INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS FUKAYA CATEGORY AND TORIC GEOMETRY LECTURES

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1. March 4

I want to start to talk about the tools and methods that are available to us in symplectic topology. Let me begin with:

Floer homology

Let (M, ω) be a symplectic manifold, either closed or open with suitable nice behavior at ∞. These are always the assumptions. What Floer homology does is the following. In symplectic topology, there is a kind of creed that says everything is a Lagrangian submanifold. For those who do not know,

Definition 1.1. A submanifold $L \subset (M, \omega)$ is called Lagrangian if dim L is $\frac{1}{2}$ dim M and $i^*\omega = 0$.

We'd like to study the intersection theory of these basic objects. Suppose L_0 and L_1 intersect transversally. For now let's assume for convenience that M and all Lagrangians are closed submanifolds.

Then there will be only a finite number of intersection points. Fix a coefficient ring. Denote

$$CF(L_0, L_1) = R[L_0 \cap L_1];$$

this is a free R-module with canonical basis $L_0 \cap L_1$. We want to define a "boundary map"

$$\partial : CF(L_0, L_1) \to CF(L_0, L_1).$$

Eventually we want to regard these as graded modules with some grading. This boundary map will have grading -1.

Any linear map can be written as a matrix in our basis, so $n(q,p)$, well $\partial q = \sum_{p \in L_0 \cap L_1} n(q,p)p$.

So how do we define this $n(q,p)$? We need some kind of probe to investigate this manifold with Lagrangian submanifold. We want to look at the motion of branes in the ambient space.

For this one, we need some auxilliary structure, another essential object from the point of view of symplectic topology, the so-called almost-complex structure. After Gromov introduced pseudoholomorphic curves, it's still not clear, actually, how these things are really, how they act on the study of symplectic topology.

This almost complex structure, there are plenty of almost-complex structures J on a symplectic manifold. They are abundant. Furthermore, Gromov proved that there is a particular choice, a class of J, so-called compatible, interacting with the given symplectic form in a nice way:

i Positivity: $\omega(v, Jv) \geq 0$ with equality only when $v = 0$.
ii Symmetry: $\omega(v, Ju) = \omega(u, Jv)$.

As a consequence, the bilinear form $g_J = \omega(\ ,J\ )$ defines a Riemannian metric. Sometimes we call (M, ω, J) an almost-Kähler manifold and g_J an almost-Kähler metric.

Gromov proved that $\mathscr{J}_\omega$, the space of such J, is a *contractible* infinite-dimensional manifold.

So when you're given, we then study some equations, Cauchy–Riemann equations on a symplectic manifold. We want to find some holomorphic strips between L_0 and L_1.

Let me draw, [pictures]. We require that the map $u : Z = \mathbb{R} \times [0,1]$ to M satisfies

$$\frac{\partial u}{\partial z} + J\frac{\partial u}{\partial t} = 0$$

with $u(z, 0) \in L_0$ and $u(z, 1) \in L_1$. I should have as well that $u(-\infty) = q$ and $u(\infty) = p$.

We need some machinery of studying the existence question for solutions of this equation.

This equation at the moment is coordinate-dependent. This implicitly uses the conformal structure on the strip $z + it$. This infinite strip Z is isomorphic to the disk with two points removed. What's important is just the conformal structure. That equation can be written in a coordinate independent way. Temporarily let me denote the set of solutions of that equation by $\widetilde{M}(q,p)$. This set, in general may have no structure. But the matrix element will be defined by counting the number of elements in this set.

Now $n(q, p; J)$ will be defined by "counting" $\#\widetilde{M}(q, p; J)$, which may not even make sense because the set may not be finite.

So we have to ensure that this set of solutions has some nice structures and is in particular finite. This means there are two critical issues. The first is smoothness of the space $\widetilde{M}(q, p; J)$; the second is compactness of $\widetilde{M}(q, p; j)$. The compactness is something in PDEs, it has to do with the "a priori estimate" and the smoothness issue is related to deformation theory and transversality (of something). How do we handle these two issues? Here comes some hidden structure in this equation here. This is not an arbitrary equation. There is something hidden in the form of the equation that we have to understand, the hidden structures. For this one, we need to study so-called

Offshell formualtion of the equations

In this stage I can now talk about the players in symplectic topology. Let me make some comparisons between symplectic and differential topology. In general topology you are going to look at a differential manifold. In symplectic topology you look at (M, ω) a symplectic manifold. You compare diffeomorphisms to symplectic diffeomorphisms.

A fundamental object of study is fixed points of diffeomorphisms and for us the fixed points of symplectic diffeomorphisms. We know that this is the same thing as the intersection of the graph of ϕ with the diagonal in $M \times M$. In the symplectic case the graph of ϕ is Lagrangian with respect to $(M \times M, \pi_1^*\omega - \pi_2^*\omega)$.

So this is a natural subset of Lagrangian intersection theory.

So isotopy invariants are interesting in topology, but somewhat mysteriously it's not just symplectic isotopy but Hamiltonian isotopy that has to do with the study of these Cauchy–Riemann equations.

This is the reason that the interplay between Lagrangian intersection theory and Hamiltonian dynamics has so much traction in symplectic topology.

So invariants in the differential setting are preserved under isotopy. On the other side are invariants that are preserved under Hamiltonian isotopy. Of course, Hamiltonian and symplectic isotopy differ only when the manifold is not simply connected. It's more than that actually.

M differential manifold	(M,ω) symplectic manifold
ϕ a diffeomorphism	ϕ a symplectic diffeomorphism
$Fix\ \phi$	$Fix\ \phi$
$Graph\ \phi \cap \Delta \subset M \times M$	$Graph\ \phi \cap \Delta \subset M \times M$
intersection theory	"Lagrangian intersection theory"
isotopy	Hamiltonian isotopy
isotopy invariants	invariants preserved under Hamiltonian isotopy

Somehow size matters in symplectic topology. Here somehow, some kind of size can be measured in symplectic geometry. Let's study T^2 as a symplectic manifold with Ω the area form. Look at the meridian θ. The rotation along the longitude φ preserves the area form and so is symplectic. So look at $C \cap R_\phi(C)$, and it's empty. But you can't do this by Hamiltonian isotopy. So $C \cap \phi(C)$ is nonempty for any Hamiltonian diffeomorphism. So you never separate the circle by any Hamiltonian diffeomorphism.

What kind of invariants satisfy these properties? It turns out that Floer homology can distinguish between symplectic things that are not Hamiltonian isotopic.

You may think this won't happen if the underlying manifold is simply connected. Let S^2 be the sphere with Ω the standard area form. In this case, the size of a Lagrangian manifold matters. I'm going to look at some circle C and by the way, the sphere is simply connected, so any symplectic diffeomorphism isotopic to the identity is a Hamiltonian diffeomorphism.

So C can be decomposed into two regions B_+ and B_-.

(1) When the areas of the two pieces are different. There are area-preserving diffeomorphisms such that $\phi(C)$ has empty intersection with C.
(2) When the areas of the two pieces are the same, then Poincaré noticed that in this case, the number of intersection points of $\phi(C)$ and C is at least two for any area-preserving diffeomorphisms.

So something is different here. There is an invariant that detects these two cases. This is some kind of evidence that this property is crucial, interacts, with thes Floer homology theories.

I should now at least give you a definition of Hamiltonian isotopy.

We say $H = H(t,x) : [0,1] \times M \to \mathbb{R}$ is called a time-dependent Hamiltonian. Then

Definition 1.2. Let $h : M \to \mathbb{R}$. The *Hamiltonian vector field* X_h associated to h is given by the equation $X_h \lrcorner = omega = dh$

This implies $\mathcal{L}_{X_h}\omega = 0$.

You study a non-autonomous ODE $\dot{x} = X_{H_t}(x)$, Hamilton's equation, and this defines a Hamiltonian flow on M, $\phi_H(t) \to \phi_H^t$ from $\mathbb{R}$ to $Symp(M,\omega)$.

Each H gives rise to a Hamiltonian path ϕ_H. You may wonder what kind of path will come from these Hamiltonian; the condition that $x \lrcorner \omega$ is exact where $X = \frac{\partial \phi^t}{\partial t}(\phi^t)^{-1}$, then there exists an H such that $\phi^t = \phi^t_H$.

Definition 1.3. Finally, well, this is an awkward object, but $Ham(M, \omega)$ is the set of *Hamiltonian diffeomorphisms*, namely time one images of such paths.

[picture]

Let me give you an interesting exercise, if you want to study symplectic topology, you have to do this exercise. Show that $Ham(M, \omega)$ forms a subgroup of $Symp(M, \omega)$. This is the set of diffeomorphisms that takes a fundamental role.

Now let's return to the study of the Cauchy Riemann equations.

Remember we needed some boundary conditions from L_0 and L_1. So we have $\partial_0 Z$ and $\partial_1 Z$ of my strip or circle, these are $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$. This ∂ugives a map from the boundary to L_0 and L_1. The behaviour could be wild near the puncture, but near $\pm\infty$ the behaviour should be nice. This is where we will impose certain decay conditions near ∞. We usually impose some kind of exponential decay of the size of the derivative at ∞. Any solution for that Cauchy-Riemann equation will be like this; in the topological sense the homotopy class of such an object is classified by maps from the square. Due to this exponential decay condition, the classification of the topology of such maps can be studied by a map $\tilde{u} : [0, 1]^2 \to (M, L_0, L_1)$ with the boundary conditions (using coordinates s and t)

$$\tilde{u}(s, 0) \in L_0; \quad \tilde{u}(s, 1) \in L_1; \quad \tilde{u}(0, t) = q; \quad \tilde{u}(1, t) = p.$$

Consider $\mathcal{F}$ the set of such smooth maps. Then $\mathcal{F}(B)$ is the homotopy class of maps in the relative homotopy class B.

