INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS WINTER SCHOOL ON VOLUME CONJECTURE, CHERN–SIMONS THEORY AND KNOT CONTACT HOMOLOGY

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1. Dec 14: Rinat Kashaev: Quantum Teichmüller theory and TQFT I

Thank you very much, it's a great pleasure to be here. Thank you very uch for organizing this. This is my first visit, great country, great people. I'm looking forward to learn myself, new things. My title is about quantum Teichmüller theory and TQFT and I'd like to outline what will be the plan. I'll list a few subjects, some from the abstract.

(1) I'll start with motivation for the subject.
(2) I'll describe Teichmüller space, the main object of study.
(3) In particular I'll describe Penner coordinates
(4) and another set of coordinates, which I'll call ratio coordinates. These four subjects will probably be the subject of today's lecture.
(5) I'll discuss the groupoid of ideal triangulations, which is an algebraic formalization of the Teichmüller theory with an emphasis on the action of the mapping class group. This action is what makes Teichmüller space interesting; topologically it's just a Euclidean space
(6) I'll continue with quantization, a combination of the physical idea of the canonical quantization and something using the canonical symplectic structure on Teichmüller space.
(7) The quantum dilogarithm is one of the main things underlying this quantum theory.
(8) I'll go on to dihedral angles and symmetry. Up to now it will be a two-dimensional theory, surfaces, and spaces associated to surfaces. This will be a bridge to the three-dimensional picture.
(9) At this state we will be able to formulate a Teichmüller TQFT that will allow us to calculate invariants.
(10) Hopefully I'll discuss a four-dimensional aspect of the theory, a relatively new thing in the theory. I'm very optimistic about further development along these lines, which initially wasn't visible.
(11) Then finally I will try to convince you that the theory is really very concrete and practical in the sense that one can do concrete calculations with concrete results, calculations, and so on.
(12) Finally there will be a version of the volume conjecture, which will relate to hyperbolic volume.

So divide it by three, combine it into groups of four items, and this is three lectures. We'll see how we manage in practice. Roughly that's my intention. If you have questions, please.

1.1. **Motivation.** So let me give some motivation. Let G be a reductive Lie group and M a 3-manifold. This is just supposed to give an idea of the picture that I want to be talking about. The Chern–Simons action functional $CS_M(A) = Tr\int_M A \wedge dA + \frac{2}{3}A \wedge A \wedge A$, where A is a connection form, a 1-form on M with values in the Lie algebra of G. This is a functional of that 1-form, and then we have gauge symmetry, which consists of invariance up to some integer ambiguity with respect to the transformation which replaces A with $A_g = g^{-1}Ag + g^{-1}dg$, where g is a function $M \to G$. The most convenient way to think of this is with respect to coordinates in the Lie group. In mathematical terms, this is left invariant one-forms in the Lie group. This is again just a Lie algebra valued 1-form. You can see that this integral is invariant at least when g is in the connected component of the constant map. This is an infinite dimensional symmetry, and so modding out by it you get a finite dimensional space. The space of gauge symmetries is $\mathcal{G}$ and the space of connection 1-forms $\mathcal{A}$

The main object of interest is the partition function $Z_{G,\hbar}(M)$, and the definition is very formal

$$Z_{G,\hbar}(M) = \int_{\mathcal{A}/\mathcal{G}} e^{i\hbar CS_M A}\mathcal{D}A$$

written in this form this is very formal, but physicists can manipulate it, do asymptotic expansion when $\hbar$ is small. A rigorous definition is not this formula, so to make this rigorous is the main motivation of my lectures.

Then maybe I should say here, what will be the method of solving the problem? It will be canonical quantization of the *phase space* $\mathcal{M}_G(M)$, and what is this phase space? We can think of it in different ways. On one hand, we can think of it as the set of critical points of $CS_M(A)$ modulo $\mathcal{G}$. We treat this as a function of A and then can write that the differential is 0. We can write out criticality conditions for this action functional, and identify them as the set of flat connections modulo the gauge action. Flat means that we can write the curvature (I will not do this, it's not the main subject of my lecture) and it's zero.

Finally, the interpretation that will be used later, I'll write the phase space $R_G(M)/G$, where $R_G(M)$ is the space of group representations given by group homomorphisms from $\pi_1(M) \to G$, and we consider the quotient, only up to conjugation by G, and the last interpretation of the phase space is convenient because it's only finite dimensional. We have to be carefuly about what the quotient is.

Now I'll move on.

1.2. **Teichmüller space.** From now on G will be $PSL(2,\mathbb{R})$ and M will be $S \times \mathbb{R}$, where S is of type $S_{g,n}$, where S is the complement in a closed compact surface $\bar{S}$ of genus g of n points $V = \{p_1, \ldots, p_n\}$. [Picture]. We call V the set of n punctures.

We'll assume that we have at least one puncture, and that the Euler characteristic of S is negative. A general fact is the following, that $\mathcal{M}_G(M)$ becomes a disjoint union $\bigsqcup_{|k| \le -\chi(S)} \mathcal{M}_{G,k}(M)$, this is a disjoint union of connected components. Here k is the Euler number of the representation. This comes from $\pi_1(G) = \mathbb{Z}$. This is a Goldman result. What is important for me, Teichmüller space is one of these components, $\mathcal{T}(S)$ is $\mathcal{M}_{G,\pm\chi(S)(M)}$. We take one of the two components given by the maximum absolute value. This is called the geometric component and is characterized by the following properties.

Remark 1.1. (1) First of all $\mathcal{T}(S)$ corresponds to representations (up to conjugation) $h : \pi_1(S) \to PSL(2,\mathbb{R})$ which are faithful, discrete, and have parabolic holonomies around punctures. Here I can make a comment on the choice of sign. Parabolic holonomy means they are conjugated to $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and the sign of x is invariant under conjugation, and the sign of x corresponds to the choice of sign in our component.

This allows us to identify S with $\mathbb{H}^2/\Gamma_h$ where Γ_h is $h(\pi_1(S)) \subset PSL(2,\mathbb{R})$, the orientation preserving isometries of the hyperbolic plane $\mathrm{Isom}^+(\mathbb{H}^2)$. This lets us consider S as a hyperbolic surface, S carries a complete hyperbolic structure, so we can start doing geometric things, geodesics and so on.

In general you can't have all three of these properties outside this geometric component.

(2) The fact is that $\mathcal{T}(S)$ is topologically trivial, it's homeomorphic to $\mathbb{R}^{6g-6+2n}$. This makes the space trivial in a topological sense.

(3) There are two things that make it nontrivial, a symplectic structure which comes with any action functional if you start the action functional, the minimal action principle, the symplectic structure comes for granted. This is known here as the Weil–Petersson symplectic structure and can be defined independently of the Chern–Simons action functional $\omega_{WP} \in \Omega^2(\mathcal{T}(S))$. I won't write it out explicitly now.

(4) There is an action $\mathcal{MCG}(S)$ on $\mathcal{T}(S)$, the mapping class group of S, which is orientation-preserving homeomorphisms of S modulo the connected component of the identity $\mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S)$. It's a discrete group, and the important thing is that the Weil–Petersson form is invariant under this action.

The quotient is very important, is moduli space; it's singular. I'll be focused on the Teichmüller space.

Now I'll describe the concrete coordinate system. I'll stop here and take a break.

2. Penner coordinates

So Penner coordinates start from a compactification of another space, which is called decorated Teichmüller space and is called $\tilde{\mathcal{T}}(S)$, which is $\{(m, H_1, \dots, H_n)\}$ where m is in $\mathcal{T}(S)$ and H_i is an m-horocycle around the ith puncture. So we draw circles. Geometrically they should be horocycles, so when we think of S as a quotient space of the upper half-space. These should be arcs in the upper half space corresponding to [picture]. This has a forgetful map ϕ to $\mathcal{T}(S)$. The decorated Teichmüller space is homeomorphic to $\mathbb{R}^{6g-6+3n}$. The Penner coordinates are an explicit equivalence.

First I'll need the idea of an ideal arc.

Definition 2.1. An *ideal arc* on S is a (nontrivial) isotopy class of a simple path running between punctures.

I should say

Remark 2.1. The set of ideal arcs $\mathcal{A}(S)$, for any $m \in \mathcal{T}(S)$ and any ideal arc a in $\mathcal{A}(S)$ there exists a unique representative of a given by an m-geodesic. This is evident by writing S as a quotient of $\mathbb{H}^2$. The lift joins two points at the boundary, so just straighten it to a half-circle.

Definition 2.2. The λ-*length* is the map $\lambda : \tilde{\mathcal{T}}(M) \times \mathcal{A}(S) \to \mathbb{R}_{>0}$ which is defined by [picture]. We measure the geodesic length between the two points of the straightened a on the corresponding horocycles, call this δ. This hyperbolic $\lambda(\tilde{m}, a) = e^{\pm\delta/2}$, where we take the plus sign if the horocycles intersect nontrivially.

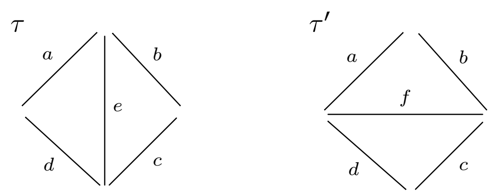
Remark 2.2. We can define the length in the picture lifted as follows [picture]

Definition 2.3. An *ideal triangulation* is a maximal subset $\tau \subset \mathcal{A}(S)$ of pairwise disjoint ideal arcs on S. We treat punctures as removed points. Then set theoretically, this means that there are disjoint representatives in the isotopy classes. If we look at the complementary regions they should be triangles.

I think now I can state the theorem of Penner, which will consist of several theorems of Penner. It has several parts.

Theorem 2.1. *(Penner)*

(1) $\phi(\tilde{m}) = \phi(\tilde{m}')$ *if and only if there is* $\alpha \in rR^V_{>0}$ *(remember* V *is the set* $p_1, \ldots, p_n$*) such that for any* a, $\lambda(\tilde{m}', a) = \lambda(\tilde{(m)}, a)\alpha(p_i)\alpha(p_j)$ *where* $\partial a = \{p_i, p_j\}$.
(2) *For any ideal triangulation* τ *the map* $\lambda_\tau : \tilde{\mathcal{T}}(S) \to \mathbb{R}^\tau_{>0}$ *defined by* $\lambda_\tau(\tilde{m})(a) = \lambda(\tilde{m}, a)$ *is a homeomorphism.*
(3) *The pullback of* ω_{WP} *under* ϕ *(I didn't define it before, but it's okay) is the sum over triangles* $\sum \frac{da\wedge db}{ab} + \frac{db\wedge dc}{bc} + \frac{dc\wedge da}{ca}$, *where* a, b, *and* c *are the signs of the triangles.*
(4) *If we have a pair of ideal triangulations that differ only as follows:*



Then $\lambda_\tau \lambda_{\tau'}^{-1}$ *is* $ef = ac + bd$, *the Ptolemy relation.*

All of the difficulty is devoted to this part of the theorem. Everything else can be restored by this relation. If you go deep into the woods, you just remember this.

Now I'll say a few words about the proof of this theorem, just the most important things, with which you can restore the rest of the proof yourself without much difficulty.

The first thing I want to say is that the inverse of λ_τ, what is this inverse map? What we do is cut out small open disks centered punctures and then get a CW complex composed of truncated triangles. Then we can orient these, there is a distinguished direction from the orientation of the surface, where things were cut out. Then we take a CW structure and consider its 1-skeleton. To give the inverse of this map, we need parallel transport along the 1-skeleton of the map. It suffices to give $PSL_2(\mathbb{R})$ matrices for each edge of this complex. Then we can realize any path in the surface, choosing a basepoint, and then any path can be deformed to an edge path of this one-skeleton, as long as the product is trivial around each of these hexagons. This will essentially be, each short curved edge we use the matrix $\begin{pmatrix} 1 & \frac{a}{bc} \\ 0 & 1 \end{pmatrix}$ where a is the opposite length and b and c the adjacent ones, these are the lengths in $\mathbb{R}^{\tau}_{>0}$. To the long edges, associate $\begin{pmatrix} 0 & a^{-1} \\ a & 0 \end{pmatrix}$. To get the horocycle, if you take a basepoint at the boundary of the disk, the horocycle, you define, the one for parabolic elemennts, the horizontal line, at height 1.

The second remark concerns the proof of part 4. This proof is very easy once you accept the first remark, about the inverse of λ_τ, for the following reason. You just look at one of the corners, the a–b corner, say, in both triangulations. You get

$$\begin{pmatrix} 1 & \frac{d}{ae} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{c}{eb} \\ 0 & 1 \end{pmatrix}$$

and this should be equal to

$$\begin{pmatrix} 1 & \frac{f}{ab} \\ 0 & 1 \end{pmatrix}$$

This gives one single relation

$$\frac{d}{ae} + \frac{c}{eb} = \frac{f}{ab}$$

and that's a restatement of the Ptolemy relation.

I have a fourth item in the plan, but I will need more than ten minutes. So probably we leave it for tomorrow.

3. Stavros Garoufalidis: Asymptotics of quantum invariants I

I do not take lectures during slide talks.

4. Sergei Gukov: Volume conjecture as a simple quantization problem: its generalization and categorification I

안녕하세요 여기에 초대해서 감사합니다.

I already gave several lecture notes described in the program. There are various exercises there. Instead I'll try to use my time wisely and efficiently so that you can take advantage of my presence.

My unofficial title is "making connections." We like to join with each other, make friends on facebook and so on. The only way to improve is not to know every detail but to join details and see connections. That will be my goal here, to say what we see in one area of mathematics has a translation somewhere else. By knowing how to translate, you can get some mileage.

I'll give you a roadmap and then try to take it easy (on myself too). We'll talk about knots and knot polynomials and three-manifold invariants. I'll connect this to quantization, really symplectic geometry.

I want to connect both of these to the recent development that goes by the names "knot homologies" and "categorification" which are two keywords that refer to the same area. We'll try to make a connection with them.

Something else that relates to this story that unfortunately will be left behind is the story of 4-manifolds, which connects to this. If pressed or if time permits I can say something, but probably I'll have to sacrifice. I encourage your input about what direction we go in. Another area I wasn't really planning to cover is that this area recently got connected to real QFT, real physics, and also string theory. That if you wish is yet another box which is very popular in non-mathematical physics. It describes real interesting physical phenomena. I could say string theory here or equally well without losing generality I could say enumerative geometry, because all the problems involving string theory in this box have to do with enumerative invariants, like most famously Gromov–Witten invariants.

These two boxes probably don't get much attention, but if I were going to connect these to Tobias Eckholm's then you'd see some connection there.

This enumerative geometry has to do with nice smooth geometries, at least in the context I have in mind, and the smooth geometries sometimes arise as limit shapes out of discrete combinatorial gadgets out of knots and three-manifolds.

I'll start with a very basic historic review of basic types of invariants. I should probably start around 89 or 90 with quantum invariants of knots and three manifolds. Here there is a very well-known and famous breakthrough of Witten and others who said by looking at a quantum version of Chern–Simons theory, you can study geometry and topology of knots, and Chern–Simons theory on M^3 is $\frac{k}{4\pi}\int_{M_3} Tr(A\wedge dA+\frac{2}{3}A^3)$. The key player is this 1-form A which is a gauge connection on a principal G-bundle over the three-manifold M^3, and as such it takes values in the adjoint representation of this group G. Another thing the theory depends on, beside M_3, is the *level* k, an integer, also called the *coupling constant* that controls how much this theory is classical or quantum. The *expansion parameter* $\hbar$ is roughly $\frac{2\pi i}{k}$, and the simple or classical limit corresponds to $\hbar$ going to 0 and from the point of view of Chern–Simons gauge theory, taking k to ∞. As such one can use this functional, I'll write it as $CS(A)$, to write the object $Z(M_3)=\int \mathcal{D}Ae^{-CS(A)}$, which will depend on all the data we specify; I make explicit the dependence on M_3 but it also depends on the group G and the integer level k or sometimes in place of k I'll use the variable $q=e^{\hbar}$ and the classical limit will correspond to $q\to 1$. You can put i instead of -1 in the exponent if you want.

Here we already see several key players right away, and it's important to keep all of them in mind. There is the dependence on the group G and the parameter k (or $\hbar$ or $q=e^{\frac{2\pi i}{k}}$).

This is the definition that Witten proposed and physicists used, but it's not useful from the point of view of mathematics because it's an infinite dimensional integral. What Atiyah and Segal set out to do is to come up with a concrete alternative. They set up a set of axioms that essentially summarized the structure of this version of field theory called topological quantum field theory. We'll use this and its generalization. That was helpful because it avoids the physical difficulties, physicists use Feynman diagrams and I should advertise this, I'm at Caltech which is where Feynman was.

