Frobenius, well, a G-Frobenius algebra, after giving a non-commutative structure the thing is a G-Frobenius algebra containing the original one as a summand, then take G-invariants. That's a new Frobenius algebra. This is something we want to get.

So that was orbifolding a Frobenius algebra.

Let's start thinking about Frobenius manifolds. The first approach changes Frobenius algebra to Frobenius manifold and then do the same thing, build a G-Frobenius manifold and take G-invariants.

So the step of building a G-Frobenius manifold is problematic. My thesis wrote about the definition of this. There are a lot of question marks, I have no idea. So I started thinking about a simpler question, a milder goal. Let M be a Frobenius manifold, think of M_0, the Frobenius algebra at the origin, suppose we know the orbifolding of this Frobenius algebra, we do that and get N_0. Can we find a compatible Frobenius manifold on N_0? This is a much milder question? I claim that there is something to be said about this question.

I'm going to talk about this question.

What does compatible mean? Let's look at my favorite example. Let A_{2n-3} go to D_n, from the Frobenius algebra to [unintelligible]that was done by Kaufmann.

The general case is very similar. Then the universal unfolding of A_3 is $\frac{1}{4}z^4 + a_2z^2 + a_1z + a_0$. What about D_3? It's $\frac{1}{2}xy^2 + \frac{1}{4}x^2 + a_2x + a_0 + a_*y$. This is a universal unfolding. We identify the polynomials with $\mathbb{C}^3$, they are (a_2, a_1, a_0) and in the other case (a_2, a_0, a_*). They have the same form in flat coordinates.

The Kaufmann orbifolding gives us a Jacobian algebra at the origin, so if we take the Jacobian algebra at the origin, his procedure give us an algebra at the origin, the metric is given by the residue pairing. If we take the Jacobian ring, then we have a family of Frobenius algebras, two families, parameterized by $\mathbb{C}^3$, semisimple generically, I'm using a coordinates, but these are not flat coordinates. So there is a way to find a flat coordinate system, I won't go into detail but I'll show you the answer. Let $a_2 = -t_2$, and $a_1 = -t_1$ and $a_0 = -t_0 + \frac{1}{2}t_2^2$ and then $a_* = -t_*$. Then the top two, [unintelligible]. Then out of this you can calculate the Φ and things, but what you want to look at is the metric potential η.

For A_3, I won't show the calculation, but it's $a_2 = -t_2$. Tdhis is in Manin's book. For D_3 you can do a similar thing and you see that it's also $a_2 = -t_2$. The metric potential matches. This is what I call compatible.

Let's generalize. My setup, I start with two semisimple Frobenius manifolds (M, g_M, Φ_M) and (N, g_N, Φ_N) and flat coordinates $\{x_M^a\}$ and $\{x_N^a\}$. These are semisimple so I have canonical coordinates $\{u_M^a\}$ and $\{u_N^a\}$ and then I'll assume I have a G action on the Frobenius algebras at the origin M_0 and N_0. Then we assume that N_0 is an orbifolding of M_0. Then N_0 should be decomposed as $N_0^{ut} \oplus N_0^{tw}$, the untwisted and twisted sectors.

This means that M_0^G is identified with N_0^{ut} and then we have several natural maps, like the projection $N \cong N_0 \to N_0^{ut} \cong M_0^G \hookrightarrow M_0 \cong M$, so we have a map $N \to M$. Then since these are semisimple, we have metric potentials η_N and η_M. We also have g_N obtained from g_M by orbifolding. So I didn't say this before, we should also, I didn't say this before, but, uh.

Okay, here's my definition of orbifolding.

Definition 8.1. N is an orbifolding of M compatible with the orbifolding M_0 to N_0 if

$$(i \circ \pi)^* \eta_M = \eta_N$$

Compatibility is expressed as a system of partial differential equations.

$$\frac{\partial}{\partial x_i} = \sum_\alpha \frac{\partial u_\alpha}{\partial x^i} \frac{\partial}{\partial u^\alpha}$$

and so

$$g_{ij} = g(\sum_\alpha \frac{\partial u_\alpha}{\partial x^i} \frac{\partial}{\partial u^\alpha}, \sum_\beta \frac{\partial u_\beta}{\partial x^i} \frac{\partial}{\partial u^\beta}) = \sum_\alpha \frac{\partial u^\alpha}{\partial x_i} \frac{\partial u^\alpha}{\partial x^j} \frac{\partial \eta}{\partial u^\alpha}$$

and so that's

$$\sum_{\alpha,\beta} \frac{\partial u^\alpha}{\partial x^i} \frac{\partial u^\alpha}{\partial x^j} \frac{\partial x^\beta}{\partial u^\alpha} \frac{\partial \eta}{\partial x^\beta}$$

Theorem 8.2. *If we have M orbifolding $M_0 \to N_0$ and a solution of this differential euation then we have an orbifolding N of M compatible with M_0 to N_0.*

The change of coordinates is given by the solution to this equation. There's another thing, this implies the flatness of the metric.

There are other trivial theorems

Theorem 8.3. *The universal unfolding of D_n is an orbifolding of the universal unfolding of A_{2n-3} compatible with the orbifolding Frobenius algebras in Kaufmann's sense.*

There's nothing to prove here, but let's check the equation. It's a fun calculation, so let me show you.

Pretend that we don't know the metrics match. The metrics are given by the orbifolding, on the levels of Frobenius algebras and pullback. We can show

$$\frac{\partial u_1}{\partial t_2}\frac{\partial u_1}{\partial t_0}\frac{\partial \eta}{\partial u_1}+\frac{\partial u_2}{\partial t_2}\frac{\partial u_2}{\partial t_0}\frac{\partial \eta}{\partial u^2}+\frac{\partial u_3}{\partial t_2}\frac{\partial u_3}{\partial t_0}\frac{\partial \eta}{\partial u^3}=g_{0,2}=1$$

Let's just check this case. So JF_{D_3} is $\langle \frac{1}{2}y^2+\frac{1}{2}x+a_2; xy+a_*\rangle$ which defines 3 points generically, (β_i,γ_i). Then I claim that $u_i = F_{D_3}|_{(\beta_i,\gamma_i)}$ solves this equation. So $u_1=\beta_1\gamma_1^2+\beta_1^2-t_2\beta_1-t_0+\frac{1}{2}t_2^2-t_*\gamma_1$.

We should use the product rule here, $\frac{\partial u_1}{\partial t_2}$ is just -1 and the same is true for $\frac{\partial u_2}{\partial t_0}$ and $\frac{\partial u_3}{\partial t_0}$. So those terms are all -1. Then we know that $\eta=-t_2$ so $\frac{\partial \eta}{\partial u_i}$ is $-\frac{\partial t_2}{\partial u_i}$ after change of variables, and this is $\frac{\partial u_1}{\partial t_2}\frac{\partial t_2}{\partial u_1}+\frac{\partial u_2}{\partial t_2}\frac{\partial t_2}{\partial u_2}+\frac{\partial u_3}{\partial t_2}\frac{\partial t_2}{\partial u_3}$ and if you look at this, the second colomn times the second row is 1. So I think my time is up. I'll stop.

9. Nov. 4: Yong-Geun Oh: Topological extension of Calabi invariants and its application

Thanks to the organizers for giving me a chance to talk here. I enjoyed this visit very much. What I'd like to talk about is some old subject about C^0 symplectic topology. Let me talk about Eliashberg–Gromov C^0 rigidity.

Consider the C^0 closure, say M is compact without boundary for now, the C^0 closure of $Symp(M,\omega)$ inside $Homeo(M)$. Then you take the intersection of $\overline{Symp(M,\omega)}$ with $Diff(M)$. The theorem of Eliashberg is that this intersection is $Symp(M,\omega)$.

If you decode what this means as a statement, this is called C^0 rigidity. If I translate this identity, it means that if ϕ_i is a sequence of smooth symplectic diffeomorphisms converging to some homeomorphism ϕ in C^0, suppose this happens to be differentiable at $x\in M$. Then $d\phi(x)$ is symplectic.

This theorem is the beginning of the whole subject of symplectic topology. The proof turns out to involve non-squeezing type theorems. Therefore, it makes sense to define symplectic homeomorphisms as the elements in that closure.

Definition 9.1. The group $Sympeo(M,\omega)$ is the C^0 closure $\overline{Symp(M,\omega)}$.

There is another group, of Hamiltonian diffeomorphisms. This is very non-orthodox, it's just a historical accident. Recall that we have the group $Ham(M,\omega)$, a subgroup of $Symp(M,\omega)$, and the definition is rather odd. The elements $\phi\in Ham(M,\omega)$ are defined as follows. Let me define a path $\mathcal{P}^{ham}(Symp(M,\omega),\mathrm{id})$. These are paths $\lambda:[0,1]\to Symp(M,\omega)$ where $\lambda=\phi_H$ for $H=H(t,x)$, with $t\in[0,1]$. So ϕ_H^t is the Hamiltonian path associated to the function H.

Then there is a natural map $Ham(M,\omega)=ev_1(\mathcal{P}^{ham})$.

Lemma 9.2. *$Ham(M,\omega)$ is a subgroup of $Symp(M,\omega)$.*

To define the C^0-analog of Ham, first we have to take a closure of $\mathcal{P}^{ham}$ inside the set of paths in homeomorphisms. We have to use a particular metric.

So $d_{Ham}(\lambda, \mu) = \max_{t\in[0,1]} \bar{d}_{C^0}(\lambda(t), \mu(t)) + ||Dev(\lambda) - Dev(\mu)||$ where $Dev(\lambda)$ is the associated normalized Hamiltonian, that is, if $\lambda = \phi^H$, then $Dev(\lambda) = H$. I normalize to define this uniquely. I should also say $\bar{d}_{C^0}(\phi, \psi) := d_{C^0}(\phi, \psi) + d_{C^0}(\phi^{-1}, \psi^{-1})$.