The Cauchy-Riemann equations are now coordinate dependent, and they can be rewritten in a coordinate-free way. Let's do that. Our derivative, it's a bundle map from $T\Sigma \to TM$. I want to regard this as a bundle map from $T\Sigma$ to u^*TM. These are maps over the same base. Using the almost complex structure J, you can decompose

$$\begin{tikzcd}
(T\Sigma, j) \arrow[rr] \arrow[dr] & & (u^*TM, J) \arrow[dl] \\
& \Sigma &
\end{tikzcd}$$

into the sum of the complex linear and anticomplex linear parts $Hom(T\Sigma, u^*TM) \oplus Hom(T\Sigma, \bar{u}^*TM)$. We can denote

$$du = \partial_{j,J} u + \bar{\partial}_{j,J} u.$$

The associated decomposition

$$\bigwedge^{1} \Sigma u^*TM = \bigwedge_{j,J}^{1,0} \Sigma u^*TM \oplus \bigwedge_{j,J}^{0,1} \Sigma u^*TM$$

all over Σ, and what are the ranks of these bundles? The rank of $\wedge^1(u^*TM) = 4n$ and the rank of $\wedge^{0,1}(u^*TM)$ is the same as for $\wedge^{1,0}(u^*TM)$, namely $2n$, which is the same as the dimension of TM because the dimension of Σ is two.

Some observations. With respect to this coordinate $Z = \mathbb{R} \times [0, 1]$,

$$\frac{\partial u}{\partial z} + J\frac{\partial u}{\partial t} = 2\bar{\partial}_{j,J} u(\frac{\partial}{\partial z}.$$

So the Cauchy-Riemann equations are just that u is (j, J) holomorphic.

This is a system of $2n$ equations and $2n$ unknowns, so it's well-posed. If the domain has bigger dimension, then there are more equations so it's overdetermined, then there may not be even local solutions. Since it is well-posed, we know from Wolff–Nijenhuis in the 1960s, they proved that these equations have many solutions locally.

The question is the global study of these. Now I haven't finished the unraveling of the hidden structure of these equations. Regard $u \mapsto \bar{\partial}_J u$ (omitting small j from now on) from $\mathcal{F}(B)$ to something, and this $\bar{\partial}_J$ is a section of a vector bundle, so to $\Omega^{(0,1)}(u^*TM)$. By definition this is $\Gamma(\bigwedge^{(0,1)}(u^*TM))$. This assignment is just a section.

I want to form the union $\mathcal{H}^{(0,1)}$, the union over $u \in \mathcal{F}(B)$ of $\mathcal{H}_u^{(0,1)}$ where the fiber is $\Omega^{(0,1)}(u^*TM)$, a vector over the infinite dimensional space $\mathcal{F}(B)$. Then this assignment defines a section of this vector bundle.

This is a quite important piece of geometric structure, and then we can see that $\widetilde{M}(p, q; B)$ is $\bar{\partial}_J^{-1}(0)$. So think of E a vector bundle over N, and regard this as a model for the infinite dimensional picture with $\mathcal{H}^{(0,1)}$ sitting over $\mathcal{F}(B)$. Then the section has a graph or image, and the moduli space is the intersection of the zero space of the bundle with the image of the section. If this intersects transversally, it's reasonable to expect that there would be good structure here. The basic machinery is the Sard-Smale theorem regarding J as a parameter.

We want to apply a parametric transversality theorem from differential topology. In this infinite dimensional setting the only theorem is for Banach manifolds, so we ned a Banach manifold structure. We need a completion of this space with respect to a suitable norm. The most common choice is Sobolev spaces.

We need Fredholm property of this section in order to use the transversality theorem. This is equivalent in the PDE framework to saying that the Cauchy-Riemann equation is a first order elliptic equation. Then the necessary completion of the map will be Fredholm and we can apply the Sard-Smale theorem.

Now let's go back to the picture. As I said, we want u to satisfy the Cauchy-Riemann equations. Assume for the moment that all the data, L_0, L_1, and q and p and the homotopy class B, this set dictates the homotopy class of the maps. To specify this data, we can give the dimension of this moduli space, linearize the section at zero. The linearizations $D\bar{\partial}_J(u)$ will be a map:

$$\Omega^0(u^*TM) \to \Omega^{(0,1)}(u^*TM).$$

This is a Fredholm operator which carries a natural Fredholm index which is the expected dimension of the moduli space.

Now compactness is another issue. Assume that the virtual dimension is 0, so we fix the intersection points and homotopy class so that the Fredholm index becomes zero. Suppose it's a nice manifold and assume that the moduli space $\widetilde{M}(q, p; B)$ is a nice compact 0-dimensional manifold. We define $n(q, p; B)$ to be the number of points.

The domain has translation symmetry, which you have to mod out to have this space be good. So this should be a 1-manifold actually. So we want it to be the number of points of $M(q, p; B) = \widetilde{M}(q, p; B)/\mathbb{R}$.

This gives a boundary map $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$, and the main question is: "is ∂^2 zero?" In general, the answer is no, even when all the transversality and compactness issues are resolved, this is not zero, especially in the open string case.

Next time I'll say why it's not zero and tell you, but at the moment I want to generalize that picture, pretending that this is, and then I'll stop.

So now temporarily assuming $\partial^2 = 0$, given a complex $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$, we can take the homology.

Floer homology is nothing but $HF(L_0, L_1) = \ker\ \partial / \mathrm{im}\partial$. To make this useful there are a lot of algebraic structures you want to associate to it, and one of these is the product structure.

How do we define this product map? This product is defined by the "pants product" or in this case the "triangle product" so we look at the disk with three punctures. We see three Lagrangians involved. This time there is a marked point at z_0, z_1, and z_2. You want to ask for one part going to L_0, one part to L_1, and one to L_2. The marked point goes to the associated intersection point. You can regard p_1 and p_2, and this defines a chain level product

$$CF(L_0, L_1) \times CF(L_1, L_2) \xrightarrow{m_2} CF(L_0, L_2).$$

This can be given by matrix elements, given again by counting these triangles. So $m_2(p_1, p_2) = \sum n(p_1, p_2; p_0; J)p_0$.

In that picture you can regard $p_1 \in CF(L_0, L_1)$, $p_2 \in CF(L_1, L_2)$, and $p_0 \in CF(L_0, L_2)$. Then $m_2(p_1, p_2)$ is

$$\sum_{p_0 \in L_0 \cap L_2} n(p_1, p_2; p_0)p_0.$$

This $n(p_1, p_2; p_0)$ itself will be defined when the given data L_0, L_1, L_2 and (p_0, p_1, p_2) (call all this data B)—now we'll look at maps, holomorphic, from a disk to M satisfying these boundary conditions. These should be written as homotopy classes of maps. So we'll encode these homotopy classes too. One way of encoding these is to introduce a formal parameter which splits n as follows.

Maybe I'll do this next time. This picture you can generalize for any angles, any number of marked points, into this, and you'll count the holomorphic polygons with associated boundary conditions, which you'll regard as a multilinear map which will define the operations m_k from $CF(L_0, L_1) \times \cdots \times CF(L_{n-1}, L_n) \to CF(L_0, L_n)$. Now the thing I'll explain next time, this starts from $n = 0$ and defines the so-called A_∞ structure.

2. March 11

Okay, so I mean, this business, because, it's usually very complicated. I should be clear at least about the setting.

Let (M, ω) be closed, with L a compact Lagrangian submanifold. For symplectic geometry, this additional data is not very important but it's very important from a physical point of view, so $\mathcal{L}$ will be a flat line bundle with an associated connection ∇.

I'll denote the pair $c = (L, \mathcal{L})$. This is the object. I'll write $\mathcal{C}$, later the Fukaya category $\mathcal{F}uk(M, \omega)$, and I'll denote the morphisms $Hom_{\mathcal{C}}(c_0, c_1) =: \mathcal{C}(c_0, c_1)$ for the pair $c_0 = (L_0, \mathcal{L}_0)$ and $c_1 = (L1, \mathcal{L}_1)$.

I'll denote $B_k\mathcal{C}(a,b)$ as the direct sum (I'll explain the notation in more detail later)

$$\bigoplus_{a=c_0,\ldots,c_k=b} \mathcal{C}[1](c_0,c_1)\otimes\cdots\otimes\mathcal{C}[1](c_{k-1},c_k)$$

I have a collection of Lagrangian submanifolds, intersecting sequentially, that's the picture. For now $k\geq 1$.

What I'm trying to do is define $m_k : B_k\mathcal{C}(a,b)\to\mathcal{C}[1](a,b)$.

When $a=b=c$, that is, when $\mathcal{C}$ has only one object c, denote $\mathcal{C}(c,c)=C$, a graded R-module, and $B_kC = C[1]^{\otimes k}$.

Then $m_k : B_kC\to C[1]$ is a multilinear map.

So divide the boundary of a disk into $k+1$ segments, and map this into the Lagrangians, count the maps u with $\bar\partial_J u = 0$, with fixed asymptotic boundary conditions, and I'll denote by $\pi_2(\vec{L},\vec{p})$ for $\vec{L}=(L_0,\ldots,L_k)$, where $\vec{p}=(p_0,\ldots,p_k)$ (at the moment assuming all pairwise intersections are transverse) as relative homotopy classes with fixed boundary conditions. The marked points are mapped into the chain of intersection points as specified by $\vec{p}$.

Each J-holomorphic map u with given boundary conditions carries such a homotopy class $[u]$. In particular it has an area. Then m_k, a priori we want to encode all such holomorphic maps of degree 0 (I won't talk about degree at the moment). We denote $\mathcal{M}_{k+1}(\vec{L},\vec{p},B)$ as the set of all such u with $[u]=B$, modulo $PSL(2,\mathbb{R})$. This moduli space has a virtual dimension, which we assume is 0 and agrees with the expected dimension.

This m_k will be defined by counting $\mathcal{M}_{k+1}$. To count this you need to worry about smoothness and compactness. Then such a map carries the homotopy class B. The associated area will all vary. The compactness theorem holds only when you give an upper bound for the area. If not, the number of such disks may be infinite. We need to use all of them; either you need to worry about convergence of areas or use formal power series.

The m_k map has the decomposition

$$m_k = \sum_{B\in\pi_2(\vec{L},\vec{p})} m_{k,B}T^{\omega(B)}$$

where T is a formal parameter.

Here $\omega(B)=\int u^*\omega$ is positive since B is realized by a J-holomorphic map. I want to exclude constant maps.