So they define a d-dimensional TQFT as a functor Z such that to a d-dimensional closed manifold it assigns a number (typically a complex number) called the partition function $Z(M_d)$. To a $(d-1)$-dimensional manifold it assigns a vector space, possibly graded, $\mathcal{H}(M_{d-1})$. It would be more uniform to call this $Z(M_{d-1})$. This could be confusing as Z changes. It's natural to call this $\mathcal{H}$ because it should be the Hilbert space of states. The point is that it's a *vector space* of states. In modern days, it's important to continue this list further. It should assign to a $(d-2)$-dimensional manifold a category $Z(M_{d-2})$, to a $(d-3)$-dimensional manifold a 2-category, and so on and so forth. Here it depends, one of my friends and colleagues says every person has a number. This number tells you at what level of n-category your brain stops functioning. I hardly get to usual categories and 2-categories have objects, morphisms, and 2-morphisms. There are various versions, and you get some interesting categorical stuff on which I'm not really an expert. Of those of you who are interested in this, if someone gives a high level talk about categories, you can ask whether you know which TQFT it's coming from. I won't go beyond the level of categories.

There are some compatibility conditions that need to be satisfied. I'll illustrate; if you wish this is a crash course on TQFT structure. I'll choose $d = 3$ and also for $d = 2$ and $d = 4$. If I'm pretending to give a self-contained overview, it will be useful, and these are perfect dimensions for low-dimensional topology. In $d = 3$, many TQFTs can be related to Chern–Simons theory, namely it means that you should be able to say, given some algebraic definition, you should be able to say, for each G and k, the theory is what you're dealing with. I said that everything should depend on G and k. In dimension 3, what happens? So $Z(M_3)$ is the number, this complicated infinite dimensional integral, which is useful for properties but not for computations. It's not really a number but a function of q or k, so depends on this additional data, and we'll come back to this dependence later.

Then if you have a 2-manifold, and in my lectures I'll call them Σ, a Riemann surface, possibly punctured, you're supposed to associate to this $\mathcal{H}(\Sigma)$ which also depends on G and k. The compatibility conditions are the following. If you have a Riemann surface Σ, we already agree that we get a Hilbert space, and physicists should think of this as the space of states. You look at translation invariant things along the time dimension. Every non-closed three-manifold is bounded by some Σ, this produces a state or vector $Z(M_3^+) \in \mathcal{H}(\partial(M_3^+))$ and if you have an M_3^- then you get another vector, and the invariant $Z(M_3)$ associated to a closed manifold M_3 made by gluing these two along the boundary is $\langle Z(M_3^+)|Z(M_3^-)\rangle$. This is a "cutting and gluing relation" and there are many of these, but this is one of the most useful. You should take inner product to glue together in this way.

This is one of the Atiyah–Segal axioms, there are quite a few more, especially if I want codimension 2 boundaries, then there will be more relations, and that will force you to be categorical. Any questions?

I'll give you one more example of an Atiyah Segal axiom and then we'll try it on something concrete. Let's take a break.

So here we're describing 3-dimensional TQFT parameterized by a group G and an integer k. I can produce a 2-dimensional TQFT, if I have my 3-dimensional TQFT, I can take $Z(S^1\times)$, this will be a $TQFT$ one dimension lower. To any Σ I'll get a number, and thinking of it as a 2-dimensional TQFT, manifolds one dimension less are circles, and that's how we can decompose our Riemann surface, and I should get a number to a Riemann surface, a Hilbert space $\mathcal{H}$ to each circle, and I can do this by saying $\mathcal{H}_{2D}(S^1)$ is $\mathcal{H}_{CS}(S^1 \times S^1)$, so I have a very simple machinery, given a higher dimensional TQFT, I can easily produce a theory one dimension lower by applying it to the circle crossed with whatever. So now I can ask a question. Do we know what this TQFT is? There is a canonical way to get a 2-dimensional TQFT. Another example in dimension 2 is a functor, I'll also call it Z_{2d} which takes Riemann surfaces to numbers and Z_{2d} associates to S^1 a vector space $\mathcal{H}(S^1)$. In two dimensional TQFTs, they're simple and interesting, they're associated naturally with Frobenius algebras. Every Riemann surface can be decomposed into pairs of pants, if you have two vectors λ and μ, and ν, then you can view this as a multiplication $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, reading left to right it's a product.

Equivalently, I'm choosing orientation so that the normal dimension goes in on λ and μ and outward on ν, but I could dualize and say that Z of this surface is in $\mathcal{H}(S^1) \otimes \mathcal{H}(S^1) \otimes \mathcal{H}(S^1)^*$; in any case, this structure is a multiplicative structures since from two boundaries you can get a third one, and this determines the entire TQFT. If you want the invariant of a closed 2-manifold, to glue you want to sum or take the inner product with respect to the Hilbert space where you're gluing. In concrete examples you can diagonalize the product, which means that Z here is nontrivial only if you have the same choice of state in $\mathcal{H}$, so if you think of this as structure constants, you're saying you have $c_{\lambda\mu\nu} = c_{\lambda\lambda\lambda}$ if $\lambda = \mu = \nu$ and 0 otherwise.

If this happens, then because the theory is topological it doesn't matter how you decompose, so Z can always be computed on a closed Riemann surface in a simple way, it's the sum over $\lambda \in \mathcal{H}$ of $(C_{\lambda\lambda\lambda})^{2g-2}$. This is a quick review or structure of a 2-dimensional TQFT. Everything is controlled by this product or pair of pants. Mathematically, very concretely, you get something universal to $2g-2$. For $g = 2$ you need two pairs of pants, shown in this picture. [picture]

What kind of TQFT is this when we take this from Chern–Simons theory? It has a name, but I'll tell you one more Atiyah–Segal axiom which tells us exactly what happens here. I'll go to another blackboard and tell you, if you have, in general, for a d-dimensional $TQFT$, the functor M_{d-1} you get $\mathcal{H}(M_{d-1})$. What about $Z(S^1 \times M_{d-1})$? You can justify this many ways. In any case, when you try to evaluate on the interval cross M_{d-1} and close it up, it's a trace, and this is $\dim \mathcal{H}(M_{d-1})$. So this is $Z(\Sigma) = \dim \mathcal{H}_{CS}(\Sigma, G, k)$. More generally, and you have to be a little more careful. If you have Fermions or spinors, what might happen is counting variables with sign. Bosonic variables have plus signs and Fermionic have minus signs, and $Z(S^1 \times M_{d-1}) = \chi(\mathcal{H}(M_{d-1}))$. My goal in the second lecture will be to calculate this $\mathcal{H}$ more concretely with G compact or even with noncompact group, and in the latter case $\mathcal{H}$ will be infinite dimensional. For Atiyah and Segal calculating the dimension or Euler characteristic by crossing with the circle is one of the fundamental axioms.

Eric Verlinde made this very concrete by connecting it with other objects. Let's take the $2d$ TQFT which comes from $3d$ Chern–Simons on a circle crossed with something. By general properties, the partition function $Z_{2d}(\Sigma) = \dim \mathcal{H}(\Sigma)$ and in the special case where $G = SU(2)$, the simplest non-Abelian group, and in these lectures this will be a common example for me, you get

$$(\frac{k+2}{2})^{g-1} \sum_{\lambda=1}^{k+1} (\sin \frac{\pi\lambda}{k+2})^{2-2g}$$

where this depends on the choice of the group, on k, and on the 3-manifold which is determined by Σ, and the formula has this structure, it's some universal ingredient raised to the power $2g-2$. You can extract $c_{\lambda\lambda\lambda}$ from this.

Now we can explore in a general example, what is the dimension of this space and how does it look like? In the case of a 2-dimensional TQFT, it associates numbers to closed surfaces, but its own Hilbert space to 1-manifolds, and we can calculate the dimension of this space. I already wrote it, λ is supposed to be a basis of the space, and $\dim \mathcal{H}_{2d} S^1$ is $k+1$. The basis vectors in this case take values from 1 to $k+1$, and span a basis there.

Let me say a word about what quantum means in this context. If k goes to ∞, then λ can be any natural number, and it sholud be regarded as labelling representations of the classical group $SU(2)$, in fact the dimension of the representation. You should think of λ being, well we started with a gauge theory in 3-dimensions, and trying to cut things, you get some data associated to the gauge group. If you have something canonical, it's probably a representation space for this group. That happens in the classical case.

What about in quantum? It truncates to the finite range where the representations only go from 1 to $k+1$. We discover that in classical Chern–Simons theory, you get the representation space of G, but in the quantum space, these are representations of so-called affine Kacs–Moody of level k, that is, $\hat{\mathfrak{g}}_k$.

Now there is another thing we see from this formula, namely that dependence on k appears not through k itself but through $k+2$. If you want to write this in terms of q, you get something like $q = e^{2\pi i}k+2$. I tried to simplify things, so I suppressed this shift by 2. If you're trying to do $G = SU(N)$ then here you would have $k+N$ in the denominator.

Now, this is the story in the late 80s or mid-90s. Then in the later 90s came along this wonderful conjecture of Rinat about the behavior of the Kashaev invariant, which, this starts with Rinat, that $\lim_{n\to\infty} \log \langle K\rangle_n n$ is the volume of $S^3\backslash K$. In the 90s this was a big puzzle and I wanted to understand where this came from. There were many things that were puzzling about it. In Chern–Simons theory it was quantum groups, affine Kacs–Moody, but the quantum dilogarithm was not in major use. But Kashaev with Murakami and Murakami (Jun is my chair here) showed that this left hand side, the $\langle K\rangle_n$ is the same as the nth colored Jones polynomial at the nth root of unity $J_{\lambda=n}(q = e^{\frac{2\pi i}{n}})$.

There are still many puzzles here. I tried to share this excitement with many people, including my former advisor Ed Witten. But if you think about it carefully, trace all the factors of i and little shifts, you'll notice that the λ defined in Chern–Simons theory in level k, it ranges from 1 to $k+1$, but you're trying to identify the order of the root of unity, which is $k+2$. But how can λ be $k+2$? That's completely not allowed in quantum Chern–Simons theory. That was one big puzzle about this. It tries to put this highest weight where it doesn't belong in Chern–Simons theory. Another puzzle, also based on the formula $q = e^{\frac{2\pi i}{k+N}}$, you always end up with expressions in q which are roots of unity with $SU(N)$. But if you think about other mathematical definitions, the Jones polynomial doesn't only evaluate at roots of unity. So is it a root of unity or not? The community of mathematical physicists were so set on Chern–Simons theory that it was psychologically hard to move away from roots of unity. This was trying to push us outside the comfortable life where Chern–Simons gauge theory lives.

I'll finish, maybe, with one last statement, with preliminary work for later. We talked a lot about this functor that assigns numbers, vector spaces, and categories, and I promised to give a quick illustration of the case $d=4$. In the case $d=4$ it's supposed to map 4-manifolds to numbers, it assigns a number to a 4-manifold, and such a number, there are several TQFTs, since I'm slowly moving through the 90s, there are Donaldson–Witten and Seiberg–Witten. What's interesting, and it'll be a little relevant for us, this Z_{4d} will assign vector spaces to 3-manifolds (and possibly knots), just by the same token that Chern–Simons theory will assign a number not just to a three-manifold but also to a knot, this will assign a Hilbert space for

$K \subset M_3$. What's interesting, most often, even though you work with a compact group, so take $G = SU(2)$, you'll find infinite dimensional vector spaces, as Stavros says there's only two options, finite or infinite, then to take Euler characteristic you have to regularize, and if you do it one way you get $-\frac{1}{12}$. Then you get vector spaces for both 3-manifolds and knots.

5. Dec 15: Tobias Eckholm: Knot contact homology – definition and calculation

I will start talking about contact homology, tie it to Chern–Simons in the second talk, and then go to the volume conjecture in the last talk. When the volume conjecture was formulated, I was around, Murakami was visiting Sweden, I remember he did this reformulation, I was just out of grad school, and he asked me to do some calculations, I did some rigorous calculation with torus knots and figure eight knots, and I asked if we should write this up, and he said no, so I never did it. Now more progress has been made. It's nice to have something to say along the lines I'm going to talk about in these lectures.

Today I'll say some basic things about knot contact homology. There will be various levels of rigor and understanding and I'll try to be clear what has been done and what hasn't. Some parts are more physical and not as rigorously proved, it's nice to point that out for the opportunity to establish closer ties and get that kind of work done.

Let's start with a knot K (or link) inside a 3-manifold. The scheme is that we would like, knot theory is about classification of such embeddings. What we're using is symplectic and contact objects naturally associated to this smooth object. So T^*M is the cotangent bundle, and we think of it as a 6-manifold, thought of as a symplectic manifold with symplectic form $\omega = dp \wedge dq$.

We can think of this as a Weinstein manifold, a noncompact symplectic manifold with some contact boundary, which is the unit cotangent bundle U^*M, the ideal contact boundary with contact 1-form $\alpha = pdq$. If you take a Riemannian metric on M, take the covectors of length 1, this is a 5-manifold. Really it's maybe better to think of the contact symplectic manifold (the disk bundle) and outside take the unit cotangent bundle cross $\mathbb{R}$.

So an interesting question is what the symplectic manifold T^*M remembers about M. That won't be the subject of these lectures so much in general. For example, a remark, it's a rather strong invariant, in general, T^*M remembers the smooth structure topology of M, for example, if Σ and Σ' are homotopy spheres of odd dimension, then if Σ and Σ' are different mod bP then $T^*\Sigma$ is not symplectomorphic to $T^*\Sigma'$. These homotopy spheres, there are many smooth structures, if you quotient by the less weird ones, the ones that bound boundary parallelizable things, then the cotangent bundle distinguishes them.

This question in general is very interesting in dimension 4. We don't know anything similar in symplectic geometry in dimension 8, so it's maybe different or maybe the same, anyway an interesting question. But that's not what we're doing here.

So what kind of objects do you associate with a submanifold in T^*M? There's the Lagrangian conormal, $\{(q,p) \in T^*M | q \in K \text{ and } p|_{T_qK} = 0\}$ so these are the covectors on the knot perpendicular to it.

There is the Legendrian conormal $\Lambda_K = L_K \cap U^*M$.

So L_K is Lagrangian, meaning ω restricted to it is zero, and Λ_K is Legendrian, meaning α restricted to it is zero.

There has been a lot of work since the 90s, Floer homology or so on, that says these are rigid. This Λ_K is a torus in a five-manifold. The knot theory there is basically trivial, any two things are isotopic if they are homotopic. But if you want things to be isotopic through Legendrians, that's nontrivial, there are machines to say when this is or isn't possible, which is what I'll talk about today.

If you deform a knot by isotopy, then the Legendrian and Lagrangian will deform by their appropriate isotopy. If we know if $\Lambda_{K'}$ and Λ_K are not Legendrian isotopic, then neither are K' and K isotopic.

So we'll use this geometry to build new invariants of knots, in particular Legendrian contact homology.

This was known as symplectic field theory, which was invented in the 90s by Eliashberg–Givental–Hofer. This Legendrian contact homology is maybe the simplest flavor and is the one where most of the calculation has been done. Many other parts of this theory can be computed through Legendrian contact homology. We'll see a little part of the other theory but it won't play a central role.

The idea is that we want to do Floer homology of the action functional which takes curves γ to $\int_\gamma \alpha$. In fact, let me somehow say a few words. If you don't have any submanifolds, working in U^*M, the critical points of this functional are Reeb orbits, the Reeb field R has the key property that $d\alpha(R_\alpha, \cdot) = 0$. This 2-form has a 1-dimensional kernel which we normalize to be 1, $\alpha(R_\alpha) = 1$. If you use this as a functional, R_α orbits will be the critical points. You should think about the homology as being concentrated on the non-negative action. When you have Λ_K, you can also have Reeb chords from Λ_K to itself, critical points of this functional on the space of paths from Λ_K to Λ_K.

Maybe it makes sense to say some words about what these things actually are in the case that we study. Basically I will say more about this but very briefly, what is Floer homology. I should say at once that it won't work out exactly as we hope. This is an attempt to do Morse homology in this setting. If you take a finite dimensional manifold and a Morse function f [picture] then you can compute the homology of the Manifold in this funny way invented by Witten to connect it to supersymmetric quantum mechanics. You generate a chain complex on the critical points and define a differential which counts rigid flow lines of $-\nabla f$ [pictures].

If you take the homology here you see you have a cycle of degree 2 and in degree 0 there is one cycle. It's easy to see that this always computes the homology. There are nonrigid ones, but you can look at the one-dimensional moduli space of flow lines that go down two levels, and that shows you that the differential squares to zero.

We have in our case a chain complex represented by Reeb orbits and then we count gradient flows.

So what are these Reeb orbits and Reeb chords in M? It's not hard to see that the Reeb vector field is the lift of the geodesic flow. I won't do it, just look at this, it's *pdq*, and then [unintelligible]. So Reeb orbits are closed geodesics. What about Reeb chords on Λ_K? They are geodesics that meet K at right angles. These are well-known objects from Riemannian geometry. Then once again, the questions about countability, for geodesics, it's exactly the same here.

