CGP WORKSHOP ON STRING FIELD THEORY OF THE B-MODEL

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1. January 6: Daniel Murfet: Fusion of defects in Landau–Ginzburg models I

Thank you very much for the invitation to speak here.

I'll be talking about fusion of defects in Landau–Ginzburg models. The notes for all three of my lectures are online, you can download them now at `www.therisingsea.org`. I'll also be demonstrating some software that can also be downloaded there.

I'll start with broadly explaining how fusion of defects in LG models might relate to other things you know about. The firs t thing I'll mention is knot homology. So many of you may have heard of Khovanov homology. This is the SL_2 version, and Khovanov–Rozansky can define this in terms of fusion of defects in LG models. I'm not going to say too much about knot homology in these lectures although I'll try to touch on it if I have time. General orbifolding I'll talk about in probably the next lecture. Carqueville–Runkel have talked about this. This, I'm interested in isolated hypersurface singularities, this lets you relate ADE singularities in a surprising fashion and makes use of the bicategory that comes out of the fusion construction. There's also topological Fukaya categories, and there are multiple groups working on this, the one that is closest related to fusion in LG models is Dyckerhoff–Kapranov. I'll say something about this in the second lecture. Finally, this is related to fusion in conformal field theories, I'll mention just two groups of names, Brunner–Roggenkamp and Davydov–Ros Camacho–Runkel. Maybe one main point is that the fusion of defects in Landau–Ginzburg models can literally be computed, I'll show you on the computer a little later.

Today:

(1) I'll start with $2D$-defect TFTs
(2) give you the basics of matrix factorizations, and
(3) and talk about the tensor product of matrix factorizations, corresponding to fusion.

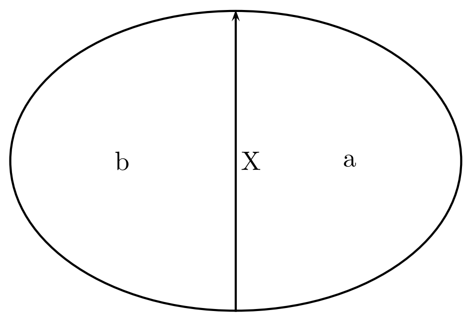
The second lecture will be about organizing this into the structure of a bicategory, and the third lecture will be hard stuff, proofs, I'll only prove very easy things in the first two. I'll use homological perturbation and Atiyah classes, I'll show you some about how this is done.

1.1. $2D$-defect TFTs.

So for the definition of $2D$-defect TFTs, let me cite three papers

- Runkel–Suszek 0808.1419
- Davydov–Kong–Runkel 1107.0495
- N. Carquiville–Runkel 1210.6363

So the data of one of these is sets D_2 (*phases*) and D_1 (*defect conditions*) and two maps $s, t : D_1 \to D_2$. I'll draw pictures like this:



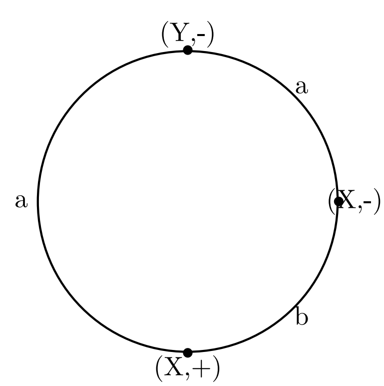
Then this gives a bordism category

Definition 1.1. The bordism category Bord has as objects oriented compact one-manifolds with points, with points decorated by elements of D_1 and $+$ or $-$ and segments labeled by D_2, so that with $+$, moving forward from a marked point marked by X the label is $s(X)$ and moving backwards $t(X)$; and the reverse for markings with $-$.

The set of circles should be ordered, and we can take as morphisms either permutations of circles or equivalence clasess of surfaces, where I draw my source on the left and the target on the right, and I draw an oriented 1-manifold with boundary, with similar labels. The 1-manifold gets labels from D_1 and the 2-dimensional pieces get labels from D_1. The boundaries have te match the boundary of the entire manifold.

The labels should be compatible. The compatibility condition also applies on the surface, looking at the labled lines should give me the right things on the boundary.

Here is an example object, where $s(X) = a$ and $t(X) = b$ and $s(Y) = t(Y) = a$:



Rather than spending any longer on that, you can look at the third paper, these things should be collared for example, or composition doesn't work.

This is just to justify the introduction of bicategories, I won't be using this formalism so much.

Definition 1.2. An (oriented) $2D$*-defect TFT* is a symmetric monoidal functor $Z : Bord \to Vect_{\mathbb{C}}$.

So Z of a circle, for X in D_1, I think of this as a map, sphere with a circle on it, with a and b on the components and X on the circle, this is a map from the empty set to itself, so an element of $\mathbb{C}$.