Remark 2.1. When $k\geq 2$, then the disk has at least three marked points. This configuration is so-called stable, in that there are no automorphisms. Here an automorphism fixes these marked points. When $k=1$, then you have two marked points, this is not stable and has one dimension of automorphisms. In this business you mod out by automorphisms. You really have to mod out by these automorphisms. If you don't mod out these automorphisms you don't get any compactness. This actually makes some of the structure different for $k=1$.

However, the m_k is supposed to map

$$\mathcal{C}[1](c_0,c_1)\otimes\cdots\mathcal{C}[1](c_{k-1},c_k)\to\mathcal{C}[1](c_0,c_k).$$

Recall that

$$\mathcal{C}(c,c') = \bigoplus_{p\in L\cap L'} Hom((\mathcal{L}_0)_p,(\mathcal{L}_1)_p))$$

so that a morphism is a pair (p, v) where v is a map between the fibers above p.

I should also describe what happens to the maps. The associated map to v, you follow the line bundle with the connection, parallel transport v_0, well, regard v_0 as a homomorphism from $\mathcal{L}_0$ to $\mathcal{L}_1$, then you do parallel transport along the first Lagrangian submanifold, and then go over, follow parallel transport, et cetera.

[much discussion]

$$P_\nabla(\partial_{k+1}D^2) \circ v_k \circ \cdots \circ P_\nabla(\partial_1 D^2).$$

So $\langle m_k(x_1, \ldots, x_k), x_0\rangle$ has the form

$$\sum_{B \in \pi_2(\vec{L}, \vec{p})} \#(\mathcal{M}_{k+1}(\vec{L}, \vec{p}; B))T^\omega(B), hol_\nabla(u(\partial D^2))$$

where $h_\nabla(u\partial D^2)$ is this map we've built.

Where does this live? Actually, let me decompose this, we have

$$\langle m_k(p_1, \ldots, p_k), p_0\rangle = \#(\mathcal{M}_{k+1}(\vec{L}, \vec{p}, B))T^{\omega(B)}$$

The second component is $hol_\nabla(u(\partial D^2))(v_1, \ldots, v_k)$. You read off the coefficients from the moduli space. They live in $\Lambda^R_{0,nov}$.

Why do we have to use the Novikov ring? The operations are defined rigorously when we extend the coefficient ring. That's the reason why from the beginning we want to consider $\mathcal{C}(a, b)$ as a module over the Novikov ring $\Lambda^R_{0,nov}$, which is

Definition 2.1. The Novikov ring Λ^R_{nov}

$$\sum_{i=1} a_i T^{\lambda_i} | a_i \in R \text{ and } \lambda_1 < \lambda_2 < \cdots, \lambda_i \to \infty(\text{ or the sum is finite})$$

Then $\Lambda^R_{0,nov}$ has all $\lambda_i \geq 0$ and $\Lambda^R_{+,nov}$ has all $\lambda_i > 0$

Now I can define

$$\mathcal{C}(a, b) = \bigoplus_{p \in L \cap L'} Hom_R(\mathcal{L}_p, \mathcal{L}'_p) \otimes \Lambda^R_{0,nov}.$$

It turns out this moduli space has, there is a certain relation, the A_∞ relation. Let me just say, in this case of Fukaya categories, there is an m_0 map. You should have only one marked point. I need $\{m_k\}_{k=0}^\infty$. So this is $\Lambda^R_{0,nov} \to \mathcal{C}[1](c, c)$ for any c.

You're looking at a Lagrangian submanifold, and a holomorphic disk with an evaluation map $\mathcal{M}_1 \to L$. The evaluation cycle depends on the homotopy class, so it's $ev_0 = \sum_{B \in \pi_2(M,L)} [ev_{0,B}]T^{\omega(B)}$ where $ev_{0,B} : \mathcal{M}_1(B) \to L$. To define this m_0 you have to extend to the case where the Lagrangian manifolds intersect cleanly. Then it's a little more complicated.

I want to finish how this A_∞ relation appears and next time I'll talk about some homological algebra.

There are two cases. Let's talk first about the unfiltered case. This is the classical case of Stasheff. In this case, the m_k tart from 1. Now the A_∞ relation on the algebra (the category with only one element). This is due to Stasheff. The relation, in my sign convention is,

$$\sum_{j=1}^{n} \sum_{i=1}^{n-j+1} (-1)^{|x_1|' + \cdots + |x_{i-1}|'} m_{n-j+1}(x_1, \ldots, x_{i-1}, m_j(x_i, \ldots, x_{i+j-1}), x_{i+j}, \ldots, x_n).$$

This structure comes from reading off the boundary of the moduli space. If you think of a moduli space of disks with circles on the boundary. If you try to compactify this, the boundary will happen by bubbling off some disks, and the relation corresponds to a degeneration, you can put parentheses in between. This term that appears is the summation over all possible nontrivial parenthesization of letters.

There is another component in the compactification. In the compactification of the disk with two marked points, it's different from the other case, because we modded out by $\mathbb{R}$-translation in the definition. There is another possible compactification, which is related to the unstable component, it's different from how we treat the stable case. This corresponds to m_1.

I'll talk about this in more detail next time. Let's not talk about why or how it appears. Roughly the relation is based on the sum of the boundary components. The relation is nothing but the fact that the oriented sum of the sign of any compact one manifold is zero. To define this operation, we required the moduli space has dimension 0. You need to look at the moduli space of one-dimensional components. That can be described by this pinching off. The origin of all this relation is essentially, it follows from that.

Now let me talk about the filtered case. We're going to look at m_k from 0 to ∞. You look at the same sum but you start with $j = 0$ instead of $j = 1$.

For each n you get one relation. I'll look at the case where $n = 1$, we have in the unfiltered case $m_1 m_1(x) = 0$ so that m_1 is a differential and we can define homology of C with respect to m_1. What is $n = 2$? There we have three, $m_1 m_2(x,y) + m_2(m_1(x),y) + (-1)^{|x_1|'} m_2(x, m_1(y)) = 0$. If you define homology, you only look at the cycles. If you apply this to cycles, you get, m_2 defines a product in homology. If you put cycles in, the result is a cycle.

In the second case, for a curved A_∞ algebra, it's one with $m_0 \neq 0$. In the filtered case, this comes in very naturally. Let's see what the consequences are in this case.

Since we start with m_0, we should look at $n = 0$, this corresponds to the case $m_1(m_0(1)) = 0$. That means that $m_0(1)$ is a cycle if m_1 turns out to be a boundary. You'll take partitions of 2. So $m_1 m_1(x) + m_2(m_0(1), x) - m_2(x, m_0(1)) = 0$. This is an interesting relation. So if $m_0(1)$ is not zero, this doesn't have to be a differential (unless $m_0(1)$ is in the center). There are special cases. To mention the special case, I should talk about one definition.

For a unit, we have to extend our discussion for the clean intersection case. I don't have time, maybe it's a good time to stop. Next time I'll introduce the unit and its consequence.

3. March 18

So let $R = \mathbb{C}$ and T a formal variable. Define

$$\Lambda_{nov} = \{\sum_{i=1}^{\infty} aT^{\lambda_i} | a_i \in \mathbb{C}, \lambda_i \to_i \infty\}.$$

We'll define $\Lambda_{0,nov}$ so that the $\lambda_i \geq 0$ and $\Lambda_{+,nov}$ so that $\lambda_i > 0$. There is a vauation $v : \Lambda_{nov} \to \mathbb{R}$ given by $v(\sum a_i T_i^{\lambda}) = \lambda_1$. This satisfies the properties of a non-Archimedean valuation. We have $v(\alpha_1\alpha_2) = v(\alpha_1) + v(\alpha_2)$ and $v(\alpha_1 + \alpha_2) \geq \min\{v(\alpha_1), v(\alpha_2)\}$.

By convention $v(0) = +\infty$.

Definition 3.1. A filtered A_∞ category $\mathcal{C}$ consists of

(1) A class of objects $Ob(\mathcal{C})$,
(2) A set $\mathcal{C}(c_1,c_2)$ of morphisms which is a filtered $\Lambda_{0,nov}$-module carrying a filtration given by $\ell : \mathcal{C}(c_1,c_2) \to \mathbb{R}$. Denote

$$F^\lambda(c_1,c_2) = \{x \in \mathcal{C}(c_1,c_2) | \ell(x) \geq \lambda\}.$$

(3) A set $\{m_k\}_{k=0}^\infty$ of linear maps

$$m_k : \mathcal{C}[1](c_0,c_1) \otimes \cdots \otimes \mathcal{C}[1](c_{k-1},c_k) \to \mathcal{C}[1](c_0,c_k)$$

of degree 1 for $k \geq 1$ and (specializing to $k = 0$) $m_0 : \Lambda_{0,nov} \to \mathcal{C}[1](c,c)$ with $\ell(m_0(1))$ strictly greater than 0. so that
(a)

$$m_k(F^{\lambda_1}\mathcal{C}[1](c_0,c_1) \otimes \cdots \otimes F^{\lambda_k}\mathcal{C}[1](c_{k-1},c_k)) \subset F^{\lambda_1+\cdots+\lambda_k}\mathcal{C}[1](c_0,c_k)$$

(b) The collection $\{m_k\}$ satisfies the A_∞ relations

Definition 3.2. Let $c \in Ob(\mathcal{C})$. We say an element $\mathbf{e}_c \in \mathcal{C}^0(c,c) = \mathcal{C}[1]^{-1}(c,c)$ is a *unit* if it satisfies the following:
(1) $m_2(\mathbf{e}_c, x_1) = x_1$ for any $x_1 \in \mathcal{C}[1](c,c_1)$,
(2) $m_2(x_2,\mathbf{e}_c) = (-1)^{|x_2|'+1}x_2$ for any $x_2 \in \mathcal{C}[1](c_2,c)$,
(3) $m_{k+\ell+1}(x_1,\ldots,x_k,\mathbf{e}_c,y_1,\ldots,y_\ell) = 0$ whenever this makes sense for $k+\ell \neq 1$.
In particular $m_1(\mathbf{e}_c) = 0$.

Definition 3.3. An A_∞ category with one object is called an A_∞ algebra.