Now consider the symplectization $\mathbb{R} \times U^*M, d(e^t\alpha)$, which is a symplectic manifold, symplectomorphic to T^*M minus the zero section. So we need an almost complex structure. We pick such a structure $\mathcal{J}$ on $\mathbb{R} \times U^*M$ such that $\mathcal{J}$ takes the kernel of $d\alpha$, it's a complex structure on $\ker\alpha$, it fixes this kernel and is compatible with $d\alpha$ and $J\partial_t = R_\alpha$. Compatibility means $d\alpha$ is positive on $\mathcal{J}$-complex lines.

We also pair these two. What does this give us? A complex structure that is translation invariant. If I translate in the $\mathbb{R}$-direction I get the same complex structure.

Let me take a little bit of time to explain why we need to consider fairly complicated holomorphic curves. What would be the first attempt to define the "Legendrian Floer homology?" We'd try to take a chain complex generated by the critical points of this action functional, by Reeb chords, and then try to define a differential by counting holomorphic strips interpolating between the two. I should draw another picture [pictures].

Let me go to the first attempt to define this. Let me assume for now that there are no Reeb orbits, only Reeb chords (for example the ambient manifold is $\mathbb{R}^3$). Let's first note that by my choice of complex structure, if I take a Reeb chord c and multiply it by $\mathbb{R}$, this is a $\mathcal{J}$-holomorphic strip with boundary on $\mathbb{R} \times \Lambda_K$. This says that Reeb is tangent in the chord direction and t is the other tangent direction, so this has a complex tangent plane.

The differential, we'd like to define ∂c by counting $\mathcal{J}$-holomorphic strips which, at positive ∞, the strip is asymptotic to $\mathbb{R} \times c$ and at $-\infty$ asymptotic to $\mathbb{R} \times b$. We'd like these to be rigid so the only ambiguity is the shift in the $\mathbb{R}$ direction.

So the main theorem you need to do something like this is the theorem that guarantees, the sort of compactness theorem, due to Bourgeois, Eliashberg, [unintelligible], and it says that any sequence of finite energy holomorphic curves converges to a several-level building (holomorphic curve). Let me not make this precise. If we know this, it seems that maybe we have $\partial^2 = 0$ because if we look at the boundary, [pictures].

How to fix this? One sees that maybe this is the only problem, and you can't insist that your chain complex is just an ordinary linear complex. As you see you need things like products of Reeb chords, if you borrow from physics, you need more particle states.

Before taking a break, I should give a definition for this.

Definition 5.1. The Legendrian differential graded algebra is $\mathcal{A}(\Lambda_K) = \mathbb{C}\langle\text{Reeb chords}\rangle$, a unital non-commutative algebra in general, generated by monomials, a typical thing looks like sums of products of Reeb chords where the order matters.

Then we want to define the differential, and now I should be a little bit more precise in terms of compactness and so on. The differential $\partial : \mathcal{A} \to \mathcal{A}$ is defined as follows. Basically, on products it satisfies the Leibniz rule, and then we just need to define it on generators.

$$\partial a = \sum |\mathcal{M}(a, b_1 \dots b_m)| b_1 \dots b_m$$

where we're summing over moduli spaces of dimension 1 with one positive boundary at a and negative boundary at $b_1, \dots, b_m$. I count the number of copies of $\mathbb{R}$ in this moduli space.

I should say what $\mathcal{M}(a, b_1 \dots b_m)$ is, it's the space of maps u from the $(n+1)$-punctured disk to $\mathbb{R} \times U^*M$ such that $du + \mathcal{J} dui = 0$, with one puncture positive and the others negative. The boundary components map to $\mathbb{R} \times \Lambda_K$.

Now $\partial^2 = 0$ works by the compactness theorem. If we look at the boundary of this moduli space, then it is equal to splittings into dimension 1 [pictures].

We'll now take a break but let me say a couple of words. There are some refinements. We can count with more refined coefficients, taking into account the homotopy or homology class of the disk itself. What I said so far is a slight lie, you have to throw in orbits. It'll be important for the future of the subject. Once things are all defined I'll try to explain how to compute this thing, get more concrete.

This still has some problems. I'd like to repair this lie. The key to this theory, in the presence of orbits other things can happen. If there are orbits, we could have something else happen, like [picture]. What this tells you is that you have to consider this algebra as a module over the corresponding orbit algebra. Let me be brief about this because it won't enter very much into what we are doing. There is a contact homology differential graded algebra associated to U^*M, which I'll write $Q(U^*M)$ generated by Reeb orbits with a differential which counts similar curves but the asymptotics are orbits. You see that there is a difference between the disk and the sphere. In the sphere there is no good way of ordering these outputs. You get a signed commutative algebra, and again for the same reason you get $\partial^2 = 0$.

Let me add again some pieces of information. The Reeb orbits, these algebras are actually graded. Just like in the Morse theory, the expected dimension of the space of flow lines between two critical points is the difference between their indices. Likewise there is the Conley–Zehnder index (plus $n-3$ or something like that) and something similar for chords. In our case $(n = 3)$ the grading is the Morse grading on geodesics. In other words, the chords and orbits are critical points of the length functional, it has infinitely many points in one direction but only finitely many in the other.

These Legendrians from the geometric setup are extra nice. We automatically have everything bounded from below. This makes life much easier. If you want to calculate homology in degree zero, you just need to look at [unintelligible].

The degree 0 orbit algebra is simple for rational homology spheres. In fact, for rational homology spheres, $Q_0(U^*M)$ is the algebra generated by conjugacy classes of π_1, homotopy classes of loops. For S^3 this is just $\mathbb{C}$. For $\mathbb{RP}^3$ there is an extra class. We'll eventually discuss relations to other manifolds. When you crush the zero section you expect a source for deformation for [unintelligible].

Let's take M for the time being to be $\mathbb{R}^3$ or S^3. If you look at this thing, our first attempt, we're somehow counting these curves as is. If we have two of these, we sum them. In fact, we can make a more refined curve count. These almost represent classes in $H_2(U^*M, \Lambda_K)$. The boundary of the curve lies in Λ_K and the interior of the curve lies in U^*M (except for the Reeb chords). In order to really get this, well, you need to add the Reeb chords somehow. I'll choose for each Reeb chord some kind of capping disk, a formal thing that I keep fixed for all time. Then indeed I get really somehow an element in this relative homotopy group. The choice of capping disk will just be a change of variables in the algebra.

So we should upgrade the algebra not to have coefficients in complex numbers but in some kind of group ring, $\mathbb{C}[H_2(U^*M, \Lambda_K)]$. For signs I'll be vague but you should use an index bundle over the space of maps. You can put there an index bundle and if you can orient that then you can count with signs. This uses a spin structure on Λ_K. I won't go into detail about how this can be done, at least not today.

For a knot in S^3 we use the following variables. Around the knot we have a tubular neighborhood which we can think of as Λ_K, and here we have some meridian, which we call p and a longitude we call x. So we use variables e^x and e^p and there is one more variable $Q = e^t$ corresponding to the class of the fiber.

Once we've found these generators, we have the knot contact homology differential graded algebra, which is the following.

$$\mathcal{A}(\Lambda_K) = \mathbb{C}[e^{\pm x}, e^{\pm p}, Q^{\pm 1}\langle \text{Reeb chords}\rangle$$

Let's try to compute something. I'll start by computing the knot contact homology of the unknot. This is a kind of key calculation. Once you know the unknot, you can figure out other knots in terms of the unknot.

So take the unknot in $\mathbb{R}^3$. Note first that $U^*\mathbb{R}^3$ can be represented as the 1-jet space of S^2, $T^*S^2\times\mathbb{R}$. I take (x,y) to $y, x-(x\cdot y)y, (x\cdot y))$ and pdq is intertwined with $dz - pdq$. I'll draw the Legendrian in T^*S^2. For comparison, let's look at something in $\mathcal{J}^0\mathbb{R}$, [pictures].

Let me now try to explain how this leads to a calculation for any knot. What we do is take the unknot and represent any other knot as a braid around the unknot. What then happens with this Λ_K is that Λ_K lies near Λ_U. The conormal lies in a small neighborhood of the conormal of Λ_U. Then Reeb chords are as follows. There will be some short Reeb chords which I will call a_{ij} and b_{ij}. You can think of drawing the braid on an ever-increasing cone. You won't get minimizing lines except in two places. You can draw the braid so there are connections between the strings in certain places and otherwise like this [pictures]. Then you have c_{ij} and e_{ij} from the unknot.

Then the holomorphic curves behave in a controlled manner. [pictures]

Let me do one example. [pictures]

6. Sergei Gukov: Volume conjecture as a simple quantization problem: its generalization and categorification II

Let me give a brief review. To a d-manifold M_d we associate $Z(M_d)$ (a number) and to a $(d-1)$-manifold $\mathcal{H}(M_{d-1})$ a vector space.

If someone can't tell you Z or $\mathcal{H}$ then they don't fully understand the theory. That doesn't mean the theory is bad, it means we have more work.

There are many different theories, and sometimes in the same dimension there are many different TQFTs.

In dimension 2, well, 2-manifolds are Riemann surfaces, and the number you'll associate is $\sum_\lambda (c_{\lambda\lambda\lambda})^{2g-2}$, where $c_\lambda\lambda\lambda$ is what we associate with the pair of pants.

Whoever has a TQFT should give me Z for every Riemann surface and also what $\mathcal{H}$ is. It often is some sort of cohomology. Often $\mathcal{H}$ is cohomology of *something*. There are many different versions even in two dimensions. I can pick any space here, $H^*(\mathbb{CP}^n)$, $\langle x\rangle/x^{n+1} = 0$. This is because cohomology rings are Frobenius algebras. So $\mathbb{CP}^n$ is too simple so maybe you want to consider more complicated things, like the Grassmannian, which gives Chern–Simons theory. You can pick any space or any cohomology. They all define two dimensional TQFTs. It's a good idea to ask if the Hilbert space is the cohomology ring of something. In higher dimensions, you may have to consider exotic versions of cohomology. In two dimensions, all simple things are always based on something like that. If the space of states is infinite dimensional, invent a space with two dimensional cohomology.

In dimension 3, we talked about Chern–Simons theory, a very special TQFT, which associates to M_3 numbers that depend on G and k (or equivalently instead of k, use q or $\hbar$), and so this is at the level of what we associate to a closed 3-manifold.

You shouldn't forget the whole tower. There's a Hilbert space for a Riemann surface, a category for a circle, and a 2-category for a point. If someone gives you a TQFT, you're legally allowed to ask all of these questions.

Yesterday I told you something about this Hilbert space in the case that $G = SU(2)$. More generally, you can say that the dimension of $\mathcal{H}_{CS}(\Sigma)$ is $Z_{2d}(\Sigma)$ where I implement this construction, where I can go from a high dimensional to low dimensional one by crossing with S^1. Yesterday I gave you an explicit example where λ ran from 1 to $k+1$ that gives the dimension of the Hilbert space in the theory one dimension higher.

This phenomenon is a relation between TQFTs across various dimensions. If you have a d-dimensional TQFT I described how to make a $(d-1)$-dimensional theory which has precisely this property, so that $Z_{d-1}(M_{d-1}) = \chi(\mathcal{H}_d)$. This process of going down is easy. If you have something bigger, you can just go down. This is analogous to differentiate. It's easy to differentiate. It's like a certificate. This is called decategorification. The reason is that you are lowering the categorical level. You can reach 2-categories in a 3-dimensional TQFT. But in 2-dimensional theories you can only get to ordinary categories.

The interesting operation is the inverse one, categorification, like integration, it's an art. It's hard to construct such a thing. This is called categorification. In practice, if you have some number, now you want to realize this as an Euler characteristic.

Yesterday we saw an example of this, a relation between the Chern–Simons theory and the $2d$-theory like this. This uses the product structure of the Grassmannian.

In three dimensions, there is a Chern–Simons theory. We'll talk a lot today about Z, my job is to describe at some point how to give a vector space given a choice of gauge group and Riemann surface. I gave as a disclaimer that I haven't explained it. If someone claims to have a TQFT, you have a full right to ask about the vector space one dimension lower.

This is homework for Sergei, and I'll do it, either later today or Thursday, to describe this vector space for $G = SU(N)$, how it depends on the level k, and also the answer for $G = SL(2,\mathbb{C})$.

Again you can ask how this structure changes if you change these knobs. In the two cases it will be finite or infinite dimensional, in fact it's already computed for the $2d$ version. If you ask this for the group $OSp(2|1)$, the simplest supergroup. What is the dimension of this Hilbert space and what is the analogue of the Grassmannian? This is a place where people focus on the three-manifolds but don't answer the simpler question, about the vector space which is computed by this simpler TQFT. So the question is, what is this for OSp. This is homework, maybe not overnight, but probably over the weekend. I'll offer a bottle of wine or a big jar of 김치 to whoever solves it. I want the multiplication rule. I want to know the product in the theory. If I replace $SU(n)$, what do I replace the Grassmannian with? I'll also ask, what about $SL(2,\mathbb{R})$.

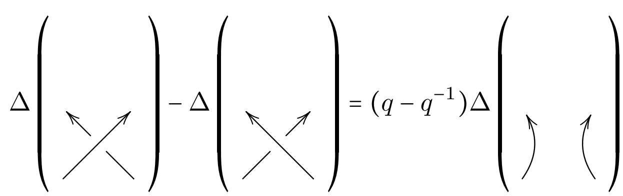
Moving on to $4d$-TQFTs and emphasizing that they exist, these are more rare and complicated. Going down is easy. Going up is hard. This is why you want to reduce to 2-dimensions first. Anyway, going to dimension 4, there are two main examples of TQFT. One goes back to the 80s, and it's the theory of Donaldson and Witten which associates to M_4 the number $Z(M_4)$, the Donaldson polynomial, which is $\#\{F_A^+ = 0\}$, modulo gauge. The space associated to M_3 is Floer homology, invented by Andreas Floer, who unfortunately committed suicide. Sometimes there are tricks to do, apart from giving a Fields medal to Simon Donaldson, these are rather hard to compute and really work with. I won't say anything else about them. However, there is another version, (I should say, there are variants of the Donaldson story) one of which is the variant of Seiberg–Witten, and that's what we'll talk about. It associates to a closed 4-manifold a number and more importantly it associates to a 3-manifold (maybe with a decoration, a knot) some vector space which is HM, the monopole homology.

If I decategorify, there should be a theory that computes $\chi(\mathcal{H}(M_3 \supset K))$, that turns out to be the Alexander polynomial Δ which can be defined for knots and three-manifolds, and can be related to other invariants.

These two theories are supposed to be equivalent and package the same information, both at the level of Z and $\mathcal{H}$. One is two copies of the other (for $SU(2)$) This can be checked in every example but proving it is very difficult. To answer your question, I'll use the mysterious property $DW = SW$. Then it will be something like two copies of Δ. Proving this mathematically is a great outstanding problem. If you make progress, let's not talk about a jar of 김치, you're going to collect the Fields medal yourself. So this is a theorem of Taubes.

In all of these cases, there is an Euler characteristic, so you have to understand this function first and then go back. You're doing something first in lower dimension and then upgrading it.

Let me first define the Alexander polynomial for knots. For knots in S^3, this is fairly easy. You take, you can define it using a skein relation [Alexander skein relation]



Then computing Δ of the trefoil is straightforward, it's $q + q^{-1} - 1$ or something like that.

[missed some]

$$\sum_{i,j} \dim \mathcal{H}^{i,j}(M_3 \supset K)(-1)^i q^j = \Delta(M_3 \supset K, q)$$

Let's take a short break and then continue the three dimensional case.

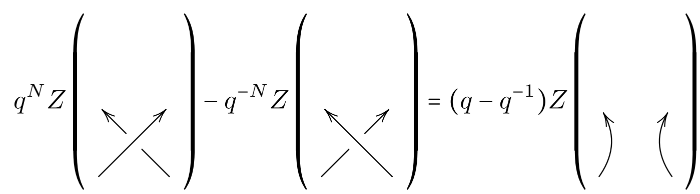
I'm done with the world of $2d$ and $4d$ TQFT. The $2d$ is basically always cohomology of something. From $4d$ I want to borrow just categorification and decategorification.

In Chern–Simons if you have M_3 and a knot in there, to this we associate a number $Z(M_3 \supset K, q, G, \lambda)$ where λ is a color, a decoration, a choice of representation of G, the way you compute the infinite dimensional thing, you take the trace of the holonomy of the gauge connection along K, and here it's the question, which trace? It's in the representation λ.

If you write this without thinking then λ can be any representation of G, but one thing we learned last time is that classically, λ is a representation of G but in a full quantum theory, for $G = SU(2)$, the representation λ is truncated, it's much smaller, it depends on k or q, where $q = e^{\frac{2\pi i}{k+N}}$, and then λ should be a representation of $\hat{\mathfrak{g}}_k$, the affine Kacs–Moody, so for instance λ is in the range $1, \dots, k+1$ for $G = SU(2)$. This is interesting, and if k becomes large you get any representation, the classical answer.