Denote by $\mathcal{P}^{ham}(Sympeo(M, \omega), \mathrm{id})$ the closure of $\mathcal{P}^{ham}(Symp(M, \omega), \mathrm{id})$.

One of the main theorems in this subject is

Theorem 9.3. *(Muller–O., Viterbo,[unintelligible]-Seyfaddim) Suppose H_i satisfy that*

1. *ϕ_{H_i} converge in C^0,*
2. *H_i converge in $L^{1,\infty}$, that's L^1 in time and L^∞ in space.*

Then ϕ_{H_i} converges to the identity if and only if H_i converges to 0.

In the smooth case, you can recover the Hamiltonian from the flow by taking derivatives. We can now define a continuous Hamiltonian path associated to H, you look at the limit of the Hamiltonians in this topology, and there is a canonically define continuous path inside sympeomorphisms, and vice versa.

I call this the C^0 Hamiltonian topology. Once you define this particular set of paths inside the group of homeomorphisms, you can define Hamiltonian homeomorphisms, which I'll abbreviate hameomorphisms. You evaluate $Hameo(M, \omega)$ is the evaluation of $\mathcal{P}^{ham}(Sympeo(M, \omega), \mathrm{id})$ at time 1.

Theorem 9.4. *(Müller–Oh) This forms a normal subgroup of $Sympeo(M, \omega)$.*

The main question is whether this is a proper subgroup or not.

This question becomes particularly interesting if you restrict to the 2-dimensional case.

Theorem 9.5. *(Smoothing theorem) Let (M, ω) be two dimensional. Then $Homeo^\omega(\Sigma)$ is the closure of $Diff^\omega(\Sigma)$.*

I gave the proof of this some time ago (O.–Sikorav) although I think it was known to experts.

But as we know, symplectomorphisms in 2-dimensions are area-preserving diffeomorphisms. So

Corollary 9.6. *$Hameo(\Sigma, \omega)$ is a normal subgroup of $Homeo^\omega(\Sigma)$.*

This was a well-known question.

Theorem 9.7. *(O.) $Hameo(D^2, \partial D^2)$ is proper in $Homeo^\omega(D^2, \partial D^2) = Sympeo(D^2, \partial D^2)$.*

Let me talk about the boundary case. They denote by $Homeo^\omega(D^2, \partial D^2)$ the group of area-preserving homeomorphisms supported on the interior of D^2.

For example, for any $\phi^i n Homeo^\omega(D^2, \partial D^2)$ there is $\eta > 0$ so that the support of ϕ is in $D^2(1 - \eta)$. This is some kind of direct limit topology over compact sets in the interior.

The main ingredient of this proof is the socalled Calabi homomorphism on D^2.

There are two ways of defining it. In both cases, say you look at $\phi \in Diff^\omega(D^2, \partial D^2)$, and we know this is the same as $Ham(D^2, \partial D^2)$ in the smooth case.

Suppose now that $\phi = \phi^1_H$, and suppose that the support of H, the union of $SuppH_t$ is contained in the interior of D^2.

The Calabi invariant is

$$Cal(\phi) := \int_0^1 \int_{D^2} H(t,\omega)\omega dt$$

and the proposition is that the right hand side does not depend on H as long as $\phi_H^1 = \phi$. This uses Stokes' theorem so it needs differentiability.

For the second definition, we can write $\omega = d\alpha$ on D^2, and then $\phi^*\omega = \omega$ implies $d(\phi^*\alpha - \alpha) = 0$. Further $\phi^*\alpha - \alpha$ has support in the interior of the disk. But $H^1(D^2, \partial D^2)$ is zero so this is dh_α, and this is a function with support in the interior of the disk.

The Calabi invariant is then

$$\frac{1}{2}\int_{D_2} h_\alpha \omega^n$$

The second definition also involves taking a derivative of the diffeomorphism ϕ.

It's well known that this Calabi homomorphism cannot be extended to area-preserving homeomorphisms. People in dynamical systems have tried to see whether, let me see, one of the consequences of the presence of this homeomorhism is the following.

The kernel of the Calabi homomorphism is a proper normal subgroup. One of the fundamental theorems is that this kernel is simple.

The main question was, is $Homeo^\omega(D^2, \partial D^2)$ simple? As a corollary of my result, the answer is negative. It's not simple.

You'll get a contradiction using this extended Calabi invariant.

The main ingredient is an extension of $Cal : Diff^\omega(D^2, \partial D^2) = Ham(D^2, \partial D^2) \to \mathbb{R}$ to a homeomorphism $\overline{Cal} : Hameo(D^2, \partial D^2) \to \mathbb{R}$.

Then non-simpleness follows by the alternatives: either $Hameo(D^2, \partial D^2)$ is proper in $Homeo^\omega(D^2, \partial D^2)$ or $Hameo(D^2, \partial D^2)$ is equal to $Homeo^\omega(D^2, \partial D^2)$ and the kernel of $\overline{Cal}$ is a proper normal subgroup.

I gave a construction before of such a wild homeomorphism assuming that there was an extension of the Calabi invariant.

An example of an area-preserving homeomorphism not contained in $Hameo(D^2, \partial D^2)$, was explained to me by Fathi before, under this assumption that this extension exists.

Roughly, you take the rotation supported on annuli, and make the rotation spin faster and faster as you go to the center of the disk. [picture]

You first start with a Hamiltonian path $\mathcal{P}^{ham}(Sympeo(D^2, \partial D^2))$. I'm going to define the path version

$$\overline{Cal}^{path}(\lambda) = \int_0^1 \int_M Dev(\lambda)\omega^n dt$$

and the main question is whether this descends to $Hameo(M, \omega)$. That is, does the right hand side depend on the path or just on $\lambda(1)$? This does not hold in general but the main theorem, let's say in the disk case,

Theorem 9.8. *(O.) The question is answered affimatively for D^2.*

So say for a topological Hamiltonian loop, $\lambda : [0,1] \to Homeo(D^2, \partial D^2)$ with $\lambda(1) = \mathrm{id}$. The question is whether the integral

$$\int_0^1 \int_{D^2} Dev(\lambda)\omega dt = 0?$$

In the smooth case this is just Stokes' formula. If λ is differentiable, then there are several facts. $Diff^{\omega}(D^2, \partial D^2)$ is contractible and so simply connected. So any smooth Hamiltonian loop can be contracted to the constant loop. Using that and Stokes' formula proves the right hand side vanishes. As I said, if the path is not differentiable, then Stokes' formula is no longer available. However, the analog still works.

Theorem 9.9. *(O.) The Alexander isotopy in the homeomorphism category can be defined in hameomorphisms.*

One hopes to use this contractibility to prove the vanishing.

Here comes all theose constructions in symplectic geometry to avoid Stokes' theorem.

To overcome the non-availability, I need many constructions in symplectic geometry.

The first one is, the Lagrangization of Hamiltonian isotopy. Given $\phi_H^t : M \to M$, we have the groph of ϕ_H^t in $(M, \omega) \times (M, -\omega)$ and this is a Lagrangian isotopy.

I want to put, well, the graph of ϕ_H^t is $\{(\phi_H^t(y), y) | y \in M\}$. For $(D^2, \partial D^2)$, embed this into S^2 where ϕ_H^t is the identity on one half and 0 on the other half.

The graph of ϕ_H^t avoids the antdiagonal, it's in $S^2 \times S^2 - \bar{\Delta}$. You can regard this inside $T^*\Omega$, the image contained completely in the unit disk bundle.

The second construction is so-called Lagrangian suspension. You can think of the whole isotopy as a Lagrangian submanifold, a Lagrangian isotopy and then suspend, take $(t, \mathbf{q}) \in [0,1] \times \Delta \to T^([0,1]) \times T^*\Delta$ where $(t, \mathbf{q}) \mapsto (t, -\pi_1^* H(t, \phi_H^t(\mathbf{q}), \phi_H^t(\mathbf{q})))$.

The problem is, you take this approximation sequence, that smooth Hamiltonian path may not be closed. That's a stumbling block. The problem is, the approximation sequence $\phi_{H_i}^t$ may not be a loop. As it is, I cannot use the Lagrangian intersection theorem. The essential step is the so-called odd-doubling. You double this one and go back. This whole thing is contractible. Then you can apply a version of Lagrangian intersection theorems. There is some ingredient of transfer, some rearrangement theorem of Hamiltonian mass. I'm sorry, I'm a little bit rushed. Then you can prove this by contradiction. You assume the integral is nonzero and get a contradiction. This whole construction replaces Stokes'. I'll stop here.

10. Fumihiko Sanda: Computations of quantum cohomology from Fukaya categories

Let me start by talking about A_∞ categories.

An A_∞-category consists of the following data.

First, a set of objects. Then, for X and Y a $\mathbb{Z}/2\mathbb{Z}$-graded vector space over $\mathbf{k}$ $A(X, Y)$.

Then $m_k \in Hom^{2-k}(A(X_0, \dots X_k), A(X_0, X_k))$ where the domain here is $A(X_0, X_1) \otimes \cdots \otimes A(X_{k-1}, X_k)$.

The m_k satisfy the so-called A_∞ relations, and we have $e_x \in A(X, X)$, a strict unit. If we have $m_k = 0$ for $k \geq 3$ then this is a category and e_X is a strict unit.

From an A_∞ category, we can define two invariants.

The first is Hochschild cohomology $HH^*(A)$, which is a $\mathbb{Z}/2\mathbb{Z}$-graded ring. This is defined as the cohomology of this complex

$$\prod Hom(A(X_0, \dots, X_k), A(X_0, X_k)$$

with respect to a differential I won't define. Second, you have Hochschild homology, which is a $\mathbb{Z}/2\mathbb{Z}$-module over HH^*A, and the complex is $\bigoplus A(X_0, \dots, X_k, X_0)$.