Let me recall the first two A_∞ relations, restricted to $C = \mathcal{C}(c,c)$.

$$m_1(m_0(1)) = 0;$$

$$m_1m_1(x) + m_2(m_0(1),x) + (-1)^{|x|'}m_2(x,m_0(1)) = 0.$$

Here is a key observation for this whole theory:

Proposition 3.1. *Let c be a unital filtered A_∞ algebra. Suppose $m_0(1) = \lambda e$ for some $\lambda \in \Lambda_{0,nov}$. Then $m_1^2 = 0$.*

Proof. There is a very nice cancellation. If we substitute λe into the A_∞ relation, we get that

$$m_1m_1(x) + m_2(\lambda e,x) + (-1)^{|x|'}m_2(x,\lambda e) = 0$$

which, we can pull out λ and get

$$m_1m_1(x) + \lambda(m_2(e,x) + (-1)^{|x|'}m_2(x,e)) = 0$$

which is

$$m_1m_1(x) + \lambda(x + (-1)^{|x|'}(-1)^{|x|'+1}x) = 0$$

and so therefore $m_1^2(x) = 0$ for all x. □

A corollary is that in this case you can define the cohomology of m_1.

We want to develop a deformation theory for A_∞ algebras. You can change this A_∞ structure for any element $b \in C[1]^0 = C^1$ with $\ell(b) > 0$ (this is an important condition) we define a new m_k map $m_k^b : C[1]^{\otimes k} \to C[1]$ by the following:

$$m_k^b(x_1,\ldots,x_k) = \sum_* m_*(b,\ldots,b,x_1,b,\ldots,b,x_2,\ldots x_k,b,\ldots,b).$$

Remember that in practice you consider the disk with $k+1$ points on the boundary. Then m_k inserts objects at k points and read out at the remaining disk. So you insert b between the marked points a finite number of times.

Proposition 3.2. *The collection $\{m_k^b\}_{k=0}^{\infty}$ is also an A_∞ algebra.*

In other words, it satisfies the A_∞ relations.

In particular,

$$m_0^b(1) = \sum_{k=0}^{\infty} m_k(b, \ldots, b).$$

This is an infinite sum so you have to worry about convergence. This is one reason we require the level of b to be strictly positive. Then this is well defined as a formal power series.

Definition 3.4. We say that a filtered unital A_∞ algebra is *weakly unobstructed* if the equation $m_0^b(1) \equiv 0 \mod \Lambda_{0,nov}\{\mathbf{e}\}e$ has a solution b. Here e is a degree 2 grading parameter

Definition 3.5. We define $H^b(C) := H(C, m^b)$ for $b \in \widetilde{M}(C, m)$ where b (called a *weak bounding cochain*) is an element in the set of solutions to the A_∞ Maurer-Cartan equation.

3.1. **Geometric realization.** Now I'll go back to (M, ω) a symplectic manifold with L a closed Lagrangian. We denote by $C(L)$ the A_∞ algebra we will construct.

Roughly this is $C(L, L)$, but we always assume these are transversal but we need $C(L, L)$. In Floer homology, this corresponds to the Floer chain complex for the clean intersection case.

Then the generator of that module is not finite, so we have to define this on the singular or de Rham complex.

Now let's say that $C(L)$ is a singular chain complex of L over $\Lambda_{0,nov}$. Now I want to define the m_k maps $C(L)[1]^{\otimes k} \to C(L)[1]$. I'm going to use C for $C[L]$. We denote

$$\mathcal{M}_{k+1}(L, \beta)$$

to be the set of holomorphic disks in (M, L) together with boundary marked points, with homology class β, up to $PSL(2, \mathbb{R})$.

In general, this is the Gromov set $\widetilde{M}(L, \beta)$, the set of holomorphic disks in (M, L), the virtual dimension is $n + \mu_L(\beta)$ where $\mu_L(\beta)$ is the Maslov index of β. This is a topological index that I'm going to skip, this is a special case of the Atiyah–Singer theorem. Hence the dimension of $\mathcal{M}(L, \beta)$ is $n + mu_L(\beta) - 3$. Now you add the marked points and get

$$\dim \mathcal{M}_{k+1}(L, \beta) = n + \mu_L(\beta) + (k - 2).$$

Decompose m_k as

$$\sum_\beta m_{k,\beta} T^{\omega(\beta)} e^{(\mu_L \beta)/2}$$

I'll completely ignore this but you should keep this to treat the filtered case.

Now I will define $m_{k,\beta} : C(L)_{\mathbb{C}}^{\otimes k} \to C(L)$ and define what it is. This is defined by

$$m_{k,\beta}(p_1, \ldots, p_k)$$

is

$$\mathcal{M}_{k+1}(L, \beta) \times (p_1, \ldots, p_k), ev_0$$

where the fiber product is taken by evaluation $(ev_1, \ldots, ev_k)$. By definition, this can be regarded as a chain on $\mathcal{M}$.

Now let's count the degree, what is the grading? We define the degree of P as the codimension of P which is $n - \dim P$.

The codimension of $m_{k,\beta}(P_1, \dots P_k)$ in L is, well, the fiber product is a subset of $\mathcal{M}_{k+1}(L,\beta) \times_{ev_+} (P_1, \dots, P_k)$. You regard ev_+ as a map $\mathcal{M}_{k+1}(L,\beta)$ to $L \times \dots \times L$, k times. You're given a subset $P_1 \times \dots \times P_k$. Assuming this is transverse, you can compute the codimension. The codimension of $ev_+^{-1}(P_1, \dots, P_k)$ should be the same as the codimension of $P_1 \times \dots \times P_k$ in L^k which is $kn - \sum \dim P_j$.

Therefore the dimension of $\mathcal{M}_{k,\beta}(P_1, \dots, P_k)$ is the dimension of $\mathcal{M}_{k+1}(L,\beta) - (kn - \sum \dim P_j)$ but we know that dimension, which is $n + u_L(\beta) + k - 2 - \sum \deg P_j$ where we're grading P_j by codimension. So this can be written $n + \mu(\beta) - 2 - \sum_{j=1}^k (\deg P_j)'$.

We want to regard this as a chain in L. So under this evaluation map at the zeroth marked point, the degree of $m_{k,\beta}(P_1, \dots, P_k)$ as a chain in L is $\dim L - (n + \mu_L(\beta) - 2 - \sum_{j=1}^k (\deg P_j)')$ so we get

$$\deg' m_{k,\beta}(P_1, \dots, P_k) = 1 - \mu_L(\beta) + \sum^k |P_j|'.$$

Therefore the degree shifted degree of $m_{k,\beta}(P_1, \dots, P_k)$, multiplying by the formal parameter $T^{\omega(\beta)} e^{\mu_L(\beta)/2}$, I get the whole overall degree $1 + \sum |P_j|'$. Hence the $m_{k,\beta}$ is shifted degree 1 so I can add them over all β to get degree one overall.

This is using the topology of the disk. If you apply to higher genus Riemann surfaces, it's governed by the Euler characteristic of the domain Riemann surface.

Now we can specialize our definition of weakly unobstructed objects. The conclusion is that each Lagrangian submanifold naturally carries a filtered A_∞ algebra structure. In general, $m_0(1)$ is not zero, it has a very nice geometric interpretation. The homotopy unit of $(C(L), \{m_k\})$ is the Poincaré dual of the fundamental class $[L]$ of L.

What is the meaning of $m_0(1)$? By definition, we know that $m_0(1)$ is this

$$\sum_{\beta \in \pi_2(M,L)} m_{0,\beta}(1) T^{\omega(\beta)} e^{\mu(\beta)/2}$$

where

$$m_{0,\beta}(1) = [M_1(L,\beta), ev_0]$$

You have a chain provided by the boundary values of a holomorphic disk. [picture]. Then this is the so-called one-point invariant.

[example in pictures.]

Now here is a definition

Definition 3.6. Assume L is oriented and relatively spin (I won't talk about this) and furthermore as a consequence of orientability the Maslov index is even. Then we can associate $(C[L], \{m_k\})$ as before, a filtered A_∞ algebra over the Novikov ring $\Lambda_{0,nov}$. This carries a natural level structure $\ell : C(L) \to \mathbb{R}_0$ the filtration function. Okay then we say that L is weakly unobstructed if the A_∞ Maurer–Cartan equation has a solution.

That means that you can actually define this Floer homology. We denote the Floer homology $HF^b(L, \Lambda_{0,nov})$ as $H(L, m_1^b)$ and call it the b-deformed Floer homology.

One theorem is

Theorem 3.1. *There is a canonical isomorphism* $\Phi_H : HF^b(L,\Lambda_{nov}) \to HF^{(\phi_H^1)_* b}(\phi_H^1 L, \Lambda_{nov})$ *for any time-dependent Haimiltonian* $H = H(t,x)$.

If your original thing is unobstructed then the pushforward is unobstructed. This will play the role of the generators of the Fukaya category.

In conclusion, the Fukaya category $Fuk(M,\omega)$ is generated by (L,b) where L is a weakly unobstructed Lagrangian and b is a bounding cochain, which can be regarded as a $\Lambda_{0,nov}$-local system.

By definition, L is weakly unobstructed if $m_0^b(1)$ can be written as λ_b times the fundamental class of L for some $\lambda_b \in \Lambda_{+,nov}$ and some b. So you deform by this bounding cochain b. Suppose you have such a b. Then you consider all possible such b, $\widetilde{M}(L)$, and regard $b \mapsto \lambda_b$ as a function $\widetilde{M}(L) \to \Lambda_{0,nov}$ called PO. This we call the *FOOO potential function*. This is the rigorous analogue of the Landau–Ginzburg potential function in the toric case. Next time maybe I'll talk about this a little bit. It looks a little complicated but fortunately all of this construction can be done explicitly for toric things.

Next week I'll be away so I'll continue after two weeks.

4. April 1

Okay, so let me briefly recall what we're doing here. So C is a filtered graded $\Lambda_{0,nov}$-module. We're given this A_∞ structure $\{m_k\}_{k=0}^\infty$. We are given $b \in C[1]^0 = C^1$ and $\ell(b) > 0$. Then our A_∞ Maurer–Cartan equation is

$$\sum_{k=0}^\infty m_k(b,\dots,b) \equiv 0 \mod \Lambda_{+,nov}$$

where is the unit.