Then we ask the following question (bless you), you have Z the number which depends on all these things, you can ask how $Z(M_3 \supset K, q, G, \lambda)$ depends on various things. It depends on the knot K, for instance. This leads you to consider classes of knots for which this gives you the same answer. It helps to organize knots into special classes but not so easy. You can ask how it depends on $G = SU(N)$ for varying N. This, and then for most of the time we'll stick to knots or links in S^3. Unlike the previous one which is hard to answer, this has a simple answer. You do computations for various knots and try to compare your results for different values of N. You do many computations and realize the answer depends in a nice regular way, you can write a nice function of 2 variables $a = q^N$ and q, and the dependence gives you an answer for different ranks. Let me illustrate how this goes.

In fact for knots in S^3, the colored (with the simplest possible color, λ the fundamental representation), invariant Z behaves in a very simple way



and then replacing q^N ith a we get the HOMFLY-PT polynomial at $a = q^N$. I think in this case people write P instead of Z.

You can do this, package all the ranks together, that's what's going on.

Computations then are really easy, and the answer for the trefoil knot is as follows, P of the trefoil knot is $aq^{-1} + aq - a^2$.

For homework, compute $P(a, q)$ for your favorite knot and see what happens, and you see already very quickly a lot of things that were computed by hard work for Chern–Simons theory. SOmething interesting happens setting $a = q$ and $a = q^{-1}$. First of all, do it for many different knots, and then try to explain. Maybe just think about it.

Okay, how does this depend on color or the choice of λ? This is a cool thing. It's precisely the volume conjecture. This is a little more interesting. If you're doing practical computations for the trefoil, you quickly spot the dependence on N, then you can take $SU(2)$ and try to play with representations. Here you're not going to notice anything simple right away. You'll see for different λ that nothing simple happens right away. This will be more subtle than regularity. Even though the number of terms grows, there is a way to tame the zoo. As an example from combinatorics, suppose you want to study a problem that involves counting three dimensional partitions. Take a positive octant in Euclidean space, then start putting boxes in the corner of the room. [puts box in corner]. This is a 3-dimensional partition, π, you can make π. If you're familiar with young tableaux, this is two dimensional boxes fitting in a positive quadrant. So start counting these guys, you can easily introduce a generating function, $\sum_\pi q^{|\pi|}$, it's always a good example to put the first few terms, it's

$$1 + q + 3q^2 + 6q^3 + \cdots$$

and this is $\prod_k (1 - q^k)^{-k}$, related to the trilog, and you can wonder whether there's some sort of thing controlling this. In combinatorics you have a so-called "limit shape" by letting the norm of π go to ∞. If you scale things appropriately, you get a nice beautiful algebraic curve which is $1 + e^x + e^p = 0$. If your size of representation gets bigger and bigger, what's the limit shape for our invariant?

So now what we're trying to do, I need my box, it has all the chalk in it. Back to real life. If you ask, first of all, how P_λ depends on λ, then for starters, you can ask about partitions that are a single row or column of boxes, and this is, how about the size of $P_\lambda(K; a, q)$ as the number of boxes n goes to ∞? Well, this goes to ∞. Motivated by limit shape, you rescale, so $q = e^\hbar$, set this to 1, and this is very much like the classical limit $\hbar \to 0$, and it would be the classical limit except you are letting the representation get bigger. The representation gets bigger as the q goes to 1. The only thing you have to know is how to correlate the growth of the partition and the shrinking of the boxes to get something sensible.

If you try to compute holonomy of the gauge field A around a strand decorated by λ, this holonomy has eigenvalues that are roughly q^λ (in this case q^n). Therefore you should take q^λ fixed. LEt me say that you want $x = q^n$ fixed. Once you have this input from gauge theory, you can say, aha, I want to take the classical limit, where the representation becomes bigger and bigger, but take the logarithm and divide by n,

$$\frac{\log |P_n(K; a, q)|}{n}$$

and take the limit as $n \to \infty$. So the tricky part is to engineer what kind of limit you want to take. The variable a should be fixed, but there are tricky things, we want q to go to 1 and $n \to \infty$ but fix q^n. So $\hbar\lambda$ should also remain fixed, it's $\log x$ in this case. Then what should be the right hand side of this?

It should be a function $V(K; a, x)$ depending on the variables we kept fixed. We can let x be a complex variable. The Kashaev invariant is $a = q^2$, the colored Jones polynomial, and the fixed value is 1. That's the original form of the Kashaev conjecture. Now you can play with details of this limit and keep the variable a which started life in the HOMFLY-PT polynomial.

What's more interesting, I'll finish with more homework, sorry, there's lots of homework but let me give a little more, I have some too, I should tell you about the vector space that I promised.

The function V is a close analogue of the volume function, $V(K; a, x)$, it's computed from a nice algebraic curve, I'll call it a "spectral curve" or "limit shape" using motivation from combinatorics, the zero set of a polynomial $A(x, y; a)$ in $\mathbb{C}^*_x \times \mathbb{C}^*_y$.

The idea is on this zero locus you have to integrate a differential 1-form from a fixed base point to the point of interest. This, integrated against $d\theta$, will give you the volume form. I'll talk about that next time.

Let me give another homework assignment and finish there.

Study similar asymptotics (compute analogous volume function) for the gadget $P_n(T)$, T the trefoil, where n is the number of boxes in a single row, which is

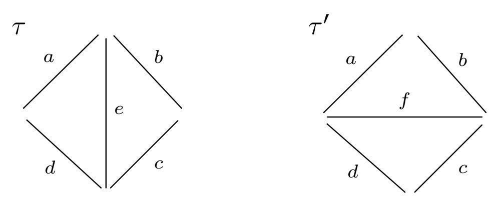
$$\sum_{k=0}^{n-1} a^{n-1}t^{2k}q^{n(k-1)+1}\frac{(q^{n-1},q^{-1})_k(-atq^{-1}q)_k}{(q,q)_k}$$

and you'll get a volume function $V(x;a,t)$, and if you wish, it will be a volume function which depends on three coordinates. The coordinate x has an interpretation. Computing the function is fairly easy, but I really offer perhaps many bottle or a whole case of champagne or fine wine for anyone who finds a definition of this volume. For $t=-1$ this is exactly the augmentation polynomial that Tobias will talk about it. There are other specializations. This is a nice classical object, comes from an algebraic curve, something nice, I believe it has a nice mathematical definition. Therefore I offer the whole case of champagne or fine wine. I'll tell you t next time, I'll talk more about the meaning of this polynomial and what it is. This depends on many variables, q, a, and t. What I'm trying to hint at is that this is a very powerful invariant of the trefoil, and this invariant is really cool and exciting and you can see how it relates to colored Jones, Alexander, and many other things. I'll tell you next time.

7. Rinat Kashaev: Quantum Teichmüller theory and TQFT II

Let me recall what I was talking about yesterday. I was talking about something where Sergei talked a lot, I had $\pi_1(S)\to PSL(2,\mathbb{R})$, and if I call this $\mathcal{M}(S)$ when I mod out by conjugation by $PSL(2,\mathbb{R})$. So we have $\mathcal{M}(S)=\bigsqcup_{|k|\le-\chi(S)}\mathcal{M}_k(S)$ and we defined the Teichmüller space $\mathcal{T}(S)=\mathcal{M}_{\pm\chi(S)}(S)$ and then decorated Teichmüller space $\tilde{\mathcal{T}}(S)$, which came with a a projection ϕ to Teichmüller space, and we had the set of ideal triangulations $\Delta(S)$. Recall S is of type $S_{g,n}$ which is $\bar{S}_g\backslash V$, and $V=\{p_1,\dots,p_n\}$, and my restrictions are that V is nonempty and $2-2g-n=\chi(S)<0$. Then, with this restriction on S we have the ideal triangulations and the identification, the Penner parameterization, we choose $\tau\in\Delta(S)$, an ideal triangulation is a subset of ideal arcs $\mathcal{A}(S)$, a maximal set of pairwise disjoint such arcs. Then λ_τ sends $\tilde{\mathcal{T}}(S)\to\mathbb{R}^\tau_{>0}$ so we parameterized this decorated Teichmüller space by assigning positive real numbers on all edges of the triangulation.

We had properties that if we have a flip in τ to τ' related by changing one edge



then $\lambda_\tau(\tilde{m})$ and $\lambda_{\tau'}(\tilde{m})$ are related by the Ptolemy relation $ef=ac+bd$. This is the most important relation in the whole theory, everything is encoded in this single relation.

A part of this, we also noticed in these coordinates, $\phi^*\omega_{WP}$, the pullback of the Weil–Petersson symplectic form, is $\sum \frac{da\wedge db}{ab} + \frac{db\wedge dc}{bc} + \frac{dc\wedge da}{ca}$, and to be strictly correct this should be pulled back further by λ_τ^{-1}, but this is a homeomorphism so we can omit that.

This is briefly what I was talking about yesterday. Now the goal is to quantize things. To do that, I'll use another set of coordinates, called ratio coordinates, and what we do is the following.

7.1. **Ratio coordinates.**

Definition 7.1. A *dotted ideal triangulation* is an ideal triangulation where each triangle has a distinguished corner.

Before we had just a triangle, and now we put a dot in one corner.

Now we define a map $r_{\dot\tau} : \mathbb{R}^{\tau}_{>0} \to \mathbb{R}^{2\dot\tau_2}_{>0}$, and if I write a dot it's a dotted ideal triangulation and if I don't write a dot then I take the ideal triangulation that is the image under forgetting the dots. Here τ_2 is the set of triangles.

We just put two ratios inside. We have the dot in a triangle, and inside we put $\frac{b}{c}$ and then $\frac{a}{c}$. We take them in a certain order, first the right side and then left side, to call them the first and second component. That's the definition of $r_{\dot\tau}$. This is a new space, this map is neither surjective nor injective. We can always think in terms of vector spaces by taking logarithms, we can deal with $\mathbb{R}$ to some power, so we can think in terms of group homomorphisms. This map becomes, even without that, any power of $\mathbb{R}_{>0}$ is a group, and this is an Abelian group homomorphism.

Now what I want to say is the following. Well, let me define a second map, $s_{\dot\tau}$, this goes from $\mathbb{R}^{2\dot\tau_2}_{>0} \to H^1(S_1, \mathbb{R}_{>0})$, and we define this, $f \in \mathbb{R}^{2\dot{t}}$ should, to the class of a closed loop, assign a number, $s_{\dot\tau}(f)(\gamma) \in \mathbb{R}_{>0}$, we define by putting γ in generic position with respect to triangulation and then calculate $\prod_{i=1}^{L} s(\gamma_i)$ where γ is given by $\gamma_1 \cdots \gamma_L$, and then γ intersects a sequence of triangles. These arcs of intersection are denoted by γ_i. [picture].

We say how we calculate $s(\gamma_i)$, these are positive numbers, where $s(\gamma_i)$ depends on how the dot is places with respect to γ_i. There are six possibilities taking into account orientation, but orientation just makes inverse so I won't write this case.

It's x_i/y_i or y or x for f_i of a triangle being (x_i, y_i), depending on which corner is clipped relative to the dotted corner. [pictures]. The product around the triangle is one.

The following is an exact sequence:

$$1 \to \mathbb{R}_{>0} \xrightarrow{\text{const}} \mathbb{R}^{\tau}_{>0} \xrightarrow{r_{\dot\tau}} \mathbb{R}^{2\dot\tau_2} \xrightarrow{s_{\dot\tau}} H^1(S_1, \mathbb{R}_{>0}) \to 1$$

This is an easy theorem. Let me remark.

Remark 7.1. (1) $\mathbb{R}^{2\dot\tau_2}_{>0} \cong P\widetilde{\mathcal{T}}(S) \times H^1(S, \mathbb{R})$ is *not* canonical.

(2) $\mathbb{R}^{2\dot\tau}_{>0}$ is the "Teichmüller" component of $\mathcal{M}_G(S)$ with $G = GL(2,\mathbb{R})/\{\pm 1\}$, this is $PSL(2,\mathbb{R})\times\mathbb{R}$. This space is like a moduli space of distinguished components for that group. If we start with that group, we'll just be describing by assigning a pair inside each triangle. In principle we're not obliged to mod out by ± 1 but we have to choose a sign of the determinant, we have two signs and we have to decide on them.

For us, it is not important what is the global thing about this space. What we'll be doing next is specifying a symplectic structure. Now $\mathbb{R}^{2\dot{\tau}_2}_{>0}$ is even dimensional and each triangle has a paired set of coordinates.

Let me erase here and put the second part of the easy theorem, which is that, let $\omega_{\dot{\tau}} = \sum_{\text{triangles}} \frac{dydx}{yx}$, like before but now each triangle contributes one single term from the coordinates. Then $r^*_{\dot{\tau}}\omega_{\dot{\tau}} = (\lambda_\tau^{-1})^*\phi^*\omega_{WP}$.

The second part is very nice because now we can forget about decorated Teichmüller space, which wasn't symplectic because we were pulling back a symplectic form to a fibration, it was degenerate on the fibers.

So what happens if, for any $\dot{\tau}$ and $\dot{\tau}'$ in $\dot{\Delta}(S)$, what do we have? We have maps

$$\mathbb{R}^{\tau}_{>0} \xrightarrow{r_{\dot{\tau}}} \mathbb{R}^{2\dot{\tau}_2}_{>0}$$

$$\mathbb{R}^{\tau'}_{>0} \xrightarrow{r_{\dot{\tau}'}} \mathbb{R}^{2\dot{\tau}'_2}_{>0}$$

and I should say that we can combine any two triangulations by recursive Ptolemy relations, so I can complete this, the theorem says that we can complete this to a commutative diagram

$$\begin{tikzcd}
 & \mathbb{R}^{\tau}_{>0} \arrow[r, "r_{\dot{\tau}}"] & \mathbb{R}^{2\dot{\tau}_2}_{>0} \\
\tilde{\mathcal{T}}(S) \arrow[ur, "\lambda_\tau"] \arrow[dr, "\lambda_{\dot{\tau}}"'] & & \\
 & \mathbb{R}^{\tau'}_{>0} \arrow[uu, "\lambda_\tau \circ \lambda^{-1}_{\tau'}"] \arrow[r, "r_{\dot{\tau}'}"] & \mathbb{R}^{2\dot{\tau}'_2}_{>0} \arrow[uu, "\beta_{\dot{\tau},\dot{\tau}'}"']
\end{tikzcd}.$$

Let's take a break.

Let me just sayabout this β, it's unique for the following reason, let me calculate in two particular cases. The first case is when we change coordinates. We have τ and τ' where we change corners in just one triangle. Then [pictures] we have the same λ coordinates, but the map $r_{\dot{\tau}}$ differs, it's either $(\frac{b}{c}, \frac{a}{c})$ or $(\frac{c}{a}, \frac{b}{a})$. So what is $\beta_{\dot{\tau},\dot{\tau}'}$? So (x, y) goes to $(\frac{y}{x}, \frac{1}{x})$, that's how β is determined. It's calculated uniquely.

Now I'm producing the map more or less.

The other case is the Ptolemy relation [pictures]

Now what I want to discuss is the combinatorics of all ideal triangulations and so on.

7.2. **Groupoid of ideal triangulations.** Let me just do a general construction, I'd like to start there because it's more easy to understand. Given a group G, but not the group G, the gauge group, just any group, freely acting on a set X. A free action means that X is a total space of a principal G-bundle. Then we can associate to this a groupoid. Define the groupoid $\mathcal{G}_{G,X}$, this is a category with all morphisms invertible. The objects of this groupoid are G-orbits in X, the base set of the principal bundle. This is a set of objects. What are the morphisms? I won't describe the morphisms between two objects, let me give the full set of morphisms of all objects. This is G-orbits of the diagonal action in $X \times X$. Set theoretically it's just this. Then we should say what is the algebraic structure of this. What is the product of two composable morphisms?

So $[x, y]$ is the orbit of (x, y). This is an orbit, that means that this is the same as $[gx, gy]$ for any $g \in G$. Then we say that $[x, y]$ and $[u, v]$ are composable if the orbit of y is the orbit of $[u]$. Then $[x, y] \cdot [u, v] = [x, gv]$ where g is the unique element that sends u to y. So u and y are in the same orbit, there's an element, but since the action is free, the element is unique. That's the product. Then $[x, y]^{-1} = [y, x]$ and the identity of $[x]$ is $[x, x]$.

Any G-principal bundle has an underlying groupoid in this sense.

Remark 7.2.
1. $\mathcal{G}_{G,X}$ is connected, between any two elements the morphism set is nonempty.
2. $Mor_{\mathcal{G}_{G,X}}([x], [x]) \cong G$.

You see that connected groupoids are classified easily, it's like vector spaces, so basically it's the same as numbers, but that's not a reason to not study vector spaces.

So here it's the object set and the group at one object. It's a general fact for any connected groupoid. There is no canonical way of representing it. That's why it's similar to vector spaces.