If A is *proper*, i.e., the dimension of $H^*A(X,Y)$ is finite, then we can define a pairing $\langle , \rangle_{shk}$ on Hochschild homology to the base field.

Theorem 10.1. *(Shklyarov) If A is* smooth *then $\langle , \rangle_{shk}$ is nondegenerate.*

Definition 10.2. An A_∞ category is called *cyclic* if the dimension of $A(X,Y)$ is finite and there is a nondegenerate even degree pairing $A(X,Y) \otimes A(Y,X) \to \mathbf{k}$ such that

$$\langle x_0, m_k(x_1, \dots x_k)\rangle = \pm\langle x_k, m_k(x_0, \dots, x_{k-1})$$

If A is cyclic then $HH^*(A) \cong HH_*(A)^\vee$ and we can define $Z : HH_*(A) \to HH^*(A)$.

Corollary 10.3. *(Shklyarov) If A is smooth then Z is an isomorphism.*

11. Lagrangian intersection Floer theory

Let (X, ω) be a compact sypmlectic manfifold, and L a Lagrangian, compact, oriented, spin, without boundary, and let E be a rank one complex flat vector bundle on L. Then Λ is the ring $\sum a_i T^{\lambda_i}$ where $a_i \in \mathbb{C}$, $\lambda_i \in \mathbb{R}$, and $\lambda_1 < \lambda_2 < \cdots \to \infty$. There is Λ_+ sitting inside of this where $\lambda_i > 0$.

Theorem 11.1. *(Fukaya–Oh–Ohta–Ono) There exists $m_K^E : H^*(L, \Lambda)^{\otimes k} \to H^*(L, \Lambda)$ for $0 \le k$ such that $(H^*(L; \Lambda), m_k^E, \langle , \rangle_{PP}, e_L)$ is a gapped filtered cyclic A_∞ algebra.*

I won't define these things but if m_0^E is 0 then this is a cyclic A_∞ algebra.

Definition 11.2. Let $b \in H^1(L, \Lambda_+)$. Then b is weak Maurer–Cartan if and only if there is a $W(L,b)$ in Λ such that

$$\sum_{k=0}^{\infty} m_k^E(b, \dots, b) = V(L,b)e_L$$

By using a weak Maurer–Cartan element, we can deform m_k^E and construct an A_∞ algebra $H^*(L, \Omega), m_k^{E,b}$, with $k \geq 1$.

(1) Now we assume that there is a cyclic A_∞ category A_c where c is a constant and the objects of A_c are $\{(L_1, E_1, b_1), \cdots (L_k, b_k, E_k)\}$ such that $W(E_i, b_i)$ is defined by holomorphic disk counting. We assume $L_i = L_j$ or $L_i \pitchfork L_j$.

We define the Fukaya category as $\prod A_c$.

(2) We assume the existence of a closed open and an open closed map $\hat{q} : QH^*(X) \to HH^*(A)$, a ring homomorphism and $\hat{p} : HH_*(A) \to QH^*(A)$, a $QH^*(X)$-module homomorphism, such that $\hat{p}$ preserves the pairings.

We assume the following diagram is commutative.

$$\begin{tikzcd} QH^* \arrow[r, "\hat{q}"] \arrow[d, "\cong"] & HH^*(A) \arrow[d, "\cong"'] \\ (QH^*)^\vee \arrow[r, "\hat{p}^\vee"] & HH_*(A)^\vee. \end{tikzcd}$$

This implies that $Z = \hat{q} \circ \hat{p}$.

Next I'll explain my main lemma.

Lemma 11.3. *If A is smooth, then there exists an idempotent e_A in $QH^*(X)$ such that $QH^*(X)e_A \cong HH^*(A)$ as a ring.*

The proof is very easy. Since A is smooth, Z is an isomorphism. So there exists α in $HH_*(A)$ such that $Z(\alpha) = e_{HH^*(A)}$. So $\hat{p}(\alpha) * \hat{p}(\alpha) = \hat{p}(\hat{q} \circ \hat{p}(\alpha) \cap \alpha)$ since $\hat{p}$ is a module map. Then this is $\hat{p}(\alpha)$. So we define the idempotent e_A as $\hat{p}(\alpha)$. Since $Z = \hat{q} \circ \hat{p}$, this implies $\hat{p}$ is injective.

If im$\hat{p} \subset QH^*e_A$ then $\hat{p} : HH_* \to QH^*e_A$ is surjective since it's a module map that hits this idempotent.

So let $\beta \in HH_*(A)$. Then $\hat{p}(\beta) = e_A\hat{p}(\beta) + (1 - e_A)\hat{p}(\beta)$. Then $e_A\hat{p}(\beta) = \hat{q}(\alpha) * \hat{q}(\beta) = \hat{p}(\hat{q}\hat{p}(\alpha) \cap \beta) = \hat{p}(\beta)$. So $\hat{p}(\beta)$ is $e_A\hat{p}(\beta)$ is in in QH^*e_A.

The next lemma is to check smoothness, I think due to Shklyrov.

Lemma 11.4. *If $H^*(A)$ is smooth then A is formal. Higher products vanish, and $HH^*(A)$ is isomorphic to the center of $H^*(A)$.*

Let me give an example.

If $H^*(A)$ is equivalent to Cl_n, then A is formal and smooth. Moreover, $HH^*(A) \cong \Lambda$.

Let me talk abut a decomposition of A, due to Benson–Iyengar–Kunse, Seidel. [missed some]

Lemma 11.5. *For e and e' in $QH^*(X)$, with $ee' = 0$ then for $X \in A_e$, and $Y \in A_{e'}$, $H^*(A(X,Y)) = 0$ so $H^0(perf(A) \cong \prod_{prim\ idem} H^0(A_e)$.*

Conjecture 11.6. *Let $B \subset perf A$, and B is smooth then there is e_B so B splits [unintelligible]A_{e_B}.*

My main lemma almost completely proves this.

Finally, I did an example of computation.

Let X be a $\mathbb{P}^2 \times \mathbb{P}^1$ and C a degree three curve in $X := \mathbb{P}^2 \times \{0\}$. Let $\tilde{X}$ be a blowup of X along C. So C should also satsify some technical condition.

Proposition 11.7.
$$QH^*\tilde{X} \cong \underbrace{\Lambda \times \cdots \times \Lambda}_{6} \times H^*(C, \Lambda)$$
as a $\mathbb{Z}/2\mathbb{Z}$-graded ring.

By AAK, $\tilde{X}$ has a Lagrangian torus fibration. We can compute "potential function" W and by direct computation, it has 6 nondegenerate critical points. So the Floer homology, there exist 6 objects corresponding to the six nondegenerate critical points. Then [unintelligible]is isomorphic to the Clifford algebra. So there are six of these and the complement is a very small, this Frobenius algebra. By a simple algebraic argument we can conclude this proposition.

12. Nov. 5: Chung-I Ho: Minimal genus problems in 4-manifolds

[I do not take notes at slide talks]

13. Chris Wendl: Tight contact structures on connected sums need not be contact connected sums

Let me thank the organizers. It's wonderful to have the opportunity to speak at an event like this so far from home. The title is basically the theorem. This is joint with K. Niedengrüger and P. Ghiggini.

There hasn't been that much talk about contact structures so let me get us on the same page. M is an odd dimensional manifold and ξ is a hyperplane field that is the kernel of a 1-form. $d\alpha$ restricted to ξ is nondegenerate. If you're used to thinking about symplectic manifolds, these are the things that arise at the boundaries of symplectic manifolds. If I have a symplectic manifold W and its boundary is M along with the convexity theorem that there's a vector field directed outward at the boundary. If this is a Liouville vector field, then $d\lambda = \omega$, for $\lambda := \iota_V \omega$, then α which is $\lambda|_{TM}$ satisfies the nondegeneracy condition.

Up to isotopy, then you get a uniquely defined contact structure at the boundary of a convex symplectic manifold.

Another important example of such an object, if I take a symplectic manifold that looks like a trivial cobordism $[0,1] \times M$, and write $\omega = d(e^t\alpha)$, if α is contact then this will be symplectic. It has a Liouville vector field that points along the cylinder. You can more generally talk about cobordisms between different contact manifolds.

Now, contact manifolds have a connected sum operation. Let me draw a picture of it.

Start by constructing the trivial cobordism for each of them. Attach a 1-handle to this trivial cobordism. I claim that you can do this in a way that naturally produces a connected sum of the contact structures. On this handle, you choose a natural Morse function with a critical point at the intersection of the core and the cocore. This is Weinstein's handlebody construction, you can do this with a k-handle whenever $k \leq n$, but I'm doing the case $k = 1$ which is the connect sum. You have this gradient of a Morse function. The stable and unstable parts become coisotropic and isotropic. Your vector fields point inward at the handle attachment and outward at the new boundary. The behavior at the corners lets you round them in a natural way. That's the contact connect sum viewed in a symplectic light.

I need to recall a few facts about contact structures in dimension 3. The first fact is true in all dimensions but we've only known them for a couple of years. There's a dichotomy into tight and overtwisted contact structures. There are those that are flexible (overtwisted) and those that are rigid (tight). The classification of overtwisted structures is easy. The classification restricts to homotopy-theoretic data. There's a hyperplane field, maybe with a symplectic bundle in higher dimensions, these are "almost-contact" structures. For overtwisted contact structures, up to isotopy they are in correpondence with almost-contact structures up to homotopy. For what's left, the classification is a hard problem.