Then $\widetilde{\mathcal{M}}^{weak}(\mathcal{C})$ is the set of solutions of the Maurer–Cartan equation, and we call any such b a *weak bounding cochain*. We'll mod out by gauge invariance. I'll need to talk a bit about the homotopy theory to talk about gauge equivalence, I might do that next time.

This is defined on $\widetilde{\mathcal{M}}^{weak}(C)$, and we denote the quotient by $\mathcal{M}^{weak}(C)$. Then we define $PO(b)$ be the relation

$$m(e^b) = PO(b)$$

where

$$m(e^b) := \sum_{k=0}^\infty m_k(b,\dots,b)$$

; that is, $e^b = \sum b^{\otimes k}$. Since this involves different tensor powers, you apply the appropriate m_k. Then this notation makes sense.

This is a simple way of denoting these things.

Then I continue this notation. We apply the above to the module $C = C(L,\Lambda_{0,nov})$ for a Lagrangian submanifold $L \subset (M,\omega)$. Now we'll look at pairs (L_0, L_1). This corresponds to the study of A_∞ bimodules. Let me first study the case when L_0 and L_1 are the same.

We want to define some operator δ_{b_0,b_1}, a deformation of m_1 using our bounding cochains. So this will be $C[1]$ to $C[1]$. So

$$\delta_{b_0,b_1}(x) = \sum_{k_0,k_1} m_{k_0+k_1+1}(b_0,\dots,b_0,x,b_1,\dots,b_1) = m(e^{b_0} x e^{b_1}).$$

Let's compute its square. We use the relation $m \circ m = 0$, so that $\sum (-1)^{(?)} m(\dots, m(\dots), \dots) = 0$.

Then our notation was that $\widehat{m}_k$ is the coderivation of the bar complex $BC \to BC$. This is how we define $\widehat{m}_k$. We apply this to n elements, this is (up to sign) $x_1, \dots, x_i \otimes m_k(x_{i+1}, \dots, x_{i+k}) \otimes \dots$ Then I define $\hat{d}$ as $\sum \hat{m}_k$. If I write this without a hat, it's the projection. So I want to compute this δ^2. Any questions?

[some discussion]

So let's compute.

Lemma 4.1. *For any b_0 and b_1 in $\widetilde{M}^{weak}(C)$, we have*

$$\delta_{b_0,b_1} \circ \delta_{b_1,b_0}(x) = (PO(b_1) - PO(b_0))x.$$

Corollary 4.1. *If $b_0 = b_1 = b$ then $\delta^2_{b,b} = 0$.*

Proof of Lemma. This follows from the A_∞ relation $m \circ \widehat{m} = 0$. So $0 = m \circ \widehat{m}(e^{b_0} x e^{b_1})$.

There is a simple formula. Let's compute $\widehat{m}(e^{b_0} x e^{b_1})$. You get

$$\widehat{m}(e^{b_0}) x e^{b_1} + e^{b_0} \delta_{b_0,b_1}(x) e^{b_1} + (-1)^{|x|'} e^{b_0} x \widehat{m}(e^{b_1}).$$

Therefore, we get

$$0 = m\left(\widehat{m}(e^{b_0}) x e^{b_1} + (-1)^{|x|'} e^{b_0} x \widehat{m}(e^{b_1}) + e^{b_0} \delta_{b_0,b_1}(x) e^{b_1}\right).$$

When we apply to the last part we get $\delta^2_{b_0,b_1}(x)$. When we apply in the other case, we get

$$m(PO(b_0) x e^{b_1} + (-1)^{|x|'} PO(b_1) e^{b_0} x).$$

By the unit properties, this all vanishes except $PO(b_0) m_2(,x) + (-1)^{|x|'} PO(b_1) m_2(x,)$. Now I can apply the unital property. My total sum then becomes

$$PO(b_0)x - PO(b_1)x + \delta^2_{b_0,b_1}(x).$$

$\square$

Definition 4.1. The Floer homology $HF((L,b))$ is $HF(C[1], \delta_{b,b})$.

Now consider the case of L_0 and L_1. They intersect transversally.

Why do we need to extend the coefficient ring? Let's motivate why we have to consider bimodules over $L_0 \cap L_1$. Let's study $\Lambda_{0,nov}(L_0 \cap L_1)$ and we extend this to a $(BC(L_0), BC(L_1))$-bimodule. We want to define a differential δ

$$\delta : \Lambda_{nov}(L_0 \cap L_1) \to \Lambda_{nov}(L_0 \cap L_1)$$

by considering the space of holomorphic strips of this type, where $\frac{\partial u}{\partial z} + J \frac{\partial u}{\partial t} = 0$, with $u(z,0) \in L_0$, with $u(z,1)$ in L_1, and with $u(\pm\infty) \in L_0 \cap L_1$. Recall that we had a moduli space of these $\mathcal{M}(q,p,B)$ for $B \in \pi_2(q,p,L_0,L_1)$. We counted the size of this thing when $\mu(B) = 1$ (that is, the dimension of $\mathcal{M}(q,p,B=0)$).

Assuming $\mathcal{M}(q,p,B)$ is smooth, this can be achieved for generic choice of J when L_0 and L_1 are orientable. Still we need to worry about compactness or we can't count.

In this case of $\mu(B) = 1$, the only possible way, the splitting does not occur. The only possible source is bubbling of holomorphic disks. Assuming transversality, each thing must have positive index (if it's orientable). So if both are orientable, this cannot occur either. If we have $u_0 + w$ where u_0 is the principal part and w is the bubble part.

When $q \neq p4$, since u_0 is not zero. Since index is additive, this cannot occur.

The same holds in index two. This is a side remark.

So our map Δ is compct and we can define

$$\delta(q) = \sum_p \sum_B \#\mathcal{M}(q,p,B)T^{\omega(B)}p$$

In this example from the Riemann mapping theorem [picture], so $\widetilde{\mathcal{M}}(q,p,B_+)$ is reparameterizations of the obvious strip in the picture, and similarly for $\widetilde{\mathcal{M}}(q,p,B_-)$.

Now let's look at the index two case. Here we have $\mathcal{M}(q,q,B)$ where B is the concatenation of $B_-\#B_+$. Both splitting and bubbling are possible because [picture]. The bubbling is very limited. This is the image picture. What's going on in the domain? In the domain, it's easier to regard it as a disk with two punctures. We use the stable map type compactification.

[more pictures]

In summary, this, we need to count the number of such $\#(\mathcal{M}(q,q,B))$ of index 2. This is nothing but a holomorphic disk whose boundary is on L_1 and passes through a point. This is the one-point open Gromov-Witten invariant of genus 0. It doesn't depend on the choice of q.

For the good case (e.g., monotone Lagrangian submanifolds), that open Gromov-Witten invariant does not depend on q. In general the story is more complicated.

So the conclusion is the following. In general, $\delta^2(q)$ is, well, we can write $\langle \delta^2 q, p\rangle$, and the off-diagonal part is zero. If $q = p$ it's $PO_{L_1}(b_1) - PO_{L_0}(b_0)$. So δ_{b_0,b_1} is a diagonal matrix, in fact a multiple of the identity.

Studying this kind of equation leads to so-called matrix factorization (in general). That's one consequence. A second consequence is that when $PO_{L_1}(b_1) = PO_{L_0}(b_0)$, the Floer homology is defined. If you consider the Fukaya category generated by these Lagrangian submanifolds, only those with the same potential values interact. Otherwise they don't interact.

In $Fuk(M,\omega)$, generated by weakly unobstructed Lagrangian submanifolds (this is the right object, this is the definition) together with a bounding cochain is decomposed into the sum of categories $Fuk^\lambda(M,\omega)$.

Let me at least once describe this one geometrically.

[pictures]

So what is an A_∞-BC_1-BC_2-bimodule? Recall that an A_∞ structure was described by $\{m_k\}$. This bimodule structure will be described by n_{k_0,k_1} where k_0 and k_1 are nonnegative.

You put as many bubbles as you want in BC_1 and BC_0. So $nk_1, k_0 : B_{k_1}(C_1) \otimes D \otimes B_{k_0}(C_0) \to D$.

Whenever you are given such a multilinear map you can extend it to a bicomodule homomorphism. We canonically extend this to a bicomodule homomorphism $\widehat{n}_{k_1,k_0} : B(C_1) \otimes D \otimes B(C_0) \to B(C_1) \otimes D \otimes B(C_0)$.

Let me be more precise. So

$$\widehat{n_{k_1,k_0}}(Q_1 \otimes \cdots \otimes Q_{n_1} \otimes \langle p\rangle \otimes P_1 \otimes \cdots \otimes P_n)$$

is the sum over applying n and m in all possible places.

In the geometric picture, what is

$$\langle n_{k_1,k_2}(P_1, \ldots, P_{k_1}|q|Q_1, \ldots, Q_{k_2}, p\rangle ?$$

We have points in order on the boundary of the holomorphic strip, the P on one side and the Q on another.

The associated moduli space, this is

$$\sum \langle n_{k_1,k_2,B}(P_1,\dots P_{k_1}|q|Q_1,\dots Q_{k_2}),p\rangle T^{\omega(B)} \in \Lambda_{0,nov}$$

This number is the moduli space of holomorphic disks with k_1 boundary punctures on side and k_0 on the other side (maybe I'll use k_0), with one limit at q and one at p. I want the class B and I'll take the fiber product with $P_1\times\cdots\times P_{k_1}\times Q_1\times\cdots\times Q_{k_2}$. There's an evaluation map on one and the other side. Count the number of points in this fiber product, pulled back over evaluation.

That's not quite right. This is a moduli space in general. We have two evaluation maps in general. Let me be more precise.

You have evaluation maps

$$ev_{L_0,i}:\mathcal{M}_{k_0,k_1}(q,p,B)\to L_0$$

and

$$ev_{L_1,j}:\mathcal{M}_{k_0,k_1}(q,p,B)\to L_1$$

and I am taking the products of these.