In our case we choose $G = MCG(S)$. Now what is X? It's almost dotted ideal triangulations, but not quite, it's pairs $(\dot{\tau}, u)$ where $\dot{\tau}$ is a dotted ideal triangulation and u is an ordering on $\dot{\tau}_2$, an integer inside each triangle to keep them in order.

Due to this, the action, just a fact that G acts on X freely. If we take just an ideal triangulation without anything, it might not be free, there might be nontrivial mapping classes that don't change the triangulation. So if we put one arrow then it becomes free. Let me call this $\tilde{\tau}$, for this we have much more than one additional arrow on one edge. What is important is that now the action is free. Consequently, we have $\mathcal{G}_{G,X}$ the groupoid of decorated ideal triangulations. That's an algebraic object. The mapping class group is encoded in the groupoid, and we have the orbits, and each orbit is represented by the combinatorial type of the triangulation. So there are finitely many objects. This is a pretty simple object in a sense, a group plus a finite set.

What is interesting, since any group has a presentation, likewise we can ask for a presentation for a groupoid. We have to say what are the generators. By composing them we need any morphism in the groupoid, and then we need relations.

Here is a difficult theorem

Theorem 7.1. *(Teschner,Hyun Kyu Kim) Let $G = MCG(S)$ and $X = \tilde{\Delta}(S)$. Then $\mathcal{G}_{G,X}$ has the following presentation, with generators*

- *symmetric group elements along with*
- *ρ_i which rotates the dot of the ith triangle counterclockwise, and*
- *ω_{ij} which interchanges a Ptolemy relation for adjacent i and j.*

For relations, write $[x_1, \dots x_n]$ for $[x_1, x_2][x_2, x_3] \dots [x_{n-1}, x_n]$. Then we have

- *permutation group relations,*
- *$\rho_i^3 = 1$,*
- *$\omega_{ij}\omega_{ik}\omega_{jk} = \omega_{jk}\omega_{ij}$*
- *$\omega_{ij}\rho_i\omega_{ji} = (\rho_i\rho_j)(ij)$, and*
- *the trivial relations, where elements commute with permutations.*

That's the presentation

So this is a difficult theorem. Just to finish I need a few more minutes.

What is the conclusion out of this theorem? There is an important consequence. It tells us to construct a representation of $\mathcal{G}_{MCG(S),\tilde{\Delta}(S)}$ it suffices to find a triple (V,T,A) where V is an object of a symmetric monoidal category, A is an automorphism of V, and T is an automorphism of $V \otimes V$ such that $A^3 = \mathrm{id}_V$, that $T_{12}T_{13}T_{23} = T_{23}T_{12}$ in $\mathrm{Aut}(V \otimes V \otimes V)$, and $T_{12}A_1T_{21} = A_1A_2P_{12}$ in $\mathrm{Aut}(V \otimes V)$. We can rewrite this last one $TA_1P_{(12)}T = A \otimes A$.

The corollary states that as soon as you have this equation, you have a representation of the groupoid. You don't need surfaces or anything. It's exactly like the Turaev theorem for braid groups, if you have Yang–Baxter. That's what these three relations that replace Yang–Baxter do.

The calculation we did for β in two places. In our case, β will be [unintelligible]. Then A will be the change of coordinates, and T will be the flip of coordinates. To quantize Teichmüller theory, all we have to do is quantize this algebra, no surfaces are needed.

8. Dec 16: Stavros Garoufalidis: Asymptotics of quantum invariants II

So I want to, last time I showed computations, and now I want to tell you what we were calculating. The first part today will be boring, with definitions, theorems, even proofs, the most boring.

The Jones polynomial of a link $J_L(q)$ is a Laurent polynomial in $q^{\pm\frac{1}{2}}$ defined by the following system of linear equations:

$$qJ_{\nearrow\!\!\!\!\nwarrow}(q) - q^{-1}J_{\nwarrow\!\!\!\!\nearrow}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_{\uparrow\;\uparrow}(q)$$

and $J_{\bigcirc}(q) = [2] = q^{\frac{1}{2}} + q^{-\frac{1}{2}}$.

In general,

$$[n] = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{\frac{n-1}{2}} + q^{\frac{n-3}{2}} + \cdots + q^{\frac{-n+3}{2}} + q^{\frac{-n+1}{2}}$$

In fact, the Jones polynomial is a polynomial in $q^{\pm 1}$, which matters when I apply it to nth roots of unity.

The *colored Jones polynomial* $J_{K,N}(q)$ is

$$\sum_{j=0}^{\frac{N-1}{2}} \binom{N-1-j}{j} J_{K^{(j)}}$$

where $K^{(j)}$ is the jth parallel of K with 0 framing. So $J_K(1) = 1$, $J_{K,2} = J_K(q)$, the normal Jones polynomial. $J_{K,3} = J_{KK} - 1$. Then $J_{K,4} = J_{KKK}(Q) - 2J_K$, and the last one I'll write is $J_{K,5} = J_{KKKK} - 3J_KK - 2$. This is the colored Jones polynomial but I want to renormalize it to

$$J'_{K,N} = \frac{J_{K,N}}{[N]}$$

for all $N = 1, 2, \ldots$ Then the Kashaev invariant for K in S^3 is $\langle K \rangle_N = J'_{K,N}(e^{\frac{2\pi i}{N}}$. [unintelligible]was proved by Murakami and Murakami.

Are you happy about this? I will not spoil your state of happiness but continue to the volume conjecture, that (at least for N odd and K hyperbolic) you have

$$\lim_{N\to\infty} \frac{\log|\langle K\rangle_N|}{N} = \frac{vol(K)}{2\pi}$$

This is the volume conjecture of Kashaev. There are versions of this to all orders formulated by Gukov, also [unintelligible]and myself.

Is he here? No? I can say anything I want then! I have license and certificate too.

So I'll say $\langle K\rangle_N \sim N^{\frac{3}{2}} e^{CN} \varphi_{K,1}(\frac{2\pi i}{N})$ where $\varphi_{K,1}(\hbar) = \tau_K \varphi^+_{K,1}(\hbar)$ and $\tau_K^2 \in F_K$ and $\varphi^+_{K,1}(\hbar) \in 1 + F_K[[\hbar]]$ and C is the complex volume $\frac{i\ vol(K)+CS}{2\pi i} \in \mathbb{C}/4\pi^2\mathbb{Z}$. Here F_K is the trace field.

Don Zagier and I proved this to all orders for 4_1.

I'm not happy, let me tell you why. I gave you a rigorous but entirely useless definition of the colored Jones polynomial. If you want to compute the Jones polynomial with the algorithm I just erased, you need something like, if K has c crossings, you need something like 2^c operations. Then $J_{K,N}(q$ requires 2^{cN^2} operations. We're interested in $N = 2500$ so this is like, not possible even with computers. This is also an algorithm, not a formula. I hope you know the difference.

So this is useless. On the other hand, Kashaev gave a finite state sum. For the trefoil,

$$\langle 3_1\rangle_N = \sum_{n=0}^{\infty} (q)_n | q = e^{\frac{2\pi i}{n}}$$

where $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$.

The corollary of the formula is that $\langle 3_1\rangle_N$ can be computed in $O(N)$ steps, linear time. I'll give you a theorem we observed with Zagier: this is true for all knots. But it does not follow from a state formula. The truth is that the complexity of computing the Kashaev invariant is linear for a very big O.

Now I'll give you the figure eight which is far more interesting.

$$\langle 4_1\rangle_N = \sum_{n=0}^{\infty} (q)_n (q_n^{-1}\Big|_{q=e^{\frac{2\pi i}{n}}}$$

and I'll give you

$$\langle 5_2\rangle_N = \sum_{0\le k\le m\le N-1} q^{-(m+1)k} \frac{(q)_m^2}{(q^{-1})_k} | q = e^{\frac{2\pi i}{n}}$$

so this looks like it's quadratic but you can set up a recursion in two steps—we did this to experiment and try to guess these elements of the trace field.

Then here's the question. Let's say for 4_1, compute the first three terms of $\varphi^+_{4_1,1}$, or even the constant term.

How do you do this, even working numerically? I want a way to compute these coefficients from the formula. If I tell you the Fibonacci sequence, you can ask about the asymptotics. You don't have a recursion relating the Nth and $(N-1)$st Kashaev invariants. This is related to quantum modular forms, but I won't talk about that yet. But I'll give you a conjectural answer to the question.

8.1. **Back to hyperbolic geometry.** My input will be an ideal triangulation of $S^3\backslash K$, so my building blocks will be ideal tetrahedra, the convex hull of 4 poitns in $\partial\mathbb{H}^3$. My points will be 0, 1, ∞, and some point z. The opposite edges will get the same variables, z between 0 and ∞, $z' = \frac{1}{1-z}$, $z'' = (z')' = 1 - \frac{1}{z}$, and as homework, check that $z''' = z$, that $zz'z'' = -1$, and $z'' + z^{-1} = 1$.

If you truncate this tetrahedron, you get a little triangle in a Euclidean plane. At some point yesterday Rinat used dots, we have a similar choice here, the choice of z versus z' versus z''.

Given an ideal triangulation with N tetrahedra (don't hate me for reusing N, this is a different N), there are N ideal edges, so for example, for 4_1, $N = 2$ and they are z_1 and z_2 and I get relations, for edge 1 in will be $z_1^2 z_1'' z_2^2 z_2'' = 1$, for the second edge, $z_1'^2 z_1'' z_2'^2 z_2'' = 1$ and a meridian relation $z_1'^{-1} z_2 = 1$. The two edge relations are related by some version of the z and z' and z'' equation, so one of those is redundant. The solution to such a system of equations can describe the unique hyperbolic structure of the knot complement.

I'll choose one edge, replace that edge's equation with the meridian equation, and eliminate exactly one of the three z_i, z_i', and z_i'' from each tetrahedron using $zz'z'' = -1$.

For the figure eight, if you remove the second edge equation and eliminate z_1' and z_2', you get the equations

$$\begin{cases} z_1^2 z_2^2 z_1'' z_2'' = 1 \\ z_1 z_2 z_2'' = -1 \end{cases}$$

Then for $A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, and $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ then this can be written symbolically $z^A z''^B = (-1)^v$ so this is $\prod_j z_j^{A_{ij}} z_j''^{B_{ij}} = (-1)^{v_j}$ for all i.

Definition 8.1. τ_τ^2 is the determinant of $A \operatorname{diag}(z'') + B \operatorname{diag}(\frac{1}{z})) z^{f''} z''^{-f}$ where $Af + Bf'' = v$.

This depends on $\gamma = (A, B, v, z)$, which is NZ datum.

Theorem 8.1. *τ_τ^2 is invariant of choices and independent of 2 and 3 moves. Therefore it's a topological invariant. Then $\tau_{4_1}^2$ is $\sqrt{-3}$ and $F_{4_1} = \mathbb{Q}(\sqrt{-3})$.*

Conjecture 8.1. τ_K is the nonAbelian $SL(2,\mathbb{C})$ torsion of K using the discrete faithful representation and because this is not acyclic, and there's a choice of meridian, this is something defined by Porti and studied by Dubois but more importantly computed by Dunfield using a presentation of $\pi_1(S^3 - K)$.

We checked this exactly for about 800 knots and numerically for about 2000.

That's a first conjecture. The second conjecture is

Conjecture 8.2. τ_K is the constant term in the asymptotics of the Kashaev invariant.

That's checked only for the eight knots we've checked.

Now we have an interpretation as a torsion, a one-loop invariant, torsion with respect to the hyperbolic representation. But what about the other terms in the expansion, the other terms in $\varphi_{K,1}^+$. This used a bunch of time to get to the one-loop term, which is anomalous. One-loops are always anomalous. The higher loop invariants, that is, $\varphi_{K,1}^+(h) = 1 + s_2\hbar + s_3\hbar^2 + \cdots$, and if you're unhappy that s_2 is the term of $\hbar$, blame the physicists, this is the 2-loop invariant.

I'll give you a formula for this entire series. This has a building block, which is some version of some kind of quantum dilogarithm.

$$\psi_\hbar(x,z) = exp(\sum_{n,k,n+k-2>0} \hbar^{n+\frac{k}{2}-1}\frac{(-x)^n B_n}{n!k!} Li_{2-n-k}(z^{-1}))$$

where B_n is the nth Bernoulli number and $Li_m(x) = \sum_{k=1}^\infty \frac{x^k}{k^m}$, the *mth polylog.* I can write this as

$$(qze^{\hbar^{\frac{1}{2}}x};q)_\infty$$

for $q = e^\hbar$. Assume the determinant is nonzero in the NZ datum which is true most of the time, and let $H = -B^{-1}A + \mathrm{diag}(z')$. For 4_1 we have, well $B^{-1}A$ is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

So there is a quadratic term here

$$f_{\tau,\hbar}(x,z) = exp(\frac{-\hbar^{\frac{1}{2}}x^T B^{-1}v}{2} + \frac{h}{8}f^T B^{-1}Af)\prod_{i=1}^N \psi_\hbar(x_i,z_i).$$

Then

$$\langle f_h(x)\rangle = \frac{\int dx e^{-\frac{1}{2}x^T Hx} f_h(x)}{\int dx e^{-\frac{1}{2}x^T Hx}} \in \mathbb{Q}[[\hbar]]$$

Definition 8.2.

$$\varphi^+_\tau(\hbar) = \langle f_{\tau,\hbar}(x,z)\rangle = exp(s_2 h + s_3 h^2 + \cdots)$$

Conjecture 8.3. This $\varphi^+_\tau(\hbar)$ is independent of τ.

Conjecture 8.4. It is the power series that appears in the asypmtotic expansion of $\langle K\rangle_N$ to all orders at $q = 1$.

This is a Feynman diagram definition, this isn't a convergent integral, but you use some formal integration process that terminates, and you can compute 2-loops, 3-loops, whatever.

So to compute S_2 you use the following Feynman diagrams [pictures]

So s_2 is the coefficient of some expression in $\hbar$. Here's what I do. I put an i or j on every vertex. I put a matrix $\pi_{ij} = (H^{-1})_{ij}$ on each edge. So I put [formula based on pictures]

So this is a finite expression, a rational function of N "z" variables. If I do $2-3$ moves, then 2 and 3 shapes go back and forth. The gluing equations are equations among rational functions. You can compute this explicitly maybe for s_2, this is a nice problem.

There is nothing infinite, no kind of strangeness. I have to say, however, that, what will happen for s_3? Well, s_3 is a bunch more graphs. How many more? As many as the Feynman diagram method tells us. If you list the graphs for s_3, we did that, there are exactly 40 which contribute.

Can you trust the result? One of these coefficients we had wrong and it was a nightmare to debug it. How do I know that I made a mistake? This expression is supposed to be invariant under moving around the dots, or removing one edge instead of the other. If I make a mistake, even for the figure eight, I have nine different expressions. What I'm trying to say is that these formulas are way overdetermined so by consistency you can make sure that what you have is independent of the choices.

I asked some students of mine, one of them unfortunately expired, meaning he chose a different advisor. For n loops how many graphs do you need?

n loops	2	3	4	5	6
# graphs	6	40	331	3700	53758

So we did calculations up to 8 crossings, I'll show you some answers later. This answer is at least effective. It requires no guessing. Construct a tetrahedral decomposition or have Snappea do it, and then [unintelligible].

Let me ask a question. What is a geometric interpretation of the 2-loop invariant s_2. Some questions you can learn to ask even if you don't know about ideal triangulations or anything. "What is the geometric interpretation of what you just defined."

[some discussion]

I'm posing this question, is there a geometric invariant. If you find an invariant that's computable, I'll compute it 20,000 times.

Okay, so. If you can categorify the volume, $V(a,x,t)$, then of course you would do $s_2(a,x,t)$. But what does it mean is a different question.

Now I'm changing gears to quantum modularity conjecture of Zagier. This is about quantum modular forms, whatever they are, and this contains the Kashaev invariant of any hyperbolic knot.

So first, let's extend the Kashaev invariant to all roots of unity. So $\alpha = \frac{a}{b}$, a rational number. I take the Jones polynomial $J_{k,b}(e^{2\pi i \frac{a}{b}})$. Now F_K is defined on all complex roots of unity.

Do you see a problem with this definition? Since I didn't say a and b are coprime, there is a problem. This uses a theorem of Habiro, not a simple theorem, that $J_7(e^{2\pi i \frac{3}{7}}) = J_{14}(e^{2\pi i \frac{6}{14}})$. It says that this is well-defined.

I'm not going to worry too much about it.

So let's see. Let's say that we want to look at the asymptotics of $F(\frac{N}{7N+1})$, let's write it $J_{7N+1}(e^{2\pi i \frac{N}{7N+1}})$, which is the same as $\langle K\rangle_{7N+1}|_{q=q^N}$. So what's the answer to that? Let's first try the simpler question, what is $\lim_{N\to\infty}\frac{N}{7N+1}$? That's well known, it's $\frac{1}{7}$. So we're looking for asymptotics of the Kashaev invariant spread out at all roots of unity around some particular root of unity.