The second thing I need to recall, in dimension 3, there is a result from 1997 due to V. Colin, that tells you what every tight structure on a connect sum is, if $(M_1 \# M_2, \xi)$ is tight then $\xi = \xi_1 \# \xi_2$. There's more to it than that, there's a whole prime decomposition theorem. I want to tell you why there can't be any such theorem in higher dimensions.

First, a remark. Colin's theorem is false for overtwisted contact structures. This is easy to see from homotopy theory. An arbitrary overtwisted structure on a connected sum, there's a 2-sphere at the center of the tube. You want to know if you can cut there and glue in balls, and extend across these balls. That's not always possible from the homotopy point of view. There's a nontrivial statement in the background, if you have a contact structure in that kind of homotopy class, it can't be tight, it'll have to be overtwisted.

So it's only possible to have this theorem in the tight case. A natural question is, is this true in higher dimensions? The answer is no.

Theorem 13.1. *Suppose $n \geq 3$ I have (M^{2n-1}, φ) an almost-contact manifold, say it's not a homotopy sphere but it admits a Morse function with unique local maxima and minima and otherwise critical points of index $n-1$ and n only.*

The connected sum $-M\#M$ admits a Stein-fillable (by a standard theorem this implies tight) contact structure ξ such that ξ is homotopic to the almost-contact structure $\bar{\varphi}\#\varphi$. Here $\bar{\varphi}$ is the same data but automatically has a different coorientation. However, ξ is not equivalent to $\xi_1\#\xi_2$ for any two contact structures ξ_1 on $-M$ and ξ_2 on M.

This critical point condition, obviously spheres have this, but you could take an S^{n-1}-bundle over S^n. So unit cotangent bundles of spheres satisfy this condition.

So there are more tight contact structures. It defeats the purpose of my talk, but this is not surprising. The higher you go in dimension, the more room you have for exotic structures, whatever kind of structure you're talking about. In dimension 3 it doesn't work now because of convex surface theory, so this is also bad news for people who want to do convex surface theory in higher dimensions.

To explain the proof I want to give a little background on Stein fillings.

There are a number of nice results about Stein things. Eliashberg showed in 1990 that in dimension 3, it's possible to say what all the Stein fillings of a connected sum are.

Namely, every Stein filling of the connected sum $M_1\#M_2$ is obtained in a kind of obvious way, but it only exists if both M_1 and M_2 are Stein fillable themselves, you just take those fillings and attach a Weinstein 1-handle.

[picture to discuss proof]

So let me mention another recent development related to this. A little over a year ago, maybe two years ago by now, there was this paper by Bowden–Crowley–Stipsicz. They said that Eliashberg's theorem was not true topologically in higher dimensions. In particular there exist Stein-fillable contact structures ξ on a connected sum $M_1\#M_2$ so that ξ is homotopic to a connect sum of almost-contact structures but these almost-contact manifolds are not Stein fillable.

This was meant to be a contradiction, although it's not quite that. I'm only specifying the homotopy class of ξ as a contcat structure. Our theorem implies ξ is actually not a contact connect sum. Eliashberg's theorem still could be true in higher dimensions.

So when we started this project, we actually wanted to ask a related question, is Eliashberg's theorem even plausible topologically in higher dimensions? One can phrase this the following way. You can view this as classifying uniqueness of symplectic fillings. In dimension 4 we can use intersection theory of holomorphic curves. There's a famous theorem of Eliashberg–Floer–McDuff that classifies the homotopy type of cobordisms. We wanted to ask a similar homotopy-theoretic question, something like "if I have a Stein filling of a connected sum," (and thus contains a belt sphere) "then must the belt sphere be nulhomotopic?"

That's the minimal thing that has to be true if you're a boundary connected sum. This seemed like a reasonable thing to try to prove and we didn't succeed, but we proved something similar.

What might you more generally expect? In general, I don't have to just think about connected sums, I can think about Weinstein handles in general, index k

surgery on a contact manifold, you attach $\mathbb{D}^k \times \mathbb{D}^{2n-k}$. Your new thing has a belt sphere which looks like S^{2n-k-1}. Eliashberg used this as a boundary condition for pseudoholomorphic curves. It needs to have dimension at least n. It could contain totally real submanifolds if and only if its dimension is at least n, so if $k \leq n-1$, what we call subcritical surgery. What we're talking about is fillings of manifolds made by subcritical surgery.

You have lots of good reasons to think that subcritical surgery is reasonable, it's the trivial case of this kind of surgery.

Definition 13.2. In (M^{2n-1}, ξ), a *Legendrian open book*, or Lob, is a closed submanifold L^n of M with an open book decomposition $\pi : L \backslash B \to S^1$, where B is codimension 2, the *binding*, and the pages are Legendrian.

This is a completely different notion than Giroux's. We only have this n-dimensional submanifold. In Giroux, they're symplectic subthings.

Let me give you a more concrete example that we've already seen. For M^3, I could take $L = S^2$, and B could be S^0, and the latitudes [sic?] connecting the poles are the pages.

Whenever you have a Legendrian open book, you get a "Bishop family" of holomorphic disks $u : (\mathbb{D}^2, \partial\mathbb{D}^2)$ which map to the bottom half of the symplectization, $((-\infty, 0] \times M, \{0\} \times (L\backslash B))$.

The other important observation is that there's a strong level of topological control over the boundaries of these disks, because of the maximum principle. The holomorphic disks, in the r direction, there are never extrema except at the boundary. This translates to $u|_{\partial\mathbb{D}}$ is transverse to the pages of the Lob. In particular, the Bishop disks, the boundaries go exactly once around the family of pages, and that's true always because of transversality. That's a very strong condition; in particular it prevents bubbling.

The main geometric observation which is not especially deep, but non-obvious enough that it took us a long time, is, suppose (M^{2n-1}, ξ) contains the belt sphere S^{n+m}_{belt} of a contact surgery of index $k = n - 1 - m$. Here $m \geq 0$. Then after a suitable deformation, the belt sphere is foliated by an m-dimensional family of Lobs. So I have a smooth family of boundary conditions for J-holomorphic disks.

Then we do analysis. Assume S^{n+m}_{belt} is in (M, ξ) which is the boundary of (W, ω), assumed to be symplectically aspherical, $[\omega]|_{\pi_2(W)} = 0$. Then for a generic tame J on W, there exists a compact oriented $n + m - 1$-dimensional moduli space $\overline{M}$ of J-holomorphic disks $u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (W, S_{belt})$ such that $\partial\overline{M} \cong S^{n+m-2}$ and the evaluation map (putting a marked point on my holomorphic disks) $(\overline{M} \times \mathbb{D}^2, \partial(\overline{M} \times \mathbb{D}^2)) \to (W, S_{belt})$ which takes $(u, z) \mapsto u(z)$, and the evaluation map is a diffeomorphism on some neighborhood of the preimage of the binding, the unions of the bindings of all those Lobs, so in particular its restriction to the boundary is a map of degree 1 to the belt sphere.

This fact about being a diffeomorphism in some small neighborhood is a uniqueness result about holomorphic disks in some neighborhood of the binding. This traces back to Eliashberg.

So the boundary of the moduli space has this degree 1 to the sphere. Now I can prove the main theorem.

The construction here is something we borrow from Bowden–Crowley–Stipsicz. We have M as in the statement. We get rid of the top critical point by removing a ball. Then we choose a Morse function on that complement M^* with critical points of index 0, $n-1$, and n. We can have the gradient of that function point outward at the boundary.

Let W be $[-1,1]\times M^*$. That's an object with boundary and corners. After smoothing the corners, then $\partial \cong -M\#M$ and remember I also have an almost-contact structure φ on M, which gives me one on M^*. I can turn that into an almost complex structure on W, and the boundary of an almost-complex manifold is always an almost-contact manifold.

This Morse function f can be extended to the product as a Morse function $\tilde{f}$ with critical points of index at most n and a gradient pointing outward.

Now there's a big theorem of Eliashberg that tells me that this almost complex structure is homotopic to a Stein structure on W which is filling some contact structure ξ necessarily homotopic to $\bar{\varphi}\#\varphi$.

That's the contact structure I claimed exists. I still have to show you that it's not a contact connect sum. Why not?

The crucial property of a contact connect sum, if $(-M\#M,\xi)$ were $(-M\#M,\xi_1\#\xi_2)$, then I'd have a belt sphere, which is $\{0\}\times\partial M^*$. My claim is that can't be true, because, well, consider my moduli space,

$$(\overline{M}\times\mathbb{D}^2,\partial(\overline{M}\times\mathbb{D}^2))\xrightarrow{ev}([-1,1]\times M^*,\{0\}\times\partial M^*)\xrightarrow{proj}(M^*,\partial M^*)$$

This is a map of degree 1 between compact manifolds with boundary. I can tell you where to find the rest of the argument. This is almost what happens when you prove the Eliashberg–Floer–McDuff theorem. By their argument, the result of their theorem was that the filling has to be contractible. Here I'm talking about th map from the moduli space to M^*, this shows that M^* is weakly contractible, and thus contractible, and so a ball. But my initial assumption is that M was not a homotopy sphere. This is a contradiction.

I'll stop there. Thank you.

14. Bohui Chen: Quantization of Kirwan morphism

Thanks for the introduction and to the organizers for the invitation and the local organizers for their hard work in making this conference happen. This is joint with Bai-Ling Wang in Australia. I'll start by talking about the background, the Kirwan morphism, I'll start by talking about (X,ω) a compact symplectic manifold, and suppose G is a compact Lie group which has a Hamiltonian action. We have $\mu : X\to\mathfrak{k}^*$, the moment map. If we have τ a central element, regular for the moment map, then we can take the symplectic reduction, $\mu^{-1}(\tau)/G$, and there's a natural symplectic form on this one. I'll call this quotient Z_τ and the level set before the quotient Y_τ. This Z_τ is usually denoted $X//_\tau G$.