Then $\widehat{n}$ is the sum of $\widehat{n}_{k_0,k_1}$. This is a map from $BC_1\otimes D\otimes BC_0$, and it satisfies $\widehat{n}^2=0$. Taking Floer cohomology corresponds to, the question now, my $\widehat{n}_{0,0}$, does it square to zero? Not necessarily? We can make it zero by deforming it along the weak bounding cochain b_0 and b_1 by the formula I wrote before, so

$$n_{b_0,b_1}(q)=n(e^{b_0}qe^{b_1}).$$

Then you compute that $\widehat{n}_{b_0,b_1}(q)=(PO(b_1)-PO(b_0))q$. Next time I'll look at the toric case.

5. April 8

Today I will not be that long. We talked about this potential function. Let me recall what and how this potential function $\widetilde{M}^{weak}(C)$ was defined, in the geometric situation of $C=C(L,\Lambda_{0,nov})$.

Basically, for weakly unobstructed Lagrangian submanifold $L\subset(M,\omega)$, (and here weakly unobstructed means, well, L is called weakly unobstructed if the Maurer-Cartan equation $\sum_{k=0}^{\infty}m_k(b,\dots,b)\equiv 0 \mod PD[L]\cdot e$ has a solution b where $deg(b)=1$ so $deg'(b)=0$ and the valuation of b is strictly positive; e is the grading parameter.)

For such solution b we deform m_k to

$$m_k^b(x_1,\dots x_k)=\sum m(b,\dots,b,x_1,b,\dots,b,x_2,b,\dots,b,x_k,b,\dots,b).$$

In particular, $m_1^b(x)=m(e^bxe^b)$.

Then $m_1^b\circ m_1^b=0$ so the Floer cohomology $H^*(L),m_1^b$ is defined and this is the deformed Floer cohomology. We denote $HF((L,b),\Lambda_{0,nov}):=H^*(C(L),m_1^b)$.

Recall the 2nd A_∞ relation $m_1^2(x)+m_2(m_0(1),x)+(-1)^{|x|'}m_2(x,m_0(1))$.

This is the second relation for the curved case. The assumption for weakly unobstructed, and the same equation holds for the deformed one. Then weakly unobstructed means that $m_0^b(1)=\sum m_k(b,\dots,b)\equiv 0 \mod \mathbf{e}e$, where $\mathbf{e}$ is the unit.

On the other hand, the definition of the unit, it's $(-1)^{|x|'+1}m_2(x,\mathbf{e})=x=m_2(\mathbf{e},x)$.

Recall that $m_{k,\beta}(x_1,\dots,x_k) = [\mathcal{M}_{k+1}(\beta) \times_{ev_+} (p_1,\dots,p_k), ev_0]$

So evaluating on the unit is putting the fundamental chain in for one p_k. If you put more one, it's a degenerate constraint, the dimension doesn't match outside dimension two. So $[\mathcal{M}_2(\beta) \times (p \times L), ev_0] \cong [M_1(p), ev_0]$.

So $PO_L(B)$ is nothing but, we have the equations $m_0^b(1) = PO_L(b)\mathbf{e}e$ or

$$\sum_{k=0}^{\infty} m(b,\dots,b) = PO_L(b)\mathbf{e}e.$$

Denote $\widetilde{M}^{weak}(L)$ as the set of Maurer Cartan elements in $C^1(L,\Lambda_{0,nov})$. Then $PO_L : \widetilde{M}^{weak}(L) \to \Lambda_{0,nov}$. But this is too big. So we need gauge equivalence.

Proposition 5.1. *If $b \sim b'$ then $PO_L(b) = PO_L(b')$.*

Define $M^{weak}(L) := \widetilde{M}^{weak}(L)/\sim$.

Proposition 5.2. *In the toric case there is a natural embedding $H^1(L,\Lambda_{+,nov}) \overset{i_L}{\hookrightarrow} \mathcal{M}^{weak}(L)$. But $H^1(L,\Lambda_{+,nov}) \subset (\Lambda_{+,nov})^n$ which modulo convergence issues is nothing but $\mathbb{C}^n$. Denote $W_L = PO_L \circ i_L : H^1(L,\Lambda_{+,nov}) \to \Lambda_{0,nov}$. This is the Landau–Ginzburg potential function of physicists.*

In the toric case all of these constructions are explicitly contstructible.

Definition 5.1. b is called gauge equivalent to b' if there exists a family $b(t)$ and $c(t)$ for t running from 0 to 1 satisfying $\frac{db}{dt} + m_1^{b(t)}c(t) = 0$. There is an explicit way to define this. So $b(t)$ has degree $deg' = 0$ and $c(t)$ has degree $deg' = -1$. There is a way to represent these in terms of differential forms. You can write this as like $b(t) + c(t) \wedge dt$.

Example 5.1 (Circles in S^2)**.**

(1) One Lagrangian L, a circle in S^2. You can think about a circle corresponding to height x. You know that $H^1(S^1,\mathbb{Z}) = \mathbb{Z}$. Write $b \in H^1(S^1,\mathbb{Z})$ as xe_1, where e_1 is a $\mathbb{Z}$-basis for $H^1(S^1,\mathbb{Z})$. Later we'll use $y = e^x$.

I want to write down, compute the potential function for that circle. So any circle in the sphere is Hamiltonian equivalent to a circle at constant height.

You recall $PO_L(x)$, by definition we need to compute, we'll need to study $m_0^b(1)$. Let's consider $b = xe_1$, where e_1 is Poincaré dual to the point. We need to study $ev_0\mathcal{M}_2(\beta) \times \{p\} \to L$ for a generic point $p \in L$. A priori we need to consider $ev_0 : \mathcal{M}_{k+1}(\beta) \times \{p\} \times \cdots \times \{p\} \to L$ for all possible k and β such that the dimension of the associated is one.

Denote by $\beta^{\pm}$ the simple class associated to the simple upper and lower semispheres. All other homotopy classes will be $k\beta_0^{\pm}$ and by dimension counting they will all be zero. The Maslov index of β_0 is 2. Therefore all the others have larger Maslov index, but the evaluation cycle has lower dimension. This is why the only contribution arises from $\mathcal{M}_2(\beta)$.

This story happens again for the Fano toric case.

Anyway, so by looking at the marked point moving around the circle, you get

$$m_{1,\beta}^{\pm}(e_1) = \pm PD(\mathbf{e})$$

and

$$PO_L(b)\mathbf{e}e = m_{1,\beta^+}(b)T^{\omega(\beta^+)} - m_{1,\beta^-}(b)T^{\omega(b-)} = x\mathbf{e}eT^{A^+} - x\mathbf{e}eT^{A-}.$$

This is $x\mathbf{e}e(T^{A^+} - T^{4\pi - A^+})$. Indeed, $PO_L(B)$ vanishes if $A^+ = A^-$.

(2) This is the story for one Lagrangian. Let's do a pair of Lagrangians, not equal. Write them as L_0 and L_1 we will compute the Floer differential $CF(L_0, L_1) \to CF(L_0, L_1)$. WE compute $\delta\langle p\rangle = ?\langle p\rangle + ?\langle q\rangle$. The diagonal element of this matrix $\langle\delta\langle p\rangle, p\rangle = 0$ and likewise for q. Because we want $\mu_B(q,p) = 1$, the only possible such B is again the simple one. We only have to compute when the starting point and ending point are different. We want to find B and B' such that $\mu_B(q,p) = 1$ or $\mu_{B'}(p,q) = 1$. There are two holomorphic strips of index one, with different areas.

We compute $\langle\delta p, q\rangle = T^{A_0 - C} - T^{A_1 - C}$. On the other hand, we have two different disks, an inside and outside, and we get $\langle\delta q, p\rangle = T^C - T^{4\pi - A_0 - A_1 + C}$.

Let's compute $\delta^2 p$. This is

$$\delta^p = (T^c - T^{4\pi - A_0 - A_1 + C})(T^{A_0 - C} - T^{A_1 - C}).$$

This shows that δ^2 is a diagonal matrix. This implies that these vanish only when the areas of the two circles are the same, so that's the only time the Floer homology is defined.

What is the cohomology? If the circle is smaller than half of the area, you can do a Hamiltonian isotopy that displaces the circle so the homology is zero. If it's half the area, then the rank is two.

This story on the sphere can be generalized to the toric case. Next time I'll start talking about the toric case.

6. April 15: Toric geometry

Let me start with complex structures. Let N be a lattice of rank n so $\mathbb{Z}^n$. Let M be the dual lattice $Hom_{\mathbb{Z}}(N, \mathbb{Z})$. We let $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \cong Hom_{\mathbb{R}}(N_{\mathbb{R}}, \mathbb{R}) \cong (\mathbb{R}^n)^*$.

Definition 6.1. A convex subset σ of $N_{\mathbb{R}}$ is called a *regular* k-dimensional cone if there exist k linearly independent $v_1, \ldots, v_k$ in N such that $\sigma = \{a_1 v_1 + \cdots + a_k v_k | a_i \geq 0\}$ and $\{v_1, \ldots, v_k\}$ is part of a $\mathbb{Z}$-basis of N. For example, if $N = \mathbb{Z}^2$, then the upper right quadrant is a regular cone.

Let $\sigma' < \sigma$ denote that σ' is a face of σ.

Definition 6.2. A finite system $\Sigma = \{\sigma_1, \ldots, \sigma_s\}$ is called a *complete* n-dimensional fan of regular cones if the following holds:

(1) if $\sigma \in \Sigma$ and $\sigma' < \sigma$ then $\sigma' \in \Sigma$ and
(2) $N_{\mathbb{R}}$ is the union of $\sigma \subset \Sigma$ (completeness).

So for example, take the vectors $v_1 = (1,0)$, $v_2 = (0,1)$, and $v_3 = (-1,-1)$. Then the regular cones spanned by each pair make up a fan. Let $\Sigma^{(k)}$ be the set of k-dimensional cones in Σ. Denote by $G(\Sigma)$ the set of generators of one-dimensional cones

Definition 6.3. A *primitive collection* is a finite subset $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_p}\}$ in $G(\Sigma)$ such that

(1) $\{v_{i_1}, \ldots, v_{i_p}\}$ does not generate a cone in the fan Σ but
(2) if you remove one element, they generate a cone in the fan.