Maybe it was easy to define this but we have to pay some price. I guess if we're working with $e^{2\pi i}N$, then $SU(2)$ at this root of unity is a *unitary* field theory. But now we're outside the unitary world. The point is that the roots of unity are here on the unit circle.

I'm asking [picture]

So quantum modularity predicts that $\varphi_{\mathcal{J}}(\hbar) = \varphi_J(0)\varphi_J^+(H)$ where $\varphi_{J_k}(0) = \tau\sqrt{k}\epsilon b$, with $\epsilon, b \in F(\mathcal{J}_K)$, with $\varphi_J^+(\hbar) = 1 + \hbar F(\mathcal{J}_k)[[\hbar]]$.

and the theorem with Zagier is that the quantum modularity conjecture is true for 4_1.

[couldn't understand]

9. Dec 17: Sergei Gukov: Volume conjecture as a simple quantization problem: its generalization and categorification III

So I'll follow Rinat's numbering system. This will be section 7, I'll talk more about

9.1. **Categorification.** 7 is just a random number. Categorification is an exciting process, upgrading a lower dimensional TQFT to a higher dimensional TQFT. This should take us to a new millenium, a new life. Let's try to make this very concrete. I want to combine general ideas with very concrete calculations. I'll highlight a couple of aspects and try to fit it in a more general philosophy and give the idea of how to do computations as well.

However, first I should continue a little bit with the history, I'm approaching the most recent and exciting phase, around 2000–2003, where some exciting things happened in low dimensional topology or representation theory in the last 10 or 15 years.

So what is categorification? Let me make a table.

knot polynomials	knot homology
Alexander polynomial $\Delta_K(q)$	Ozsváth–Szabo–Rasmussen's HFK (Heegaard–Floer knot homology)
Jones polynomial $J(q)$	Khovanov $Kh^{*,*}(K)$

These two were categorified in precisely this time. Each of these got upgraded to a vector space.

What is the relationship of categorification? Well I said it last time but it's

$$\Delta_K(q) = \sum_{i,j}(-1)^i q^j \dim HFK^{ij}(K)$$

The Jones polynomial has the same relationship with Khovanov homology.

Both of these happened at the same time, there was the question why these had integer coefficients, the answer was because they were the dimensions of some spaces. The real excitement came with, it's exciting to work with polynomials, but a vector space with possible more information, you can do a lot more. There was an explosion of activity based on the mere existence and definition of these two theories. You can study *maps*, which you can't do with polynomials. We'll see some of this very concretely.

There is an intermediate stage when it's useful to work with Poincaré polynomials, this could be a third column in my table. It's convenient to introduce a Poincaré polynomial by replacing (-1) with another variable, and I'll use the homology letters to indicate the Poincaré polynomial as well, so

Alexander	$HFK(K,q,t) = \sum_{i,j} t^i q^j \dim HFK^{ij}(K)$
Khovanov	$Kh(q,t) = \sum_{i,j} t^i q^j \dim Kh^{ij}(K)$

and then we recover the Alexander and Jones polynomials by specializing to $t = -1$. Both of these are polynomials with nonnegative coefficients.

These are easier to deal with than doubly graded vector spaces. They lose information about torsion, but remember the dimensions of these spaces. Even dimensions contain more information than the Euler characteristic. It would be surprising to find a class of manifolds where all information is contained in the Euler characteristic.

From the viewpoint of knot invariants, these can distinguish way more knots. This is part of the excitement of low dimensional topology. Representation theorists, you can do a lot more with vector spaces. You can connect to many aspects of representation theory. This is even cooler (in my opinion) than giving a more powerful invariant.

There were a lot more developments that I'm going to skip for you, but let me give you an idea of these two theories.

So HFK is some version of Seiberg–Witten theory adapted for knots. Just like monopole Floer homology categorifies the Alexander polynomial in a very broad setting, this is optimized for knots and links in the 3-sphere. But this is much more computable than in the gauge theory. They extracted a finite dimensional (and more computable) residue. You count some holomorphic disks (essentially the same as what Tobias is discussing). The gauge theory is hard, involved analysis, they produced something much more computable, and a lot of good results based on this theory.

Khovanov was surely motivated by TQFT, but he said, I want to do something that everyone can use. It's like Steve Jobs. I want it to be user friendly, I want to be able to explain it to graduate students. He didn't use symplectic geometry, he used combinatorics. He upgraded the skein relation from numbers to vector spaces.

It remains the iPhone of knot homology, once you start playing you cannot stop swiping up and down, clicking, it's really beautiful. This is about flavor, how they feel.

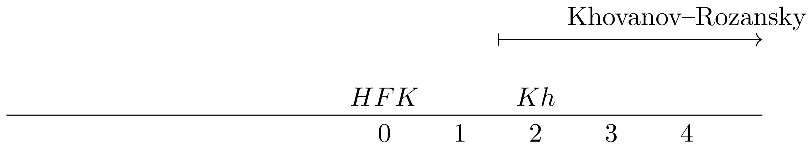
Both theories had become quite computable. Gauge theory doesn't work well enough to compute things for 5_2. People calculated these for many knots. That's why it's very attractive to a younger generation, it's very simple and concrete.

For many knots K, you suddenly have the relation that $Kh(K) = HFK(K)$. That became kind of strange. This was true for all knots of up to nine crossings, all alternating knots, other classes of knots. Even techniques for showing relations like this were impossible because the definitions were not directly related, combinatorics and symplectic geometry. I should say that the dimension of $Kh(K)$ is the dimension of $HFK(K)$. This came as a puzzle.

Over the years, people tried to come up with a combinatorial version of HFK and a holomorphic definition of Kh. So this was an interesting puzzle at the time. It had no right to be, but it was astonishingly accurate, in many cases, then you can quickly try to wonder, as we discussed the other day, $\Delta(q)$ is a specialization of HOMFLY-PT $a = 1$ and the Jones polynomial is also a specialization at $a = q^2$. So one possibility could simply be to come up with a theory that packages different ranks, we talked about $sl(n)$ theory, and that's how we came up with the HOMFLY-PT, if I were going to label these by specializations by $a = q^N$, I label these (Alexander and Jones) by $N = 0$ and $N = 2$. So you can ask if there are versions for other examples for different values of N.

So this was an obvious question back in 2003. Khovanov and Rozansky worked very hard. Khovanov also did the $N = 3$ version, the $sl(3)$ or $G = SU(3)$ version, and he said, "I don't believe in a triply graded version that categorifies all N, but there should be a version for each N." You can open his $sl(3)$ paper and this phrase was there. These theories are hard to construct, it's hard to believe that you could package all the $sl(n)$s together. I totally sympathize how he didn't expect this theory to exist.

Instead of constructing $sl(4)$, $sl(5)$, and so on, they constructed something that covers all values of N. I should add this to the diagram. [picture] There are special points we already covered



Physicists (including me) said that there should be a theory like this, we were nervous, saying things that people didn't believe. I was a little bit worried about it, but you can even extend to $-N$. Once people started developing this for different values of N, people noticed that you can also call $-N$ Khovanov–Rozansky, and this was also kind of a puzzle, it felt like you have two phases. You start with large N and decrease, here you see Khovanov, and if you go to $N = 1$, then you get a very simple trivial theory, by trivial I mean one dimensional. There is a homology categorifying $sl(1)$ but it's very boring. This mirrors the statement, I asked you to look at HOMFLY-PT for $a = q$ or $a = q^{-1}$. These are the values for $N = \pm 1$. The (not such a big) surprise, this specialization, this gives you a single monomial, the dimension of this guy is 1. Pick any knot, you always get this. The explanation from the quantum group point of view is that $sl(1)$ invariants, this should be a simple boring theory. What's nontrivial is $N = -1$. It's not terribly surprising that you get something trivial at $N = \pm 1$, or rather, one dimensional.

So complexity goes down with positive N, so $sl(2)$ theory is smaller than $sl(3)$ and so on, but HFK is highly nontrivial, and then it drops again, and then goes up, so something weird happens here.

The answer can be given in the bigger framework of HOMFLY-PT homology, but some things still need to be explored.

The answer for many of them is something that categorifies HOMFLY-PT homology and even there it's nontrivial.

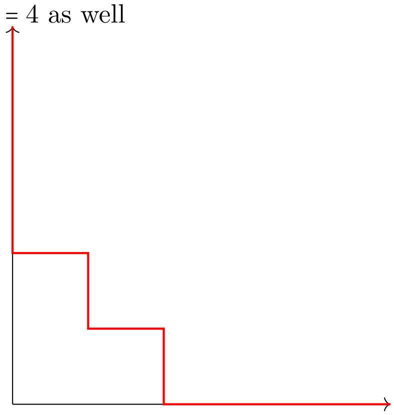
So we want get quantum sl_N knot invariant by specializing HOMFLY-PT to $a = q^N$. These special invariants come from Euler characteristic for doubly graded homologies, and you want to complete this.

$$\begin{tikzcd}
P(a,q) \arrow[d, "a"] & & H_K^{ijk} \arrow[ll, "\chi_{a,q}"'] \arrow[d, "d_N"] \\
\text{quantum } sl_N \text{ invariant} & & \text{doubly graded homology} \arrow[ll, "\chi_q"']
\end{tikzcd}$$

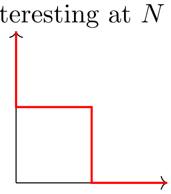
and it's triply graded, it's H_K^{ijk}, so the way it works out, you have this differential, and you get the doubly graded homology theory going from the triply graded to the doubly graded theory, this is the most interesting part of this diagram. This says that sl_N homology as a doubly graded theory, you take your HOMFLY-PT homology, and take cohomology with respect to d_N, which we'll discuss a bit later after the break.

So the degree of d_N is $(-k, kN, -1)$ for $N > 0$, and usually k is chosen to be 1. Sometimes $k = 2$ in the literature. For the other guys, it,s $(-1, N, -3)$, this is a little harder to work out, to see, but again it follows from this interesting symmetry. I'm giving you the statement, not a derivation. This is quite a useful powerful framework. Now what we can do is to play with this. First of all, this diagram is encoding a lot of structural properties of the theory, it should be consistennt with all of its specializations. If you take d_3 or d_4 you should get 3 or 4 Khovanov–Rosansky. If you take homology with respect to d_1 or d_{-1} you get something trivial, and with respect to d_0 you should get HFK. This invariant would package all the knot invariants in one package. This is like the European Union for knot invariants.

I'd like to do two things before the break. Do you like games? Tricks with cards? I'll do it after the break. I'd like to do two quick things. One thing is to explain the gap of complexity, with a bump for $N = 0$. The reason, I'm going to give the answer, all of what I've done here, everything is for a color that is a single box, a Young tableau with one box. You can study this for different λ. If you put λ^t in the square grid, like this, what you'll see is the following, that the right answer involves supergroups, where you continue to $sl(n|m)$ so $N = n - m$. Maybe I should make a bigger plot, this is an exercise where you do the following, you put your Young tableau or its transpose in the plane and draw the line bounding your Young tableau and then you start working down. [picture] When $n = 4$ and $m = 0$ I get $N = 4$ as well



and then you see interesting things happening at corners. So with one single box you get some low complexity at $N = 1$ and $N = -1$ and then something more interesting at $N = 0$.



This reflects some advice from Sir Michael Atiyah. He's great, I met him when I was very young, we stay in touch, he's great not just as a mathematician but also as a mentor and a wise man, he said "if you want to understand something, generalize it." If you do this for a single color you'll be puzzled by this small gap, if you think outside this single box, this gap widens and behaves in a particular way and it'll give you some information about what's going on.

How many of you know Khovanov homology? How about HFK? Well, after the break I'll show you how to calculate both on the back of an envelope.

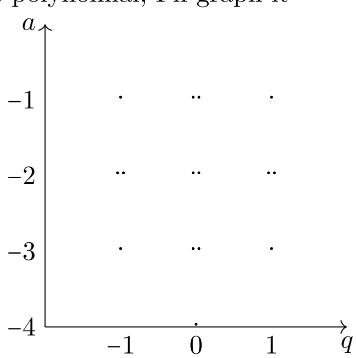
I want to show you this magic trick. I've never tried this before. I honestly don't know what will happen. I'm warning you because I'm part of the audience as well, don't be too hard on me. I'll ask for volunteers in just a second. Give me a number between one and 10. How about a knot with 7 crossings. Oh, 7_4? Okay, and what's its Jones polynomial? $-q^8 + q^7 - 2q^6 + 3q^5 - 2q^4 + 3q^3 - 2q^2 + q$. I can't guess Khovanov homology just looking at this. I need to give you $Kh(q,t)$, so that when you set $t = -1$ you get this Jones polynomial. My job is to restore t-dependence. I could guess that terms with − will be odd grading and with + will be even grading. The specialization will not be enough, I'll use the more general theory, and one thing I need from you, we'll start with N-dependence. You can ask any software to give you the sl_N invariant. Can you give me HOMFLY-PT of 7_4? With some normalization it's

$$-a^{-4} - 2a^{-3} - 2a^{-2} - 2a^{-1} + a^{-3}q^{-1} + 2a^{-2}q^{-1} + a^{-1}q^{-1} + a^{-3}q^{1} + 2a^{-2}q^{1} + a^{-1}q^{1}$$

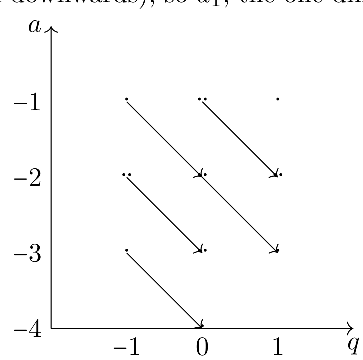
and this should be specialized to $a = q^2$ and then $q \to q^{-1}$ to give the other.

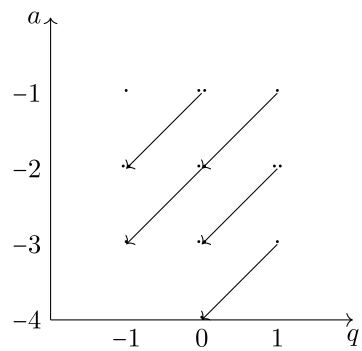
We'll use structural properties to deduce the answer. I need more helpers. I want one of you, I saw some of you, I want one of you to look up the Khovanov homology, I want someone to confirm yes or know. One of you guys can do it on the internet or otherwise. I want someone else to do this for HFK. I want one volunteer for Khovanov and one for HFK.

So I want to put these in graphical form. I can put in the q and a degree, and the polynomial, I'll graph it



First I'll get HOMFLY-PT, so to construct the big object I need t-degrees of each term. The obvious ones are very special differentials d_1 and d_{-1} which have degrees $(-1, 1, -1)$ and $(-1, -1, -3)$. So the gradings are slope -1 and slope $+1$ (both downwards), so d_1, the one dimensional homology



should be in $q = 1$ and $a = -1$. Then I know that the difference in t-degree at different ends of the arrow should be -1. On the other hand I can do the same thing for d_{-1} and get 

and now they differ by -3. So you need at least one number to constrain this guy, but you don't need to provide homological degree for every generator. Since no one volunteered, I'll guess that, and this is a complicated knot, I'll guess that we did the correct jump.

How do we compare this to the correct answer for actual Khovanov homology?

Let's see if this can really happen, we have a proposal, which is this diagram, and the total dimension will be $4+6+4+1$ if I keep all the dots, this is 15-dimensional. Here we have 15 as well. The good news is that nothing cancels. For Kh we will have $(-1,2,-1)$ and I'll just guess this is trivial; then HFK should be downward and I'll guess that is trivial as well.

My prediction is that Floer homology will be in degrees -1, 0, and 1. It will be like $0,-2,-2,-4$, and then in q degree 0, you'll have 7 generators, and then 4 generators.

So this is a highly overconstrained system. I used the boring case, that HOMFLY-PT is boring when $N=\pm1$, so this is barely anything. If I use more then I really have a strong check. Let's try to write down the answer and see what I get.

This is my HOMFLY-PT polynomial, I got a^{-4} gets t^{-5}, so I'll get a -1 sign. Overall I get

$$a^{-4}t^{-5}+2a^{-3}t^{-3}+2a^{-2}t^{-1}+2a^{-1}t^{1}+a^{-3}q^{-1}t^{-4}+2a^{-2}q^{-1}t^{-2}+a^{-1}q^{-1}+a^{-3}q^{1}t^{-2}+2a^{-2}q^{1}+a^{-1}q^{1}t^{2}$$

so we decorated the previous answer with t, even powers of t appear in positive things, and odd powers of t with the $-$ sign. We specialize to $a=q^2$ and invert as before and see the Khovanov homology. The Poincaré polynomial, from $a=q^2$

$$q^{-8}t^{-5}+2q^{-6}t^{-3}=2q^{-4}t^{-1}+2q^{-2}t+q^{-7}t^{-4}+2q^{-5}t^{-2}+q^{-3}+q^{-5}t^{-2}+2q^{-3}+q^{-1}t^{2}$$

Oh boy, I guess if I translate to this notation, I get t degrees $-5,-4,-3,-2,-1,0,1,2$. I claim that that's Khovanov.