We have a principle, a G-equivariant theory of X should imply the theory of Z_τ. So for example, we have a cohomological theory, $H^*_G(X)$ should imply some homology theory on Z_τ. How do we get this one? We have the restriction $H^*_G(Y_\tau)$, by restriction you get this, and then there's an isomorphism $H^*_G(Y_\tau)\cong H^*(Z_\tau)$, just assuming that G acts freely on Y_τ (otherwise you need to get into orbifolds). We easily construct such a map, and this map $H^*_G(X)\to H^*(Z_\tau)$ is called the Kirwan map K_τ, which is surjective, everything in the Z_τ can be seen in the equivariant theory. Another property is that this is a ring morphism. If we consider this with the cup product, then it's very easy to see that this is a ring morphism.

The main question we want to consider is, we said the G-equivariant cohomology ring of X implies the cohomology ring of Z_τ, but is it true at the quantum level? Does the quantum G-equivariant cohomology of X imply the quantum cohomology of Z_τ? We can ask the generalization where Z_τ is an orbifold.

For Z_τ, by Gromov–Witten theory, we have $QH(Z_\tau, \Lambda_\tau)$. This is a candidate for the right side. Then people try to see, do we have the equivariant cohomology for X? We have an equivariant Gromov–Witten theory, by Givental. This somehow gives you this quantization $QH^*_G(X, \Lambda_X)$. Then you can ask if there is a Kirwan map, here we emphasize the ring structure. The answer is kind of negative. I don't believe there is a ring morphism between these two guys.

Theorem 14.1. *Quantization of K_τ,*

- *We have a new quantization $\widetilde{QH}^*_{CR}(Z_\tau)$. Here I'll include the Chen–Ruan orbifold case.*
- *We have the quantization $H^*_G(X)^{CR}$*
- *We have the quantization QK_τ of the Kirwan map.*

Now I will explain the details. The starting point is the sypmlectic vortex equation. It's better to think of this as a Gromov–Witten type theorem for Z_τ based on the "ambient space" (X, ω). [didn't understand some]

So the motivation is, originally, we have $\Sigma \to Z_\tau$. Then Y_τ is a principle bundle over Z_τ. So you can pull this back to a principal bundle P on Σ and lift the map to Z_τ. There's a preferred connection on Y_τ because there's a horizontal [unintelligible]. Then you can pull back this connection and get a connection on the P side. and somehow generalize the $\bar\partial$ operator and say $\bar\partial_{u^*A}\varphi = 0$. This is the version of this being a holomorphic map from the equivariant version. You also have $\mu(\varphi) = \tau$. So the image is in the level set.

We want to generalize this equation. We consider $P \to X$ to be the G-equivariant map, and now we want A to be a parameter, a connection on P. Then the equation is

$$*_\nu F_A + \mu(\varphi) = \tau, \quad \bar\partial_A \varphi = 0.$$

Here ν is the volume form on Σ.

There's a very nice point of view on this equation, we consider the space $\mathcal{C}$ of pairs (A, φ) such that $\bar\partial_A\varphi = 0$. Then $\mathcal{G}$ is the gauge group of P. It turns out that this acts on the space of pairs. Then the moment map $\Psi : \mathcal{C} \to Lie(\mathcal{G})$ takes $(A, \varphi) \mapsto *_\nu F_A + \mu(\phi)$. Then $\{\Phi^{-1}(\tau)\}/\mathcal{G}$ is the moduli space of symplectic vortex equation. So we consider the solutions as above and then mod out by the gauge group.

The equation was introduced by Salomon and his student, and also by [unintelligible]Mundet. This appeared in physics earlier as the gauged σ-model.

If you have $\varphi : P \to X$ and it's G-equivariant, $[\varphi] \in H_2^G(X, \mathbb{Z})$. I don't want to write down the energy equation, but the energy of (A, φ) is $\langle \omega - \mu + \tau, [\varphi]\rangle^2$.

So okay, let me say, this isn't completely finished, but let me discuss Hamiltonian Gromov–Witten invariants. In this talk for simplicity I'll think of Σ as S^2 with three marked points. You regard this moduli space and map

$$ev : \overline{\mathcal{M}}_{\tau,[B],0,3}(X) \to X/G$$

where $[B] \in H_2^G(X)$.

Then there is a three point function $\psi_B^\tau : (H^*(X/G))^{\otimes j} \to \mathbb{R}$. People replace $H^*(X/G)$ with $H^*_G(X)$. If you cook up the [unintelligible]then you get an invariant theory like this.

People expect $\overline{M}_{\tau,[B],g,n}$, but the compactification has some problems. You have the phenomenon that energy is lost. They say they have several hundreds of pages for this issue. In the algebra case, for example Woodward and his collaborators have some work on this.

I want to mention our version. We have $L^2 - SVE$ and $\Sigma = (S^2, 0, 1, \infty)$ a punctured surface. If I look at this equation, the SVE, I emphasized that there's a volume form. I probably think of a punctured point and wonder what metric to put on the marked points. I want cylindrical ends. The L^2 theory means that thee energy is finite. This equation on cylindrical ends reads as a gradient flow of a certain function $\mathcal{L}_\tau$ of Bott–Morse type.

The critical points of this gradient flow are IZ_τ, the inertial set of Z_τ.

Assuming that everything works here, denote the moduli space $\overline{\mathcal{N}}^\tau_{[B],0,3}$ with an evaluation map to IZ_τ. Then $\psi_[B] : (H^*(Z_\tau))^{\otimes 3} \to \mathbb{R}$. You need to consider $0, 4$ to see that this defines a ring structure. The conclusion is that we have a Gromov–Witten type theory on Z_τ by considering such a moduli space with a cylindrical end metric. The three point functoins define the ring structure.

Now I move to the next part, the equivariant side. I should have said, we can only do this next part for G Abelian, so I'll assume for simplicity that G is S^1.

Let me review Givental's theorem. Then G acts on this moduli space, and you can consider the G-space of this, the invariants, $EG \times_G \mathcal{M}(X)$. I want the equivariant theory so I'll take $\Sigma \to X_G = S^{2\infty+1} \times_{S^1} X \to \mathbb{CP}^\infty$. Now we'll take the special case that u has the class of the point. Then it's easy to get $\mathcal{M}_{[B]}(X_G)$ which is $S^{2\infty+1} \times_{S^1} \mathcal{M}_B(X)$. We can try to generalize this picture, think of the more general case.

On the other hand, I need our picture to motivate the equivariant case. [pictures]

So I have $\mathcal{M}^\tau$ to quantize Z_τ and I want to cook up a quantization $\mathcal{C}$ of X so that there's a symplectic reduction to $\mathcal{M}^\tau$.

The idea to get the picture is very easy. For $\mathcal{M}^\tau$ we have the equations

$$* F_A + \mu(\varphi) = \tau$$

$$\bar{\partial}_A \varphi = 0$$

This is all taken modulo G', the based gauge group so that $\mathcal{G}/\mathcal{G}' = S^1$. So I take the union over τ and that gives me a new space $\mathcal{C}$, and hopefully this quantizes X.

Now I can write this one as $d\tau = 0$, so τ is any constant. Roughly speaking, this is what I want to consider. So for example, if I consider $\Sigma = (S^2, 0, 1, \infty)$, and $[B] = 0$ then [unintelligible].

Okay, this is the motivation for the construction. We do this using the L^2 theorem, so we should do that in this equation. We have to modify the equation to fit it with our picture for the L^2 case.

So $*F_A + \mu(\varphi) = \tau$ and $\bar{\partial}_A \varphi = 0$. This is a family with parameter τ. Remember that for each equation I'll consider the L^2-theory. The equation will be gradient flow of $\mathcal{L}_\tau$ on the cylinder ends.

If you think of this this way and then think of the flow converging to the critical points of $\mathcal{L}_\tau$, then this one is actually, for each $\mathcal{L}_\tau$, you have Y_τ, you put them together, there is some extra thing. I had forgotten to mention, only when τ is regular [unintelligible]. When τ is not regular, you get some extra part, because it's not Bott–Morse type. That makes this picture fail because we don't really want the extra. The family version does not really work.

The family version, for each τ I get $\mathcal{L}_\tau$, and it turns out that if I treat τ as a variable, I have this function. Then when I consider the gradient flow for this one, $Crit(\mathcal{L}) = X$. We can cook up an equation and on the cylinder ends is the gradient flow for this $\mathcal{L}$.

The gradient flow of $\mathcal{L}$ is $\begin{cases} \alpha_t + \mu(u) - \tau = 0 \\ \varphi_t + J(\varphi_\theta + X_\alpha) = 0 \ \tau_t + hol(\alpha) = 0. \end{cases}$ so then [missed a little, degenerates into questions]

Using this moduli space you really get the three-point function

$$\psi^G : (H^*_G(X)^{\otimes 3} \to H^*_G(pt)$$

and you can prove associativeity and this gives you a quantum product. Let me just, three minutes. The quantum Kirwan, on one side we use the augmented part, and this on the other side, the natural part, and this is how we define the morphism.

Let me add some final comments. We already have the equivariant homology $H^*_G(X)$ and equavariant Gromov–Witten, and then on the other side $H^*(Z_\tau)$. Then we can [unintelligible], and you can [too fast] The quantum Kirwan morphism, Woodward tried to draw a map like this [picture], we have a project to do this, anyway, I'll finish here.

15. Toru Yoshiyasu: On Lagrangian submanifolds in the complex projective spaces

First of all, I thank the organizers for giving me a chance to talk here. I'm really enjoying this stay and listening to good talks and eat good Chinese food. Let me start. The contents of this talk is as follows. In section 1, I will explain my motivation and main theorem. In section two, I'll talk about Lagrangian self-intersections, and then in section three I'll discuss the proof.