In this example, $\mathcal{P} = \{v_1, v_2, v_3\}$ is a primitive collection.

Let m be the cardinality of the set of one-dimensional generators. Let $\mathbb{C}^m$ have the standard coordinates $z_1, \ldots, z_m$.

Consider the exact sequence $0 \to \mathbb{C}^{m-n=k} \to \mathbb{C}^m \to \mathbb{C}^n \to 0$ where $e_j^* \in \mathbb{C}^n$ goes to v_j.

Now we are going to consider for each primitive collection $\mathcal{P} = \{v_{i_1}, \ldots, v_{i_p}\}$ define the $m-p$ dimensional affine subspace of $\mathbb{C}^n$ defined by $z_{i_1} = \ldots = z_{i_p} = 0$.

Denote by $Z(\Sigma)$ the union $\bigcup_{\mathcal{P}} \mathbb{A}(\mathcal{P})$ where $\mathbb{A}(\mathcal{P} = \{(z_1, \ldots z_m)\}|z_{i_1} = \ldots = z_{i_p} = 0\}$ and $U(\Sigma) = \mathbb{C}^m \backslash Z(\Sigma)$.

Note that the kernel $\mathbb{K}$ is the set

$$\{(\lambda_1 e_1^* + \cdots + \lambda_m e_m^*)|\lambda_1 v_1 + \ldots + \lambda_m v_m = 0\}.$$

Now exponentiate to get an exact sequence of tori which I'll denote $0 \to K \to T^m \to T^n \to 0$.

Physicists sometimes call $\mathbb{K}$ the charges. Associate to each $\lambda_1, \ldots, \lambda_m$ in $\mathbb{K}$ a subgroup which I'll denote $t \mapsto (t^{\lambda_1}, \ldots, t^{\lambda_n})$ and regard this as a complex torus $(\mathbb{C}^*)^m$. We denote by $D(\Sigma)$ the group generated by these one-parameter subgroups, which is $\mathbb{K} \otimes_{\mathbb{Z}} \mathbb{C}^*$. So

Lemma 6.1. *$D(\Sigma)$ acts freely on $U(\Sigma)$.*

Definition 6.4. Define $X_\Sigma \coloneqq X(\Sigma)/D(\Sigma)$ as a complex manifold whose complex dimension is n. This is the toric mannifold associated to this fan Σ.

Proposition 6.1. *Denote $U(\sigma)$ as $\{(z_1, \ldots, z_m) \in \mathbb{C}^m\}$ for which $z_j \neq 0$ for all $j \notin \{i_1, \ldots, i_k\}$. Then $U(\sigma) \subset U(\Sigma)$.*

Proof. We need to prove that $U(\sigma) \cap Z(\Sigma) = \{0\}$. If $z \in Z(\Sigma)$ then there is a $\mathcal{P}$ such that $z_{j_1,\ldots j_\ell} = 0$ for $j_* \in \mathcal{P}$. But all z_{i_*} are nonzero. So then we want that j_* is contained in $\{i_1, \ldots, i_k\}$. Then there is a primitive collection contained in σ. But we know that σ is a cone so its subfaces are cones. $\square$

Next,

Proposition 6.2.
1. *If $\sigma' < \sigma$ then $U(\sigma') \subset U(\sigma)$.*
2. *$U(\sigma_1 \cap \sigma_2) = U_(\sigma_1) \cap U(\sigma_2)$.*
3. *$U(\Sigma) = \bigcup U(\sigma)$.*

Let's look at our example [pictures].

Everything that we have made is complex geometry. What about the symplectic structure?

I want to make a description by moment maps starting from the standard T^m action on $\mathbb{C}^m$ and using the standard symplectic structure $\omega_m = \dfrac{i}{2}\sum_{i=1}^m dz_i \wedge d\bar{z}_i$.

The symplectic description of X is generated by the moment polytope $P \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$.

We start with the exact sequence $K \to T^m \to T^n \to 0$. But we will look at T^n as T^m/K.

We're given the standard action of $T^m = S^1 \times \cdots S^1$ on $\mathbb{C}^m$ and this has the moment map $\mu(z_1 ldots, z_m) = (\frac{1}{2}|z|_1^2, \ldots, \frac{1}{2}|z|_m^2)$.

Recall that $dx \wedge dy = d(\frac{1}{2}|z|^2) \wedge d\theta$.

Whenever you are given a subgroup, well, regard $0 \to k \to \mathbf{t}^n \to \mathbf{t}^n \to 0$; we can dualize this. The moment map goes from $\mathbb{C}^m$ to $(\mathbf{t}^m)^*$. The composition with the map to k^* is the moment map applied to the subgroup K. More explicitly, if we choose a basis of $\mathbb{K}$ which is $Q_1, \ldots, Q_{m-n}$.

[some discussion.] So

$$\mu_k(z_1, \ldots, z_m) = \frac{1}{2} \sum_{j=1}^{m} Q_{j1}|z_j|^2, \ldots \sum_{j=1}^{m} Q_{jk}|z_j|^2$$

which I think of as some subset of $\mathbf{t}^n$, which I have to identify somehow with its dual.

At the end of the day the moment map of the subtorus action [unintelligible].

Next we'll do symplectic reduction. The general procedure is to look at the regular values, consider a point $(\lambda_1, \ldots \lambda_m)$ in $\mathbb{R}^m$.

Consider $\mu_K^{-1}(\lambda)/K$. This is the toric manifold X on which there is an action, this T^m/K action. But that is just the torus T^n. So there is a two-step process. You have two symplectic reductions involved.

By construction, the reduced symplectic form ω carries a natural moment map $\pi : X \to \mathbb{R}^n \cong M_{\mathbb{R}} \cong \mathbf{t}_n^*$.

Denote P as the image of π. This is the standard polytope associated to a symplectic toric manifold. As it depends on λ it comes in a family depending on the choice of λ.

What is the description of the moment polytope? It's the $U \in M_{\mathbb{R}}$ [pictures] such that $\langle u, v_j \rangle - \lambda_j \geq 0$. As you change λ, the moment polytope shrinks and expands.

I'll denote by $\ell_j(u)$ the difference $\langle u, v_j \rangle - \lambda_j$. Conversely, there are the Delzant's theorems which say this polytope uniquely determines X as a T^n-equivariant symplectic manifold modulo this T^n-equivariant symplectic diffeomorphism. In fact, there is Guillemin's formula, and the symplectic form can be written purely in terms of the moment polytope.

$$\omega_P = \sqrt{-1}\partial\bar{\partial}\left(\pi^*\left(\sum_{i=1}^{m} \lambda_i \log \ell_i + \ell_\infty\right)\right)$$

where

$$\ell_\infty(u) = \sum_{i=1}^{m} \langle u, u_i \rangle = \langle u, \sum_{i=1}^{m} u_i \rangle.$$

This is the Kähler potential. I'll stop here. Next time I'll write down the potential function explicitly in terms of this data.

7. May 13

Let me give a reference, Surveys in Differential Geometry 2012, pp 229–295. The title is Lagrangian Floer theory on compact toric manifolds. I have to finish up, I don't have much time. Today will have to be very quick.

The homological mirror symmetry is abstract, what is a concrete consequence? One basic question is the following.

Question 7.1. Does homological mirror symmetry imply classical mirror symmetry?

L''ike for instance, the expectation of curve counting. If it does, how will homological mirror symmetry include this kind of information. In this regard I'll state a theorem we proved. This is about a relationship between the quantum cohomology of something and the Hochschild homology of some Fukaya category.

So classical mirror symmetry has a symplectic side and a complex side, (M,ω) mirror to $(M^\vee, J)$. There should be a quasi-isomorphism between two triangulated categories. On the left hand side, the objects should be Lagrangian submanifolds. If you want to realize these objects geometrically, you have to allow singularities. On the right hand side there is a complex of coherent sheaves. This is the reason you have to allow resolutions on the left hand side. There's no geometrically rigorous way to construct this object. The morphisms on the left side are $HF(L_1, L_2)$, the Floer homology, and on the right hand side the Ext group $Ext(\mathcal{E}_1, \mathcal{E}_2)$. Then $QH_{\mathbf{b}}(M,\omega)$, the "big quantum cohomology" should correspond to deformations of J, $H^*_{\bar\partial}(M^\vee, J)$.

In a way, Lagrangian Floer homology encodes things on the mirror side. The Fukaya category should include the information from the B side. Our construction suggests that the A-model construction includes some of the data, that's what homological mirror symmetry is about. So in fact there is the conjecture that there is $QH_{\mathbf{b}}(M,\omega)$ which is isomorphic to $HH^*(Fuk(M,\omega))$ under some conditions.

Theorem 7.1. *(AFOOO) Let X be a toric manifold. In this case, let me denote by $U_{\mathbf{b}}$ is some kind of subcategory of $Fuk(M,\omega)$ (in the triangulated context) generated by $\mathbf{b}$-balanced toric fibers $\{(L_1, b_1), \dots, (L_k, b_k)\}$. Then, first of all, $U_{\mathbf{b}}$ split-generates $Fuk(M,\omega)$, and there is an explict "open-closed" map $\mathbf{q_b}$ which induces an isomorphism on homology $QH^*_{\mathbf{b}}(M,\omega) \to HH^*(U_{\mathbf{b}})$.*

Here $\mathbf{b}$-balanced toric fiber is $L_u \coloneqq \pi^{-1}(u)$ for $u \in \operatorname{Int} P$ and b is a $\mathbf{b}$-deformed weak bounding cochain and $\mathbf{b}$ is an ambient cycle in the the cycles generated by toric divisors.