So this is one little corner that I tried not to touch because it goes too much into physics, but given a knot K or more generally a link, it should be colored, you produce a special Lagrangian submanifold L_K that can see it as a Lagrangian submanifold in T^*S^3, this is Tobias' setup. This Lagrangian is a conormal bundle, now an interesting thing is that if you ask someone in differential geometry what's so special about this manifold, that someone will tell you it's Calabi–Yau, it has a complex structure and the canonical class is trivial. You can describe this as $x^2+y^2+z^2+w^2=1$ in $\mathbb{C}^4$. That makes it easy to check that it has the right topology, and play with the Kähler form, but this makes it clear it's a complex variety, in fact a simple computation tells you $c_1=0$ so it's Calabi–Yau. Then you can ask, whatever, whoever you ask, can you tell me anything else? They'll say it's connected to another geometry among Ricci flat Calabi–Yau metrics. You can collapse the three-cycle and blow up a 2-sphere (in two different ways) and both spaces look like an $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$-bundle over $\mathbb{CP}^1$. These are connected in turn by a so-called "flop" transition, which collapses one $\mathbb{CP}^1$ and grows another. You can take the Lagrangian and try to pull it through in the other phases and it survives, and the little dots represent in the colored HOMFLY-PT, counting holomorphic curves in this phase. On the other side, you can relate it to sl_N knot homology with N fixed. On the other side of the collapse it's triply graded and gives you the HOMFLY-PT homology.

Every generator represented by this little dot is actually an embedded curve, a pseudoholomorphic curve, with a boundary condition. This Lagrangian is noncompact, he'll look at the contact structure at ∞ which will remain fixed. At the interior you have the knot. This lives in an $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$-bundle over $\mathbb{CP}^1$ and we're counting embedded pseudoholomorphic curves with boundary conditions like this [picture]. They end on the Lagrangian. Any red dot over on this side is an embedded curve. There's a whole theory about what you decorate things with, but after you set up your enumerative problem, it's a curve. For the trefoil, you have only two nontrivial differentials, you get only three interesting curves ending on the trefoil knot here. For this knot there are 15. You observe things at 3-manifold topology, you can ask what it means to have $N\to -N$? That corresponds to the flop transition because N is the volume of $\mathbb{CP}^1$. This is where communication and translation is helpful, because something nonobvious in one description is obvious in another.

Another fact is that, for example, one thing is that knot Floer homology detects genus of the knot. The $q=1$ and -1 is the Seifert genus. If you ask what q corresponds to in this enumerative geometry, it has to do with genus of the curve Σ we're counting. Therefore, there will be maximum genus one curves ending on this Lagrangian. So genus for 3-manifold invariants is the same as genus for this enumerative problem. There are so many parallels. My time is up so I'll stop here.

10. Rinat Kashaev: Quantum Teichmüller theory and TQFT III

Thank you, so let us come back to the very first lecture and recall our main goal, right? So we want to give a precise mathematical definition of the Chern–Simons partition function $Z_{G,\hbar}(M)=\int e^{\frac{i}{\hbar}GS_G(M)}\mathcal{D}A$. We've chosen $G=PSL(2,\mathbb{R})$. But everything I'm doing here does not work for compact groups. Don't try to apply this to compact gauge groups. The method does not work.

Let me recall what I said at the very end of the last lecture, which was that to construct a representation of the groupoid of ideal triangulations, which corresponds to $MCG(S)$ acting on $\tilde{\Delta}(S)$, and the remark was that to construct a representation of that groupoid $\mathcal{G}_{MCG(S),\tilde{\Delta}(S)}$, you just specify (V,T,A) where V is an object of a monoidal category (with symmetry P, A is an automorphism of V, and T is an automorphism of $V\otimes V$, such that

(1) $A^3=\mathrm{id}_V$,
(2) $T_{12}T_{13}T_{23}=T_{23}T_{12}$
(3) $TA_1P_{(12)}T=A_1A_2$

Then using the presentation of the groupoid, we get a representation. So if we solve this algebraic problem we get a representation without using the surface.

Let me give an example from Teichmüller space. What is $\mathcal{C}$? It's sets with Cartesian product and permutations. What is V? It's a set of pairs of positive numbers. What is A? It's from the β map, $A^{-1}(x,y) = (\frac{y}{x}, \frac{1}{x})$, and $T^{-1}(v_1, v_2) = (v_1 \cdot v_2, v_1 * v_2)$ where $v_1 \cdot v_2 = (x_1 x_2, x_1 y_2 + y_1)$ and $v_1 * v_2 = (\frac{x_2 y_1}{x_1 y_2 + y_1}, \frac{y_2}{x_1 y_2 + y_1})$. Effectively the whole Teichmüller theory is encodded in these two operations. What is the next step? Of course, I will not be able to do the whole plan.

10.1. **Quantization.** What is quantization? It has as input a pair, a symplectic space and a Lagrangian in it, and has as its output a vector space $H(X)$ and a vector $\psi_L \in H(X)$. It will be clear how this black box operates. If Lagrangian means, you have a half-dimension subspace to which the symplectic form restricts trivially, if the dimension of X is $2n$, then L is determined by $f_i = 0$ for $i = 1, \ldots, n$, where $f_i \in \mathbb{R}^X$. Most importantly, the Poisson brackets $\{f_i, f_j\}$ are trivial. Then the quantization consists in deforming $\mathbb{R}^X$ to an algebra $\mathcal{A}_\hbar(X)$, and $\mathcal{A}_\hbar(X)$ module $H(X)$ so that $f \in \mathbb{R}^X$ goes to $\hat{f}$ and ψ_L is determined by a system of equations $\hat{f}_i \psi_L = 0$ for all i in $\{1, \ldots, n\}$. That's a rough idea of what the black box is.

In the case, in particular, if $f : X \to X$ is a symplectomorphism, so invertible, then one can associate a Lagrangian $\Gamma(f)$ in $(X \times X, pr_1^* \omega - pr_2^* \omega)$, the graph is Lagrangian. We can do something algebraically, so f^* maps $\mathbb{R}^X$ to itself. We can also take the inverse of this. This respects composition of symplectomorphisms, then quantization, according to the first remark, goes to an algebra map $\mathcal{A}_H(X) \to \mathcal{A}_H(X)$ and $(f^{-1})^*$ goes to an algebra homomorphism a_f (if you deal with some issues I'm hiding), in fact an algebra isomorphism. You can turn our Lagrangian equations, well, ψ_L is equivalent to $\hat{f}$ in $\mathcal{A}_\hbar(X)$ such that $a_f(\hat{g}) = \hat{f}\hat{g}\hat{f}^{-1}$.

This is all possible maps g but of course they should be smooth or whatever but that's the general idea. It will happen that $\hat{f}$ is a unitary operator if $\tilde{H}$, if there is Hilbert space $\tilde{H}(X) \subset H(X)$.

Let me discuss an example. Let us discuss an operator A, the A map was $A^{-1}(x,y) = (\frac{y}{x}, \frac{1}{x})$. Let $X = (\mathbb{R}^2_{>0}, \frac{dy \wedge dx}{yx})$

Then $\mathcal{A}_\hbar(X)$ is generated by $\hat{x}$ and $\hat{y}$ satisfying $\hat{x}\hat{y} = q\hat{y}\hat{x}$ with $q = e^{i\hbar}$. In practice this is case by case dependent.

Let me say how we construct, $\varphi = A$ and then we see that A_φ acts on $\hat{x}$, so a_φ acting on $\hat{x}$ should give this, $\hat{y}\hat{x}^{-1}q^{\frac{1}{2}}$

(these are positive operators, so when you conjugate them you see that $\hat{y}\hat{x}^{-1}q^{-\frac{1}{2}}$ is self-conjugate). This is part of fine-tuning the quantization.

With this you can check, A_φ acting on $\hat{y}$ is $\hat{x}^{-1}$. Then you just look for $\hat{\varphi}\hat{x} = \hat{y}\hat{x}^{-1}q^{\frac{1}{2}}\hat{\varphi}$ and the second equation is $\hat{\varphi} = \hat{x}^{-1}\hat{\varphi}$. Then $\hat{\varphi} = e^{-\frac{\pi i}{3}} e^{3\pi i \hat{q}^2} e^{\pi i (\hat{p}+\hat{q})^2}$ with $\hat{x} = e^{2\pi b \hat{q}}$ and $\hat{y} = e^{2\pi b \hat{p}}$, you have $\hat{q}\hat{p} - \hat{p}\hat{q} = \frac{1}{2\pi\sqrt{-1}}$ and $\hbar = 2\pi b^2$.

Let me state the result, that $H(X)$ is the space of tempered distributions on $\mathbb{R}$, the dual space to Schwartz functions $S'(\mathbb{R})$ with $\hat{Q}f(x) = xf(x)$ and $\hat{p}f(x) = \frac{1}{2\pi i} f'(x)$. In particular, $S'(X)$ contains inside it the square integrable functions on the real line $L^2(\mathbb{R})$, and the restriction of $\hat{\varphi}$ to $L^2(\mathbb{R})$ is unitary. So this is $\tilde{H}(X) \subset H(X)$. This is a realization of the claims I made on the general level.

This is a simple part of the story. The complicated part is the map T which acts on twice the bigger space, and if the transformation A is linear after taking logarithms. But here we have a genuinely nonlinear system.

Unfortunately I don't have enough time to talk in detail how to quantize this part.

Let me say a few words about how we deal with the quantized algebra. The symplectic form is given by $\omega = \frac{dy_1 \wedge dx_1}{y_1 x_1} + \frac{dy_2 \wedge x_2}{y_2 x_2}$. Then a_φ acts, we have four cases, before doing that, sorry, let me say first $\mathcal{A}_\hbar(x)$ is $\mathcal{A}_\hbar(\mathbb{R}^2_{>0})^{\otimes 2}$. Now we can think of $\hat{x}_1 = \hat{x} \otimes 1$, for $\hat{y}_1 = \hat{y} \otimes 1$, for $\hat{x}_2 = 1 \otimes \hat{x}$, and for $\hat{y}_2 = 1 \otimes \hat{y}$.

So now we take $\varphi = T$, and we have to say what is a_φ, and we have to say what is $\hat{x} \otimes 1$, and let me write it here, we have to see this is $\hat{x} \otimes \hat{x}$, so what is the image of $(\hat{y}, 1)$, so we have $\hat{x} \otimes \hat{y} + \hat{y} \otimes 1$. Let me write them because we'll need them, and $a_\varphi(1 \otimes \hat{x}) = \hat{y} \otimes \hat{x} \cdot (\hat{x} \otimes \hat{y} + \hat{y} \otimes 1)^{-1}$ and $a_\varphi(1 \otimes \hat{y}) = 1 \otimes \hat{y} \cdot (\hat{x} \otimes \hat{y} + \hat{y} \otimes 1)^{-1}$. There's no normalization to the term for $1 \otimes \hat{x}$ and $1 \otimes \hat{y}$ because you have that the two terms commute with one another. So we can copy the formula from the classical case.

Then we look for the equations $\hat{\varphi}(\bullet) = a_\varphi(\bullet)\hat{\varphi}$. Then $\bullet$ can be calculated from its actions.

If we look at the equations for $\hat{x} \otimes 1$ and $\hat{y} \otimes 1$, if we redenote $a_\varphi(\hat{x} \otimes 1) = \Delta(\hat{x})$ and $a_\varphi(\hat{y} \otimes 1) = \Delta(\hat{y})$. We have an algebra homomorphism $\Delta : \mathcal{A}_\hbar(\mathbb{R}^2_{>0}) \to (\mathbb{A}_\hbar(\mathbb{R}_{>0})^2) \otimes 2$ and this is a Borel subalgebra of $U_q(SP_2)$. This is conceptually important, we started from a geometrical setting, and then had some Penner coordinates, and then we ended up with a transformation T that came naturally, and the remark is that a certain part can be understood as a morphism from an algebra to its tensor square. So we have complete information about a quantum group. But this is the Ptolemy relation. So if Ptolemy know what is [unintelligible], he would have invented quantum groups.

So I can't explain in detail how we calculate $\hat{\varphi}$ so let me just comment a little on its structure. The result of the calculation, this has no guesswork, you solve a linear system. This should be $\hat{T}$ in this example. In the first example it was $\hat{A}$. Now this is

$$\hat{T} = e^{2\pi i \hat{p}_1 \hat{q}_2} \psi(\hat{q}_1 - \hat{q}_2 + \hat{p}_2)$$

where

$$\psi(x) = \frac{1}{\Phi_b(x)}$$

and

$$\Phi_b(x) = exp\left(\frac{1}{4}\int_{\mathbb{R}+i\epsilon} \frac{e^{-2ixz}}{sinh(bz)sinh(b^{-1}z)}\frac{dz}{z}\right)$$

and then $\hat{T}$ restricted to $L^2(\mathbb{R}^2)$ is unitary which means that, it says something about ψ defined here.

10.2. **Quantum dilogarithm.** So $\Phi_b(x)$ is meromorphic on $\mathbb{C}$ with poles at $ib(\frac{1}{2} + \mathbb{Z}_{\geq 0}) + ib^{-1}(\frac{1}{2} + \mathbb{Z}_{\geq 2}) \subset \mathbb{C}$. The zeros are negatives of the poles. We have the property that

$$\overline{\Phi_b(z)} = \frac{1}{\Phi_b(\bar{z})}$$

for positive b which implies by a spectral argument that Φ_b is unitary. We also have an inversion relation

$$\Phi_b(z)\Phi_b(-z) = \Phi_b(0)^2 e^{\pi i z^2}$$

and most importantly the pentagon relation

$$\Phi_b(\hat{p})\Phi_b(\hat{q}) = \Phi_b(\hat{q})\Phi_b(\hat{p} + \hat{q})\Phi_b(\hat{p}).$$

Now

$$\frac{\Phi_b(x - ib/2)}{\Phi_b(x + ib/2)} = 1 + e^{2\pi bx}$$

and also the same for replacing b by b^{-1}. This ends the subsection, and I think we finally reach the TQFT.

10.3. **Teichmüller TQFT.** The main idea is to replace $\hat{T}$ by $\hat{T}(a,c)$ where a and c are real numbers with $a + c < \frac{1}{2}$. Then $\hat{T}(a,c)$ is structurally the same formula,

$$\hat{T}(a,c) = e^{2\pi i \hat{p}_1 \hat{q}_2} \psi_{a,c}(\hat{q}_1 - \hat{q}_2 + \hat{p}_2)$$

where

$$\psi_{a,c}(x) = \psi(x - 2c_b(a+c)e^{-4\pi i c_b a x}$$

with $c_b = \frac{i}{2}(b + b^{-1})$ and the integral kernel of $\hat{T}(a,c)$ is (switching to physical notation),

$$\langle x_1, x_1 | \hat{T}(a,c) | x_2, x_3 \rangle$$

and what is the meaning? It's that

$$(\hat{T}(a,c)f)(x_0, x_1) = \int \langle x_1, x_1 | \hat{T}(a,c) | x_2, x_3 \rangle f(x_2, x_3) dx_2 dx_3$$

This is a distributional function in four variables, it's in $S'(\mathbb{R}^4)$, and explicitly it has

$$\langle x_1, x_1 | \hat{T}(a,c) | x_2, x_3 \rangle = \int D(x_0, x_1, x_2, x_3, x_4) \tilde{\psi}'_{a,c}(x_4) dx_4$$

where

$$\tilde{\psi}_{a,c}(x) = \int e^{-2\pi i x y} \psi_{a,c}(y) dy$$

the Fourier transform (and then the integral is easy because there are delta functions and)

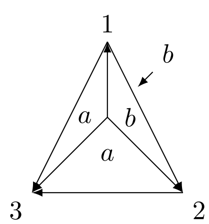
$$\tilde{\psi}'_{a,c}(x) = p\tilde{s}i_{a,c}(x) e^{-\pi i x^2}$$

so we get

$$D(x_0, \ldots, x_4) = e^{2\pi i x_0 x_4} \delta(x_0 - x_1 + x_2) \delta(x_2 - x_3 + x_4).$$

Okay, now X is a *shaped* (each tetrahedron carries dihedral angles of an ideal hyperbolic tetrahedron) *triangulation* (a Δ trinangulation in the sense of Hatcher, a CW-complex with all cells standard simplices, all vertices ordered and gluings respecting that order and orientation) of an oriented pseudo-3-manifold.

We'll associate to this a partitition function. Let me give an example [picture]



with face identifications in pairs.

This has one vertex, two edges, and one tetrahedron.