I'll discuss an existence result in $\mathbb{CP}^3$ and $\mathbb{CP}^1 \times \mathbb{CP}^2$. We consider the set of Lagrangian embeddings. It is important to study the topology of Lagrangian submanifolds. We know only a few necessary conditions on these. They are far from classification. They have some topological constraints. For example, Gromov and Viterbo and Seidel and so on. On the other hand, we can put them inside the set of Lagrangian immersions. These are classified by Gromov's h-principle. The existence of a Lagrangian immersion is reduced to, or classified by, [unintelligible].

The principle does not give an explicit characterization, but we can obtain some characterization. They are dominated by immersion and homotopy theory.

We don't know the topological classification of embeddings and so we don't know the difference between embeddings and immersions. How are they different? There are some results about this.

First, there's Maslov class rigidity. Immersed Lagrangians are not restricted in the same way. There are also non-existence results for Lagrangian embeddings, some manifolds have immersions but no embeddings.

I'm concentrating here on how the two notions are different. I'll talk about the difference between Lagrangian immersions and Lagrangian embeddings.

The following is the main theorem.

Theorem 15.1. *Let X be the complex projective space $\mathbb{CP}^3$ or $\mathbb{CP}^1 \times \mathbb{CP}^2$. with the Fubini–Study form ω_3 or $\omega_1 \times \omega_2$. Let L be a closed oriented connected 3-manifold and $f : L\#(S^1 \times S^2) \to X$ a Lagrangian immersion.*

Then there exists a Lagrangian embedding $g : L\#(S^1 \times S^2) \hookrightarrow X$ such that $g \sim f$ as a continuous map.

I have to comment on the existence of a Lagrangian embedding.

Remark 15.2. Ekholm–Eliashberg–Murphy–Smith (EEMS) proved that there exists a Lagrangia embedding of $L\#(S^1 \times S^2)$ into $\mathbb{C}^3$ (the standard symplectic 6-space). Their result says that there is a contractible such submanifold in any 6-fold. But our theorem cannot be reduced to this because it provides *non-contractible* Lagrangian embeddings.

15.1. **Lagrangian self-intersections.** Let me talk about the h-principle. This is for Lagrangian embeddings that are concave with [unintelligible]boundary.

Theorem 15.3. *(Eliashberg–Murphy, 2013) Let $n \geq 3$ and L a connected n-dimensional manifold with a negative end. Let (X, ω) be a symplectic $2n$-dimensional manifold with negative Liouville end.*

Assume that (X, ω) has Gromov width infinite if $n = 3$.

Let $f : L \hookrightarrow X$ be a proper embedding satisfying

- *the embedding is cylindrical at the negative end,*
- *$[f_0^*\omega] = 0 \in H^2(L, \mathbb{R})$ and there is $F_t : TL \to TX$ homotopy of monomorphisms such that $F_0 = df_0$ and F_1 is Lagrangian, and*
- *the asymptotic Legendrian boundary of f_0 has a loose component.*

The conclusion is that there exists an isotopy $f_t : L \to X$ which is fixed at $-\infty$ such that f_1 is a Lagranian embedding.

In their paper, they mention that the ∞ Gromov width is a technical assumption. We don't know if it's necessary or not. Actually, this is not necessary (due to the speaker).

The statement includes some unusual definitions. I will draw a picture instead of giving precise definitions.

[picture]

Negative means there exists an end component which is a cylinder. A negative Liouville end means that there is an end obtained by the negative symplectization, the product with the negative half-line. The Liouville vector field points inward from the end.

Being cylindrical at the end means it's an embedding on some contact slice and the end is a linearization of that embedding. The image of the embedding in the slice is the asymptotic Legendrian. The cohomological condition is a trivially necessary condition. Taking the differential satisfies the homotopy of monomorphism conditions, so that is also a trivial necessary condition.

Before starting a proof, I define looseness. Let $n \geq 3$ and (Y^{2n-1}, ξ) be a contact $2n-1$ manifold and (Λ) a connected Legendrian submanifold of Y. Then Λ is loose if there exists Λ' such that Λ is Legendrian isotopic to a stabilization of Λ'.

There is one more unusual definition, a *stabilization* of a Legendrian is a localization move.

For any Legendrian submanifold and any point p [unintelligible]we can take a Darboux neighborhood of p such that there is a cusp singularity at the origin in the front projection, with $\xi_{st} = \ker dz - ydx$. We move the lower branch around the upper branch [picture] and this is called a stabliization of Λ'. The meaning of stabilization is not clear. It makes a Legendrian manifold flexible. Recall [unintelligible]theory, [unintelligible]theory, Gromov's [unintelligible]theory, and Eliashberg's classification of overtwisted contact structures. Normally solving differential equations is hard. But [missed].

The looseness is crucial for Murphy's h-principle, which says loose Legendrian submanifolds are dominated by algebro-topological properties.

For general Legendrian submanifolds, they do not have such a property. I forgot the example, but there are two submanifolds which satisfy algebro-topological homotopy conditions but are not Legendrian isotopic.

Let me start a brief sketch of the proof of the main theorem.

I'll start with a modified Whitney trick, due to Eliashberg–Murphy.

- For a Whitney pair of an exact Lagrangian immersion, if the symplectic area of the Whitney disk is 0, then there exists a Darboux ball containing the Whitney pair such that [picture]. For general Lagrangian immersions, this condition is not satisfied, the symplectic area can be changed by a small perturbation of the Whitney disk. So it's a special condition.
- We have to construct a procedure to make the conditional step satisfied. There is a deformation to make the previous condition satisfied.
- The Lagrangian intersections with the Whitney pair can be cancelled by Legendrian regular homotopy if and only if their boundaries are unlinked in the Legendrian category. In general, this is false. We have to use loose Legendrians at the negative ends. We can check by using asymptotically loose Legendrian boundary and the h-principle. In general, the right-hand side statement is false. But now we can take the connect sum of a Legendrian sphere and a loose Legendrian, then the behavior of this Legendrian is dominated by algebraic properties. [missed] In the smooth category they are unlinked, and then they are also unlinked in the Legendrian category.

This is a brief sketch of the theory of Legendrian immersions. There is the infinite Gromov width. To do this they need many many Darboux points. We encounter the volume problem of the whole symplectic manifold X. They are in distinct Darboux balls, if the volume is small, we cannot take this picture.

What I improved in their deformation was to prove that the infinite width assumption was not necessary.

In the last section I prove the main theorem.

15.2. **Proof of the main theorem.** My first proposition

Proposition 15.4. *Let L be a 3-manifold and $\gamma_3 \to \mathbb{CP}^3$ the tautological line bundle. Then there exists a bijection $[L, \mathbb{CP}^3] \to H^2(L, \mathbb{Z})$ where $[d] \mapsto -h^* c_1(\gamma_3)$. The proof is elementary. First, $K(\mathbb{Z}, 2) = \mathbb{CP}^2$, which is its 6-skeleton. If we change from $\mathbb{CP}^3$ to $\mathbb{CP}^\infty$, and then $H^2(L, \mathbb{Z}) = [L, \mathbb{CP}^\infty]$. We can change the the target space.*

We want to study Lagrangian immersions and embeddings by Gromov's h-principle.

Proposition 15.5. *Let L be a closed oriented connected 3-manifold and $h : L \to \mathbb{CP}^2$ continuous. Then the following statements are equivalent.*

(1) *There exists a Lagrangian immersion* $L \to \mathbb{CP}^3$ *homotopic to* h.
(2) $[h^*\omega_3] = 0$ *in* $H^2(L, \mathbb{R})$ *and there exists a Lagrangian monomorphism* $H : TL \to T\mathbb{CP}^3$ *which is a lift of* h.
(3) $h^* c_1(\gamma_3)$ *is* 4*-torsion in* $H^2(L, \mathbb{Z})$.

Non-contractibility corresponds to this cohomology class. The contractiblity corresponds to being zero in the integral cohomology.

Let me start the proof. I'm only going to do the case when X is $\mathbb{CP}^3$, the essentials are the same but the homotopy class statement is complicated so I won't explain the details.

Proof. The first and second are equivalent by Gromov's h-principle. The second and third are equivalent by a computation of cohomology classes, $[\omega_3] = -c_1(\gamma_3) \in H^2(\mathbb{CP}^3, \mathbb{Z})$, which implies that $h^* c_1(\gamma_3)$ is torsion. The existence of H can be treated by obstruction theory. You can show that the only obstruction is $c_1 h^* T\mathbb{CP}^3$ which is $-4h^* c_1(\gamma_3) \in H^2(L, \mathbb{Z})$. □

Let me talk about about the proof of the main theorem for $X = \mathbb{CP}^3$.

Proof. We have a Lagrangian immersion $L\#(S^1 \times S^2) \to \mathbb{CP}^3$. We construct a Lagrangian embedding homotopic to f.

We restrict to $L \backslash D^3 \to \mathbb{CP}^3$. Then $[h]$ is in $H^2(L \backslash D^3, \mathbb{Z}) \cong H^2(L, \mathbb{Z})$. We can take an extension of h as a continuous map. This satisfies the third condition of the proposition, and so does the extension. Then we apply the proposition to the extension of h. There exists $h' : L \to \mathbb{CP}^3$ a Lagrangian immersion, homotopic to h. via the identification above.

By the Whitney trck, the number of double poits of h' is one. At last, applying Polterovich's surgery, we get an embedding of the connected sum into $\mathbb{CP}^3$. They are homotopic by computing that $g^* c_1(\gamma_3) = f^* c_1(\gamma_3)$.