I think I should define this open-closed map and start from there, let me call this $\mathbf{q}_{\ell,k}$, which goes $\Omega(M)^{\otimes \ell} B_x \Omega(L) \to \Omega(L)$ where $B_k\Omega(L)$ is $\bigotimes^k \Omega(L)[1]$. Then I want to write, let's say,

$$CH_*(\Omega(L)) = \bigoplus_{k=0}^{\infty} \Omega(L)^{\otimes(k+1)}$$
$$CH^*(\Omega(L)) = \bigoplus Hom(\Omega(L)^{\otimes(k+1)}, \Omega(L))$$

Then $q_{\ell,k,\beta}$ uses the moduli space

$$\mathcal{M}_{k+1,\ell}(\beta) \times_{M^\ell \times L^k} \left(\prod Q_i \times \prod P_i\right)$$

Then $q_{\ell,k,\beta}(\omega_1, \dots, \omega_\ell, \alpha_1, \dots, \alpha_k) = (ev_0)_!((ev_1^{int})^*\omega_1 \wedge \dots \wedge (ev_\ell^{int})^*\omega_e ll \wedge (ev_1)^*\alpha_1 \wedge \dots \wedge ev_k^*\alpha_k)$.

Then $\mathbf{q}_{\ell,k} \coloneqq \sum_\beta q_{\ell,k,\beta} T^{\omega(\beta)}$.

Now $\mathbf{q}_{\ell,k}$ defines for each $\mathbf{b}$ a map $\Omega(L, \Lambda_{nov})^{\otimes \ell} \otimes B_k\Omega(L;\Lambda_{nov}) \to \Omega(L;\Lambda_{nov})$ and so a map $\Omega(M) \to CH^k(\Omega(M))$. The picture in terms of holomorphic disks is [picture].

We can expand this construction imposing the requirement that each segment $z_i z_{i+1}$ lies in a different Lagrangian L_i. In this way you can categorify this construction.

Let me prepare some homological machinery. Now $CH_*(\Omega(L))$ has a natural A_∞ structure. Let me write this $\bigoplus_{k=0}^\infty \Omega(L)^{\otimes k+1})$. then $\partial_H(x_0, \dots, x_k) = \sum \pm x_0 \otimes \cdots d(x_i) \otimes \cdots x_k + \sum \pm (x_0, \dots x_i \wedge x_{i+1} \wedge \dots x_k)$.

Then I define $\partial_H = \partial_H^0 + \sum_{k,\beta} \partial_{H,k,\beta} T^{\omega(\beta)}$, where

$$\partial_H(k,\beta)(x_0, \dots, x_n) = \sum \pm (x_0, \dots m_{k,\beta}(x_i, \dots, x_{i+k-1}), \dots, x_n) + \sum \pm m_{k,\beta}(x_j, \dots, x_n, x_0, \dots, k_{i-1}) x_i, \dots, x_{j-1}$$

[pictures]

Definition 7.1. $HH_*(\Omega(L))$ is $H(CH_*(\Omega(L),\partial_H))$ and $HH^*(\Omega(L))$ is $H(CH^*(\Omega(L),\delta_H))$. Then one can categorify by putting different Lagrangian submanifolds on each boundary.

Now we can deform δ_H and the Hochschild cup product $\bigcup$ using $\mathbf{b}$, which I'll denote $\delta_H^{\mathbf{b}}$ and $\bigcup_{\mathbf{b}}$.

Theorem 7.2.

$$\mathfrak{q}_{\mathbf{b}} : QH_{\mathbf{b}}(X) \to HH^*_{\mathbf{b}}(Fuk(X))$$

is a ring isomorphism. I might want to assume some nondegeneracy condition for $\mathbf{b}$. *This is at the level of Frobenius supermanifolds.*

Now here is a very important one. There is a notion of a Kodaira–Spencer map, which we denote $KS_{\mathbf{b}} : QH^*_{\mathbf{b}}(X,\Lambda_0) \to Jac(PO_{\mathbf{b}})$. This is the so-called Kodaira–Spencer map, where $PO_{\mathbf{b}} : (\Lambda^*)^n \to \Lambda$ where n is the dimension of X. This is the $\mathbf{b}$-deformed potential function.

So now I want to give a precise definition of a Frobenius algebra. In general, $(C, \cup, \langle\ ,\ \rangle, 1)$ is a Frobenius algebra if they satisfy

(1) C is a $\mathbb{Z}/2$-graded vector space (over the Novikov field)
(2) $\cup$ is a graded commutative associative product with 1 as a unit.
(3) $\langle\ ,\ \rangle$ is a graded nondegenerate bilinear pairing of degree $n \pmod 2$, so $C^k \cong Hom_\Lambda(C^{n-k},\Lambda)$.
(4) Finally, $\langle x\cup y, z\rangle = \langle x, y\cup z\rangle$ for all x,y,z

There is a nice way to construct a Frobenius algebra using Feynman diagrams.

Let $Z(C)\in\Lambda$, and $\{e_I\}$ is a basis of C. Then $Z(C)$ is a kind of trace. We have $g_{IJ} = \langle e_I, e_J\rangle$, $g^{IJ} = (g_{IJ})^{-1}$ defined to be

$$Z(C) := \sum_I \sum_J g^{I_1J_1} g^{I_2J_2} g^{I_30} g^{J_30} \langle e_{I_1}\cup e_{I_2}, e_{I_3}\rangle\langle e_{J_1}\cup e_{J_2}, e_{J_3}\rangle.$$

Proposition 7.1. *The right hand side does not depend on the choice of basis.*

Let me define

$$Crit(PO_{\mathbf{b}}) := \left\{\vec{y}\in(\Lambda^*)^n | y_i\frac{\partial PO_{\mathbf{b}}}{\partial y_i}(\vec{y}) = 0; val_P(\vec{y}) = (val_P(y_1),\dots val_P(y_n)) \in Int\ P\right\}$$

Theorem 7.3. *There is a one to one correspondence between* $Crit(PO_{\mathbf{b}})$ *and* $\mathbf{b}$-*balanced Lagrangian fibers with* $HF_{\mathbf{b}}(L_{U,b})\neq 0$.

This forms the "exceptional collection" we denote $\mathcal{U}_{\mathbf{b}}$ the triangulated subcategory generated by these $\mathbf{b}$-balanced fibers. The theorem was that these generate the Fukaya category. This uses a version of Abouzaid's generation criterion, i.e., that $\mathfrak{q}_{\mathbf{b}}$ is injective.

Proposition 7.2. *(FOOO)If* $\vec{y}\in Crit(PO_{\mathbf{b}})$ *and* $u := val_P(\vec{y}) \in Int\ P$ *then* $m^{\mathbf{b},\vec{y}} = 0$. *Then that automatically means that* $HF_{\mathbf{b}}(L, b_{\vec{y}}) \cong H^*(L,\Lambda)$.

Now I'm ready to introduce the residue pairing. Well, first. We have that $(H^*(L(U),\Lambda), m_2^{\mathbf{b},\vec{y}}, \langle\ ,\ \rangle_{PD_{L(U)}}, 1)$ forms a Frobenius algebra for each $\mathbf{b}$. Then we can write $Z(\mathbf{b},\vec{y})$ as before. In this case, geometrically $\langle e_{I_1}\cup e_{I_2}, e_{I_3}\rangle = m_2^{\mathbf{b},\vec{y}}(e_{I_1},e_{I_2}), e_{I_3}\rangle$. Summing first over I_3 and J_3, this whole expression becomes the following

$$\sum_{I_1,J_1}\sum_{I_2,J_2} g^{I_1J_1}g^{I_2J_2}\langle m_2^{\mathbf{b},\vec{y}}(e_{I_1},e_{I_2}), vol\rangle\langle m_2^{\mathbf{b},\vec{y}}(e_{J_1},e_{J_2}), vol\rangle.$$

Then I can switch these sums to have

$$\langle m_2^{\mathbf{b},\vec{y}}(e_{I_2}, vol), e_{I_1}\rangle\langle m_2^{\mathbf{b},\vec{y}}(e_{J_2}, vol), e_{J_1}\rangle.$$

[pictures]. This is the "one-loop partition function." This picture is very important in the proof of the isomorphisms.

Now I can define the pairing. We are going to define the residue pairing on the Jacobian ring

$$\langle\ ,\ \rangle : Jac(PO^{\mathbf{b}}) \times Jac(PO^{\mathbf{b}}) \to \Lambda$$

defined by

$$\langle \mathbf{1}_{\vec{y}}, \mathbf{1}_{\vec{y}'}\rangle = \begin{cases} \frac{1}{Z(\mathbf{b},\vec{y})} & \vec{y}' = \vec{y} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 7.3.

$$Z(\mathbf{b},\vec{y}) \equiv \det \mathrm{Hess}\ PO^{\mathbf{b}}(\vec{y}) \pmod{T)^{\lambda}\Lambda_+}$$

where $\lambda = val_P(Z(\mathbf{b},\vec{y}))$

So this is an indication that if you look at some oscillatory integral with a Morse function in the exponent, there is an asymptotic that tells you [unintelligible]. Here is a key proposition.

Proposition 7.4. *(FOOO) If $PO^{\mathbf{b}}$ is Morse then actually*

$$\langle Z_1, Z_1\rangle_{PD_X} = \langle KS_{\mathbf{b}}(Z_1), KS_{\mathbf{b}}(Z_2)\rangle_{res}$$

.

Remark 7.1.
(1) the left hand side does not depend on $\mathbf{b}$.
(2) $PO^{\mathbf{b}}$ is Morse for a generic choice.
(3) We can extend the residue pairing to all of $\mathbf{b}$ by continuity. Now the residue pairing is defined as a whole family. We have a Frobenius manifold structure over $Jac(PO^{\mathbf{b}})$ sitting over $\{\mathbf{b}\}$.

Now I'll state:

Theorem 7.4. *(AFOOO) Say X is a smooth compact toric manifold. Then the $\mathbf{q_b}$ is an isomorphism*

$$QH_{\mathbf{b}}(X) \xrightarrow{\sim} HH^*(Fuk(X,\omega)).$$

We constructed a map $\mathbf{p_b} : HH^*(Fuk(M,\omega)) \to QH_{\mathbf{b}}(M)$ using the same moduli space in a different way. The key proposition is that for each $\mathbf{h} \in HH_*(Fuk_{\mathbf{b}}, Fuk_{\mathbf{b}})$ we have $\langle \mathbf{h}, \mathbf{q_b}(g)\rangle_{HH} = \langle \mathbf{p_b}(\mathbf{h}), g\rangle_{PD_X}$.