There are two types of states of X, a "face state" for $x \in \mathbb{R}^{X_2}$ and "tetrahedral states" for $t \in \mathbb{R}^{X_3}$, and two types of tetrahedra [pictures] depending on the cyclic order.

Then

$$K(X,t) = \int_{x\in\mathbb{R}^{X_2}} \frac{K(K,x,t)}{\prod_{T\in X_3} D(T,x,t)} dx$$

where

$$D(T,x,t) = \int D(x_0,\dots x_3,t)$$

if $\epsilon(T) = 1$ and the complex conjugate if $\epsilon(T) = -1$ and x_i is the value on the $\partial_i(T)$, and that finishes the definition of the *kinematical kernel* $K(X,t)$.

Now the partition function is

$$Z_\hbar(X) = \int_{\mathbb{R}^{X_3}} K(X,t)\Psi(x,t)dt$$

where

$$\Psi(T,t) = \psi_{a,b}(t(T))$$

if $\epsilon(T) = 1$ and the complex conjugate if $\epsilon(T) = -1$ this $\Psi(x,t)$ is the *dynamical kernel*. Let me remark that the kinematical kernel $K(T,t)$ is an element of $S^1(\mathbb{R}^{X_3})$, and the dynamical kernel $\Psi(T,t)$ is an element of $S(\mathbb{R}^{X_3})$, the space of test functions. Then $Z_\hbar(X) = \langle K(X), \Psi(X)\rangle$, just the evaluation because one is in the dual space of the other.

Let me formulate the theorem and you'll see why I'm cheating.

Theorem 10.1. *Let X be such that $H_2(X\backslash X_0,\mathbb{R})$ is trivial. Then*

(1) *$Z_\hbar(X)$ is a finite complex number, and*
(2) *$|Z_\hbar(X)|$ is invariant under the $2-3$ or $3-2$ shaped partner moves and gauge transformations induced by total dihedral angles around edges of X induced by Neumann–Zagier's Poisson structure.*

(shaped means you have to incorporate the presence of angles; if you remove an edge, the edge should be balanced so the total dihedral angle around it is 2π)

In words, you can write the most evident Poisson bracket, which is the NZ bracket, and if you sum up the dihedral angles from tetrahedra, [unintelligible]Poisson commutes with [unintelligible], then you induce by Poisson commuting, [unintelligible]. This doesn't depend on the gauge transformation. The first part says it's topologically invariant as soon as you balance an edge. In our example, there are two geometric edges e_0 and e_1 and the total dihedral angle which sends $\omega : B_2 \to \mathbb{R}_{>0}$, we have $\omega(e_0) = \gamma$ and $\omega(e_1) = 2\pi - \gamma$, so there's one real number, and γ commutes with itself, this is a gauge invariant quantity, and this will depond only on γ and not on α and β which can be varied. My time is over so I'll stop here.

11. Tobias Eckholm: Chern-Simons theory, open topological strings, and augmentations

[started with a bunch of stuff already on the board, couldn't keep up, calculation of the trefoil] The main subject today is augmentations. Let's think about the variables $e^{\pm x}$, $e^{\pm p}$ and $Q^{\pm 1}$ as complex numbers. Then ϵ is a map $\mathcal{A} \to \mathbb{C}$, thought of as a chain complex in degree 0. For our purposes it's not so bad, it's a graded map, it can take nonzero values on the a_{ij} only, and the chain map equation says that $\epsilon\partial - \partial\epsilon = 0$. But $\partial\epsilon = 0$ automatically. Then I want something so that $\epsilon\partial = 0$. We see we have the image of the differential, which gives me a polynomial in the a_{ij}. So now, I'm looking at these augmentations, I can pretend that the a_{ij} commute, it doesn't matter because $\mathbb{C}$ is a commutative algebra. I have a number of polynomial equations, and I'd like them to have a common root. I can ask what is the locus of the coefficients where I find such a common root. This is the *augmentation variety.* If you're awake, you see that I've committed a crime of decategorification.

You'd like to take the Zariski closure of this thing.

In effect, it's stated some place, this is in fact an algebraic variety which typically has codimension 1. I think I can prove the conjecture that this is the codimension. For any example you can compute this, it's some variety, and you can talk about the augmentation polynomial, which generates the corresponding ideal.

That is, $\mathrm{Aug}_K(e^x, e^p, Q) = 0$ is this variety.

Now we have two examples, the trefoil and the unknot.

The unknot is simple. The algebra of the unknot is generated by c and e with e in degree 2 and $\partial c = 1 - e^x - e^p + Qe^x e^p$. So you want to take 1 to 1. The augmentation polynomial is $1 - e^x - e^p + Qe^x e^p$. Why decategorify? Over this variety there lives naturally a kind of sheaf. If you know that, if you have an augmentation, then you can change variables in the algebra. If ϵ is an augmentation, change variables $a_{ij} \to a_{ij} - \epsilon(a_{ij})$ you kill the constant term. That's the point of an augmentation. Then this is zero. So then the constant term goes, and I'm left with some differential that starts out with a linear, then quadratic, then cubic, so this is like the first page of a spectral sequence, this linearized Legendrian contact homology. In the case of the unknot it's a simple thing. If you assume your differential satisfies this equation, then the linearized homology has two generators, one in degree 2 and one in degree 1.

How do we do this for the trefoil? This is not such a hard exercise, for the trefoil it's pretty simple. The first equation lets us express a_{12} and a_{21}, and then you get a polynomial expression. What is the result? I won't do it but I'll write it down.

$$\mathrm{Aug}_{3_1} = -e^{2x}e^{3p} + e^x e^{4p} + e^{2x}e^{4p} - 2e^x e^{2p}Q - e^x e^{3p}Q + e^x Q^2 - e^x e^p Q^2 + 2e^x e^{2p}Q^2 - e^p Q^3 + Q^4$$

and this is a mildly complicated expression, if I got it right it's the sum of ten monomials but not so hard to calculate from the formulas.

[discussion of computability]

I'll start with the special case $Q = 1$. You'll see at $Q = 1$,

$$\mathrm{Aug}_u = (1 - e^x)(1 - e^p)$$

and these sit in the augmentation polynomial of any knot. Then in T^*S^3, this is filled by L_K, this is topologically $S^1 \times \mathbb{R}^2$. The torus, the meridian is killed, my variables, I fill in the S^2, that kills Q, and $p = 0$, I'm claiming that defining a map $\epsilon \mathcal{A}_{Q=1} \to \mathbb{C}[e^x]$ can be done by counting holomorphic disks, much like I counted them in the differential. If I take a Reeb chord c, a general one, then I count

$$\epsilon(c) = \sum [\text{picture}]$$

counting terms with one positive puncture at c and the boundary on L_K. I read off the homology class of the boundary. This is a finite sum, a fixed sum because of finite length. I'm counting 0-dimensional moduli spaces. I claim that ϵ is a chain map, $\epsilon \circ \partial = 0$. Why is that? Look at the boundary of 1-dimensional moduli spaces. The key observation is that this Lagrangian is what they call exact. The Lagrangian condition on L_K, we had $dp \wedge dq$ and this comes from pdq. We know that pdq is closed so we can ask if it's exact and in this case it is exact.

So the proof of this claim, it means that if you're trying to form a holomorphic curve on L_K, you can try to compute its area, takyng $\int_D dp \wedge dq$, and that's $\int_{\partial D} pdq$

and then this is 0. So there's no holomorphic disks on this Lagrangian. Then the only thing that can happen is the splitting where you split off an $\mathbb{R}$-invariant disk near ∞ and some number of rigid disks below. Then [unintelligible]I get zero.

So I can put $p = 0$, keeping x, so I get an augmentation. So $(1 - e^p)$ is in the augmentation variety for every knot.

We use the other exact filling for $(1 - e^x)$, M_K, which is the complement of the knot. You can take the conormal and then join them by some Lagrange surgery, and then the topology of what you get is the topology of the knot complement, or you can take a function and [unintelligible]let the [unintelligible]go to ∞. In any case there is such a filling and if you just look at the homology, this puts the longitude to zero, so this factor is there for every knot. The augmentation variety for the unknot is the canonical one, and then these are in all knots' augmentation varieties, and this is the only knot with this particular polynomial.

Lenny first proved that the A-polynomial divides this polynomial. You can show that the degree zero contact homology at $Q = 1$ is isomorphic to a certain subring in the group ring of the fundamental group, and looking at, augmentations are maps of that. Then you can say they arise from certain representations of the fundamental group of the complement, it detects the unknot and torus knots and other things.

The main subject of the talk today is a geometric interpretation of the rest of the augmentation variety. The background is that the augmentation polynomial was found in physics before we understood why as a certain limit of the colored HOMFLY-PT polynomial. So somehow they calculated some polynomial that they initially suspected was the A-polynomial, and it turned out that it was a Q-deformation of the augmentation polynomial. You suspect that they're the same after you see it for the trefoil.

What I want to do is try to tell you the story that came from physics rather briefly, and it will in a while relate to what Sergei talked about. I want to go through it without details mainly because many details are not there in math.

The starting point of the physics story is again the Chern–Simons partition function. Remember this was

$$Z_{CS} = \int \mathcal{D}A e^{\frac{ik}{4\pi} CS(A)}$$

and we use here $U(N)$, recall

$$CS(A) = \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

and this A is a $U(N)$-connection.

We sum this up Z_{CS} depends on N and k. We integrate over gauge as Rinat explained.

This is the Chern–Simons partition function and one can now insert, when you want to go to knots, you insert, you take the trace of holonomy along K and insert this into the integral for the partition function, and then you normalize the path integral, you should get the HOMFLY-PT polynomial. But Witten's observation was that you can take this seriously, do this in perturbation theory and get some Feynman diagrams that are sums over fatgraphs. Here you decorate everything by gauge indices, which means you can fatten the surfaces and then you somehow have, if you haven't seen this before it's hard to follow, but something like this can give you $Ai^k \wedge A^k_j \wedge A^i_k$. If you look at these partition functions, what it's doing, you get the same graph as a power or something like that, and that's a small motivation for why you have the following result of Witten. Let's look at open topological strings (open Gromov–Witten theory) of holomorphic curves in T^*S^3 with boundary on S^3. What are we supposed to do? We invent a Gromov–Witten potential, also a function of N and of g_s, a string coupling constant, given a curve it has boundary components, and this potential is

$$Z_{GW}(N, g_s) = \sum C_{\chi,h} g_s^{-\chi} N^h$$

counting holomorphic curves that have boundary on S^3 and the curve has an Euler characteristic that you raise g_s to, and you have some factor N^h as well. You have such a partition function. Witten's result is that these two functions are actually equal in a certain limit, this and Z_{CS}. Witten showed (in physics style) that these things are equal. He says that the partition function of Chern–Simons is equal to the partition function of Gromov–Witten, with $g_s = e^{\frac{2\pi i}{k+N}}$. How do you show this? This hasn't been mathematically proved. We understand (from Yong-geun and company) what one *has* to prove.

But how did Witten show such a thing? He invented string field theory and used this theory to show that all the contributions to the Gromov–Witten come from constant maps and then when you do the perturbation you get Chern–Simons.

If you try to find a holomorphic disk or curve at all on S^3, there are only constants, no curves. Dimension formulas are all rigid. Things should live in a 0-dimensional moduli space. They come in a 3-dimensional family. So when you perturb it should still be rigid and you can still count it.

There is one more step to this story which is to explain how we can see insertions in the path integral in Chern–Simons. In fact, what you need to do is perhaps not so unexpected. You can put one brane on the conormal. We're doing a very similar curve count, here's S^3, and then you have a conormal and [picture].

So again we can run the Witten argument and the things we want are concentrated in the knot and we can integrate this out and if we do this count, integrating out à la Witten corresponds to inserting $\det(1 - e^{-x}\operatorname{Hol}_K)^{-1}$, and when you expand that you get these symmetric things

$$\sum tr_{S_k} \operatorname{Hol}_K .$$

And this is colored HOMFLY-PT, right? When you take,

$$\psi_K(X) = \frac{Z_{GW}(L_K)}{Z_{GW}} = \sum H_K(x) e^{-kx}.$$

So the prediction is we get something well-known counting holomorphic curves, but they're all constant. But [unintelligible]–Vafa offered a solution, and this is a physics argument, one we're further from in math. Let's take a short break.

Let me tell you about the conifold. Witten is counting curves in T^*S^3. If you look at this space at ∞, then it looks like, this is the unit cotangent bundle here, and then I pinch the S^2, drawing it like [picture]. As Sergei explained, you can crush the S^3 and go to a cone or resolve the other way so that the S^2 lives and the S^3 is crushed. If you do it in complex analysis, this is $\mathcal{O}(-1)^{\oplus 2}$ over $\mathbb{CP}^1$. So these people suggest that for, the area of $\mathbb{CP}^1 = t = Ng_s$, if we keep the ratio fixed, then counting open curves in one should correspond to counting closed curves in the other. In fact, they prove this so somehow this, maybe I don't have to write it down, there are various proofs, but [unintelligible]and on the level of path integrals you can see it's a reasonable statement.

For S^3 it was verified by doing Chern–Simons. In principle there's only one holomorphic curve, but then you have to count the covers. It agrees there, and it's a great thing, and for other manifolds there may be other things that appear.

Now let's think a little bit about what happens when we include a Lagrangian. I want to first make sure that the Lagrangian stays through the conifold transition. There's a nice way to do this for every knot. You cannot do it with an exact Lagrangian, but you can do it with a different one. The intersection is the knot itself. You can in a neighborhood get this thing that looks like the differential of a function. You shift which takes you off the zero section but it's not actually exact. Then you do the conifold transition, and the conifold transition is localized near the zero section. Now let's look at what Vafa and Iguri told us, that boundary components on S^3 shrink to nothing, so if you have another boundary component [picture] then one should stay and the other should shrink to nothing. So you get [pictures].

Then the idea is that the normalized Gromov–Witten count

$$\frac{Z_{GW}(L_K)}{Z_{GW}(X)} = \Psi_K(X)$$

Now after some shifting and some things, we actually have the wave function.

So now, I'll skip the motivation, and

$$\Psi_K(K) = exp(\frac{1}{g_s}\int pdx + \cdots)$$

if you want to integrate out short strings, those that connect L_K to itself, that's $GL(1)$ Chern–Simons on the solid torus. If you write this down, it's high school (I don't know when you learned this) mechanics on the line. When you quantize it, you get operators [unintelligible]. If you do the usual stuff you get this kind of expansion. This is from the Chern–Simons perspective.

From the Gromov–Witten perspective, $\Psi_k(X)$ is $exp(\frac{1}{g_s}W_K(X) + \cdots)$.

From this formula we see that $p = \frac{\partial W_K}{\partial x}$, and if you think about this recursion relation, you see that this gives an algebraic curve $A(e^x, e^p, Q) = 0$. So if you know now something about the colored HOMFLY-PT. It's not easy, but in these examples it gave exactly the augmentation polynomial. I don't have time but let my try to start this explanation.

The only thing, I'll start and then we'll finish next time, tomorrow. The idea here is to try to use L_K in X to define an augmentation as before.

Remember what we tried to do. Basically we want to define $\epsilon(c)$ as $\sum$ [picture], we're trying to carry out our proof. We look at 1-dimensional moduli spaces, try to look at the boundary, and if we find as before that we just have this several level splitting, we just have a chain map. We have also a new phenomenon, that we can split off holomorphic curves with boundary on [unintelligible]. The local model is [picture]. So we don't have a chain map and in some sense that's bad, but similar problems were actually solved by Yong-geun, Fukaya and company in ordinary Floer theory. There you can sometimes overcome by finding so-called bounding cochains. I'll tell this fast now and tomorrow again. We take all these disks, the boundary is on L_K. This wraps some number of times around the generator of the homology of L_K. Now at ∞ in Λ_K we wrap around some standard curve ξ. [pictures].

If we fix such a surface for the splitting disk, then we can stop this badness, when we start to bubble off, we initially don't know where to go, but now the bounding cochain goes off to ∞ and we can continue the moduli space by including objects which intersect the bounding cochains that we fixed. Now the thing is just continuous and we know what to do. There are now only such splittings. Now we count disks with insertions of the bounding cochains. These can fly off to ∞. We should decide what happens counting these as they go off to ∞.

Now we have the differential curve at ∞ and we have to count how many insertions we have. Formally there is, we attach to it some holomorphic curves down here. This is very easy because each time, for each p that I wrap around here, I intersect the red curve as many times as I wrap around. If I set e^p, if I set $p = \frac{\partial W}{\partial x}$, remember W counts disks (C) and multiplies by e^{kx}, so $\frac{\partial W}{\partial x} = \sum Cke^{kx}$, so if I substitute p by $\frac{\partial W}{\partial x}$ then this is a chain map. But the chain map is just [unintelligible]an augmentation. So this [unintelligible]augmentation variety.

I'll stop now, sorry for going over time, I'll continue tomorrow.

12. Dec 18

I did not attend the final two talks.