□

16. Nov. 6: Bohan Fang: Global mirror curve of a toric Calabi-Yau 3-fold

Thank you for the chance to speak here. This is joint with C.-C. Lie and Z. Zong. I'll be talking about toric Calabi–Yaus. So X is a toric Calabi–Yau, the defining polytope Σ and a triangulation. If $X = \mathcal{O}_{\mathbb{P}^2(-3)}$ then the defining polytope is [picture], a triangle with corners $(1, 0)$, $(0, 1)$, and $(-1, -1)$. If you look at $X = \mathbb{C}^3/\mathbb{Z}^3$, that's a polytope with no further triangulation [picture] and then the fan looks like this: [picture]

Mirror symmetry has this to say. If X is a toric variety, I think by Givental he constructed its mirror as a Landau–Ginzburg model, a holomorphic function $W : (\mathbb{C}^*)^n \to \mathbb{C}$. The combinatoric data defining this toric Calabi–Yau tells you this W has to be in the form of $X_n H(X_1, \ldots, X_{N-1})$. By [unintelligible], this can be further reduced to some noncompact Calabi–Yau, which gets to the more traditional line of thinking, that the mirror of a Calabi–Yau should be a Calabi–Yau.

What about toric Calabi–Yau threefolds? The right side, by Hori–Vafa in physics, can be further reduced to $\{H(X, Y) = 0\}$ in $(\mathbb{C}^*)^2$. This is a mirror curve. This is a quick overview of the mirror of a Calabi–Yau threefold.

Let me give some examples. For the purposes of the rest of the talk, I'll shift the polytope for some reasons I'll say later. As you know, this triangulation I gave before is equivalent to the triangle with vertices $(0,0)$, $(1,0)$, and $(3,-1)$ with center at $(0,1)$. Then $X+Y+1+qX^3Y^{-1}=0$. Then $X=[\mathbb{C}^3/\mathbb{Z}_3]$ has the same polytope but no triangulation. You still look at all possible integer lattices inside of this. You repeat what you did, but the only preferred triangle is this one. [picture] and you get $X'^3Y'^{-1}+Y'+1+q'X'=0$ and these two curves are isomorphic up to the change of variables $X=q'X'$, $Y=Y'$, and $q'^{-1}=q$. This is explained by them being crepant resolutions of the same singular variety $\mathbb{C}^3/\mathbb{Z}_3$.

How this mirror curve reflects the A-side information, mirror symmetry predictions of Gromov–Witten invariants. Okay. So X is our toric Calabi–Yau 3-fold, given by the defining polytope with a preferred triangle (the one with vertices $(0,0)$, $(1,0)$, and $(0,1)$). You can draw this so-called toric graph, which is the dual graph of this polytope. [picture]. If you look at symplectic geometry, this is the image of 1-dimensional toric divisors under a moment map $\mu'_{\mathbb{R}}$ of $T'_{\mathbb{R}}$, where $T'\subset T$, the big torus that acts on this variety, preserving the Calabi–Yau form. The $\mathbb{R}$ means its the compact part.

For $X=\mathbb{C}^3/\mathbb{Z}_3$ this looks like this (picture). What is the purpose of the preferred triangle? You pick a point p and can define a Lagrangian $L=\mu_{\mathbb{R}}^{-1}(p)+$ constant argument condition, so this is a Lagrangian inside X that is homeomorphic to $\mathbb{R}^2\times S^1$ ([unintelligible]-Vafa)

Now you can do a counting, or at least pretend (some definitions are up in the air). $N_{g,\beta,\vec{\mu}}$ counts [picture] where you have h boundary circles, genus g, $\xrightarrow{f}(X,L)$ with topological type $\vec{\mu}$, a tuple of positive integers, and $f_*[C]=\beta\in H_2(X,L;\mathbb{Z})$ and $f_*([R_i])=\mu_i\in H_1(L,\mathbb{Z})$.

This is not a definition but let's pretend that we can do this for the purpose of this talk.

The actual function that we worry about is

$$F_{g,h}(Q)=\sum_{\beta,\vec{\mu}}N_{g,\beta,\vec{\mu}}\tilde{X}_1^{\mu_1}\cdots\tilde{X}_h^{\mu_h}.$$

This is the open Gromov–Witten potential. You should add twisted insertions for orbifold points. The first prediction, anyway, is that

$$F_{0,1}=\int_{P(X)}logY\frac{dX}{X}.$$

So let's get back to the toric graph. The mirror curve looks like this [picture] and this is the preserved point and you can choose a path to a point and integrate over it. This is an indefinite integral.

This is Aganagic–Klemm–Vafa, and also F.–Liu and F.–Liu–Tseng.

Okay, so let's see what we get for the mirror curve of $K_{\mathbb{P}^2}$ and for $\mathbb{C}^3/\mathbb{Z}_3$. SO it's basically the same thing up to an isomorphism. So you have

$$F_{0,1}^{K_{\mathbb{P}^2}}(q,X)=F_{0,1}^{\mathbb{C}^3/\mathbb{Z}_3}(q',X')$$

up to change of coordinates and analaticy continuation, [something about q]

Let me state a long theorem and discuss how you can imply this kind of global picture. This mirror curve $C_q=\langle H(X,Y)=0\rangle$ and the compactification $\overline{C}_q$. We have $A_i,B_i\in H_1(\overline{C}_q,\mathbb{Z})$ with $A_i\cap A_j=B_i\cap B_j=0$ and $A_i\cap B_j=\delta_{ij}$. We have

$\omega_{0,2}$, symmetric meromorphic 2-form on $(overlineC_q)^2$, $\int_{A_i}\omega_{0,2}=0$ and you have

$$\omega_{0,2}=\frac{dxdy}{(x-y)^2}+\text{ holomorphic at diagonal (only poles)}$$

and $\omega_{g,h}$ is constructed from $\omega_{0,2}$, a symmetric meromorphic n-form on $(\overline{C}_q)^h$,

$$\omega_{g,h+1}=\sum_{P_2,dx=0} Res_{p=P_\alpha}\frac{\int_{[unintelligible]}^{p}\omega_{0,2}(P_0,\quad)}{2(y(p)-y(\bar{p}))dx(p)}$$

[missed some]

So this is a recursion, this is well-studied, it's well-defined over the compactified curve. Before I proceed, the point is that you have to remember what the ingredients you need to cook up this higher $\omega_{g,h}$. You need X and Y, the meromorphic functions on your curve with poles at your punctures, and you need A and B cycles, from that you have $\omega_{0,2}$. You need not just X and Y but also the A and B cycles, this choice.

The BKMP remodeling conjecture says that

$$\int_{P(X_1)\times\cdots\times P(X_h)}\omega_{g,h}=F_{g,h}$$

This was proved by Eynard–Orantin, F.–Lie–Zong.

So

$$F_g=\frac{1}{2-2g}\sum_{P=P_\alpha,dx(p_\alpha)=0}\omega_{g,1}\int\log Y\frac{dX}{X}.$$

[missed a little]

Now what do we have? So we have this very long, we just gave some brief introduction including the higher genus picture. Now I want to say something about how to look at how things change under different points in the moduli space of toric varieties. The statements I gave is just for one point in the moduli space. That's good, but what about, how do you fit everything into the whole moduli space of toric Calabi–Yaus? This is a global mirror symmetry thing that actually we did this discussion before about disk invariants, for example let $X=K_{\mathbb{P}^2}$ or $X=[\mathbb{C}^3/\mathbb{Z}_3]$, this is the large radius limit and this is the orbifold point. In the middle we don't know the Gromov–Witten theory.

The crepant resolution conjecture says how to try to relate these. We at least wrote down a change of coordinates between these two ponits. Now I'll say for a more general 3-fold, how to write down a mirror curve; overall they'll form a flat family of mirror curves.

To solve the problem of this global mirror curve, I'll still have X a toric Calabi–Yau 3-fold and $N=Hom(\mathbb{C}^*,T)$, and $M=Hom(T,\mathbb{C}^*)$ with $\ker e_3=T$. Then $M=Hom(T',\mathbb{C}^*)=M/\langle e_3\rangle$. If you know what I'm talking about, good, if not, that's fine, I'll give a picture.

$$0\to\mathbb{L}\to\tilde{N}=\bigoplus\mathbb{Z}\tilde{b}\to N\to 0,$$

$$0\to M\to\tilde{M}\to\mathbb{L}^\vee\to 0,$$

$$0\to M'\to\tilde{M}'\to\mathbb{L}^v\to 0.$$

So you have the secondary fan Δ in $\mathbb{L}^\vee$ and the extended secondary fan $\tilde{\Delta}$ in $\tilde{M}'$ given by $\tilde{D}_i'$, plus infinite rays of toric graphs. This is a very simple combinatoric description.

We know $K_{\mathbb{P}^2}$, the secondary toric variety is $\mathbb{P}^1$, and this D_i, actually you have four, and [pictures]

There is a fan morphism $\tilde{\Delta}$ to Δ and this gives rise to a toric morphism $\tilde{\mathcal{M}}_3 \to \mathcal{M}_3$. [pictures]

I construct S a polytope given by $\tilde{b}_i'$, say that $\tilde{N}$ is dual by a toric construction to [unintelligible], and $\tilde{N}'$ is a subthing of $\tilde{N}$ killed by the Calabi–Yau character, and this is not canonical but it's fine. This defines a line bundle up to a translation of the polytope. This gives a line bundle L on $\tilde{M}_B$, which is fiberwise ample. Then $s = \sum s_i$ is a section, with each s_i a section corresponding to an integer point in S. Define this $\overline{\mathcal{C}}$ as the zero set of this section. [pictures]

You have to pick A_i and B_i in the beginning. The standard notation, $\tau_{ij} = \int B_j \mathcal{O}_i$ and $\int_{A_j} \theta_i = \delta_{ij}$. Around a toric divisor in $\mathcal{M}_B$, you'll see $\tau \mapsto \frac{a\tau+b}{c\tau+d}$. If you compute all of these you'll get a monodromy group. For example if $X = \mathcal{O}_{\mathbb{P}^2(-3)}$, then the group Γ is a subgroup of $SL_2\mathbb{Z}$, of level three. So you need to modify it somehow.

Eynard–Orantin proposed something,

$$A_i(\tau) = A_i - \sum_j \frac{1}{\bar{\tau} - \tau} B_j(\tau)$$

and

$$B_i(\tau) = B_i - \sum_j \tau_{ij} A_j$$

and the $A_i(\tau)$ and $B_i(\tau)$ are monodromy invariant. Once you do this, then A and B are no longer geometric, and no longer holomorphic. But this fits into the physics prediction. From $A_i(\tau)$ you can construct $\tilde{\omega}_{0,2}$ and from the E–O recursion you can construct $\tilde{\omega}_{g,h}$ globally defined on $\overline{\mathcal{C}}$.

What's the relation? By some physics relation I'll explain,

$$\lim_{Im\tau \to \infty} \tilde{\omega}_{0,2} = \omega_{0,2}$$

$$\lim_{Im\tau \to \infty} \tilde{\omega}_{g,n} = \omega_{g,n}$$

Definition 16.1.

$$\tilde{F}_g = \sum Res \tilde{\omega}_{g,1} \int \log Y \frac{dX}{X}$$

then

$$\lim_{Im\tau \to \infty} \tilde{F}_g = F_g$$

Theorem 16.2. *(Calso–Coates–Iritani) F_g from Gromov–Witten has a unique anti-holomorphic ocmpletion as the constant term of a polynomial in $\frac{1}{Im\tau}$ with holomorphic coefficients that is modular invariant.*

So this is my version, I don't know if this is the same as other peoples' version. Let me stop here.

17. Gabriel C. Drummond-Cole: Chain level string topology operations

I do not take notes on my own talks.

18. Yohsuke Imagi: Simple singularity of special Lagrangian submanifolds

Before I talk about some technical things about my results, I want to comment on the relationship between [unintelligible]and my own study. Many results use some sort of Floer theory or Gromov–Witten invariants or pseudoholomorphic curves and one important theorem is that you can define a moduli space of pseudoholomorphic curves and there's a nice compactification that you can use to define some Floer homology.

The important point is, you have a nice compactification of the moduli space. Some differential geometric nice compactification goes back to Yang–Mills gauge theory in dimension 4 which is called Yang–Mills instantons, and you can also define a moduli space of instantons and define a nice compactification of the moduli space of them. This is not really symplectic geometry but is historically, [unintelligible]one motivation of my study comes from these things, and why I mentioned these things. These are all solutions to elliptic equations over some [unintelligible]manifold, in a suitable sense, they are all elliptic nonlinear equaitons and can be treated similarly in some way. The treatments are well-known but the special Lagrangians are more difficult, seriously so, and so that's why I have to restrict to only simple singularities.

This is some historical comment. I'll start with some basic definitions of special Lagrangian submanifolds.

I consider a Calabi–Yau manifold, that's a Kähler manifold and in the latter part of my talk it will have complex dimension 3. For the moment the dimension may be arbitrary. Basically I'm interested in the higher dimensional case; in the lower dimensional case they are easy. HyperKähler rotation takes them to pseudoholomorphic curves in dimension 2, so think big. So Ω is a nowhere-vanishing $(m-0)$-form on M. I suppose Ω exists. I want to define special Lagrangian submanifolds. L is special if $\Im\Omega|_L = 0$. It's an extra condition on the Lagrangian submanifold. One basic theorem by Harvey and Lawson is that special Lagrangian submanifolds are volume minimizing. This is the basic theorem. It's in the 1980s and they discovered a large class of volume-minimizing submanifolds including [unintelligible]and [unintelligible], some special classes of volume-minimizing submanifolds.

If you know their original definition, they originally assumed that the Kähler metric was Ricci flat, and the special Lagrangians are volume-minimizing with respect to the Ricci-flat metric. In the latter part I'll explain my result and I don't suppose the Kähler metric is Ricci flat. Then you have to choose a metric. This is a technical part. If this is your first time then it's okay, it's volume-minimizing with respect to some certain canonical metric.

So the equation is a kind of elliptic equation and there's some [unintelligible], anyway, some kind of elliptic thing. And as I told you, I want to put the moduli space of special Lagrangian submanifolds, I want to consider, I fix M and Ω and consider the moduli space of compact special Lagrangians, I call it $\mathcal{S}$.

One feature is that this is always unobstructed. Each component of S is [unintelligible]. I should mention McLean's name. You have a good understanding of the local structure. The global structure of the moduli space, I want to compactify the moduli space, and one way to do this is to use geometric measure theory and one way is to just use special Lagrangians and currents or varifolds, some measure-theoretic generalization, and this is one compactification,and the compactified space, I call it $\bar{\mathcal{S}}$. This is very difficult to analyze. I go back here and in the well-known case, the Yang–Mills gauge theory case in dimension 4, the instantons, you can compactify by adding some singular objects. Then [unintelligible]shows that you can [unintelligible]δ functions, compactifying the moduli space. So in this case it's very simple but in our case, these are the singular objects, the currents, but they may be very complicated and the problem is very difficult. There is no classification result, that's the basic difference.

In 2000, Joyce, [unintelligible], I., started to study some singular objects, so one approach is to just consider simple singularities. Ultimately, we want to consider all the singularities like that. What we have done is to choose some simple class of singularity and study them in detail. That's what I'll explain, what kind of complete progress I can make, that's what I'll explain.

Now I restrict to dimension 3. In this talk I consider this class of singularity, compact special Lagrangian 3-folds with one singular point. If you know minimal surface theory then at each singular point there is a tangent cone to the singular point and in general there is some multiplicity here, I suppose multiplicity 1, there's just one singular point and there's also an isolated singularity in the tangent cone, also multiplicity 1, a very simple case. What kinds of things can I put here to make this cone special Lagrangian. The tangent space at x is like $\mathbb{C}^3$ and the holomorphic volume form is $dz_1 \wedge dz_2 \wedge dz_3$. Consider the $SU(3)$ action on $\mathbb{C}^3$. This volume form should be preserved along with the complex structure and the metric. So take this maximal torus T^2, where the total rotation is 0, and you can construct the T^2 cone like this [picture]

This is Harvey and Lawson's example. This class of X is a simple class of singularity to study.

Theorem 18.1. *We can determine a neighborhood of X in $\mathcal{S}$.*

In the Yang–Mills case, Donaldson proved that the moduli space is a manifold with boundary or corners. This theorem is a local analogue of Donaldson's theorem in $\overline{\mathcal{S}}$, in the moduli space of special Lagrangian submanifolds. This is really only local.

Theorem 18.2. *There exists an example of X. More precisely, there exists a compact special Lagrangian submanifold of (M, Ω) with just one singular point. moedled on C with multiplicity* 1.

This is historically younger, so Dominic Joyce, Haskins, we did some general theory at first, but we had no examples, this was very unhealthy. There might be no examples to which we could apply the theorem. But what I'm going to talk about is that there is an example of X and that's what i'm going to talk about.

The basic idea of the theorem is originally due to Joyce and Haskins, I learned it from Dominic and also talked to Haskins several times, and I also talked to Johannes Nordström, he and Haskins were in Imperial College, London. The basic strategy is well-known to them.

In my case I should choose the Calabi–Yau very carefully. The analysis will be more complicated as well. The basic strategy is very well-known.

I start with a concrete method for constructing special Lagrangian submanifolds. The trivial example is $\mathbb{R}^n$ in $\mathbb{C}^n$, which is useful to regard as the fixed point of the involution, antiholomorphic. If I have a Calabi–Yau manifold M and an antiholomorphic involution, then its fixed points will be special Lagrangian with respect to some Ω. It's hard to construct them in any other way. This is particular to special Lagrangians. This always gives smooth objects, without singularities.

So I start with a singular Calabi–Yau. I take N and take some nodal 3-fold in $\mathbb{CP}^4$, and I need an antiholomorphic involution. I have to choose some real polynomial, and I take, $\bar{w}, \bar{z}, \bar{x}, \bar{t}$.

Now I can choose ι so that it has only one singular point, and it's also modeled on [unintelligible]. This involution method gives you a three-fold with a singularity modeled on a nodal three-fold. What I want is a special Lagrangian three-fold in a non-singular threefold. I don't want the ambient space to have singularity. So I take a projective small resolution here. It needn't exist in general for all Calabi–Yau nodal three-folds. I also choose this guy carefully so that I can take some divisors whose blowup gives some small resolution. Let me explain what I mean.

It's a well-studied object, but let me say, projective means the resolution is Kähler. Small means it contains some blowdown. There is a singular point y. The fiber over y is $\pi^{-1}(y)$, this is $\mathbb{CP}^1$, it's normal bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. You can change the Kähler form from here. You can prove the Kähler form, you can modify this so that the [unintelligible]will be Kähler too.

I can pull back this guy away from the singular point. [missed some] You can find a T^2-invariant special Lagrangian like this.

I have a T^2 cone downstairs, and I just pull back this Y, so near this $\mathbb{CP}^1$, $\pi^{-1}(Y)$ approaches this cone K, and on the other hand I have a [unintelligible]action on [unintelligible]and I construct it explicitly, the non-compact special Lagrangian approaching this K. Over Y the cone Z, some different cone C [picture] and this Z is explicit, fully explicit. This is also explicit.

This is the method going back to Taubes. What we do finally is ta deform it globally. I also rotate the tangent cone and translate this point by T^3 and then you can do some, write the special Lagrangian equation at the linear level it's Fredholm and it's also surjective and you can use the implicit function theorem and solve it analytically.

19. Ziming Nikolas Ma: Scattering in the SYZ programme

[I did not attend this talk]