Let me give an, you can get a 2-category out of a $2D$ defect TFT, it's harder to go in the other direction, the set of constraints necessary are still not known. Conjectureally, examples are D_2 being smooth Calabi–Yau varieties over $\mathbb{C}$, and the elements of D_1 are in $D^b(\mathrm{Coh}\, A \times B)$. The claim is that starting with D_2, the bounded complexes of coherent sheaves, you can define a $2D$-defect TFT and what would you need to define? Vector spaces and linear maps, and you use Grothiendieck duality and things like that.

The example I will be using, D_2 are isolated hypersurface singularities $W(x) \in \mathbb{C}[x]$ (isolated critical points) and D_1 are matrix factorizations, a region between W and V is a matrix factorization X of $V(y) - W(x)$. Given this data, a conjecture, you can get a $2D$ defect TFT, you can see that the data is close, and you need to see how to get Z out of that data. This is a way of orienting yourself with regard to the construction I'm going to do for matrix factorizations.

Let me define matrix factorizations and then fusion.

1.2. Matrix factorizations.

Let $W(x) \in \mathbb{C}[x]$ be a potential, with isolated critical points. This makes sense over any commutative ring. Then the condition, the constraint is a little more involved. I'll stay over $\mathbb{C}$ and then all I need are isolated critical points.

Definition 1.3. A *matrix factorization* X of W is a $\mathbb{Z}_2$-graded free $\mathbb{C}[x]$ module $X = X^0 \oplus X^1$ with an odd $\mathbb{C}[x]$-linear operator $d_X : X \to X$ such that $d_X^2 = W \cdot 1_X$. So choosing a basis this looks like $\begin{pmatrix} 0 & d_X^1 \\ d_X^0 & 0 \end{pmatrix}$.

I'll secretly use idempotent completions, but you could instead do things locally and have germs with some complete ring [unintelligible].

In order to get to the definition of fusion, I'll skip some stuff and come back to it in the next lecture.

Let me give some examples.

Let $W = x^3$. Then $d_X = \begin{bmatrix} 0 & x^2 \\ x & 0 \end{bmatrix}$, this is a rank one matrix factorization because the odd and even pieces have rank 1 as modules.

Then for $W = y^5 - x^3$ you can get this rank 2 factorization, a difference of two polynomials:

$$\begin{pmatrix} 0 & 0 & x^2 & -y \\ 0 & 0 & y^4 & -x \\ -x & y & 0 & 0 \\ -y^4 & x^2 & 0 & 0 \end{pmatrix}$$

If you square this you get $y^5 - x^3$.

1.3. **Fusion.** For fusion, I'll take two potentials, $W(x) \in \mathbb{C}[x]$, which always means $\mathbb{C}[x_1, \ldots, x_n]$, and we take $V(y) \in \mathbb{C}[y]$, and then two matrix factorizations Y of $V(y) - W(x)$, that is, a matrix factorization over $\mathbb{C}[x,y]$ and a matrix factorization X of $W(x)$. Then $Y \otimes_{\mathbb{C}[x]} X, d_Y \otimes 1 + 1 \otimes d_X$, this pair is a matrix factorization of $V(Y)$. A priori it'sa free $\mathbb{C}[x,y]$ module of finite rank (assuming X and Y are finite rank), and it's easy to check that the cross-terms cancel, you get $d_Y^2 + 1 \otimes d_X^2 + (d_Y \otimes 1)(1 \otimes d_X) + (1 \otimes d_X)(d_Y \otimes 1)$ which is $(V - W)1_{Y\otimes X} + W 1_{Y \otimes X} = V 1_{Y \otimes X}$. But this $d_{Y \otimes X}$ is infinite rank. I can think of this as a matrix factorization over $\mathbb{C}[Y]$ if I like, $Y \otimes X$, I can think of x as just a grading on it, if Y is rank r and X is rank s then $Y \otimes X$ is rank rs as a $\mathbb{C}[x,y]$-module, but I can think it's $\mathbb{C}[y]^{\otimes rs} \oplus x\mathbb{C}[y]^{\otimes rs} \oplus x^2 \mathbb{C}[y]^{\otimes rs} + \cdots$ (letting $\mathbb{C}[x]$ be literally one variable for simplicity).

So you could think of the x as a grading, and this just a grading shift, you get this gigantic matrix oer $cC[y]$, some infinite matrix with 1s from all the x's and then matrices in y. This could still be homotopy equivalent to a finite rank matrix factorization. I didnt define the homotopy category, but let me say

Theorem 1.1. *(Brunner–Roggenkamp); (Khovanov–Rozansky); (Dyckerhoff–M.) There exists a finite rank matrix factorization $Y * X$ of $V(y)$ such that*

$$Y * X \cong Y \otimes_{\mathbb{C}[x]} X$$

as matrix factorizations of $V(y)$ over $\mathbb{C}[y]$ in the homotopy category of matrix factorizations.

You can't do Gaussian elimination, but you can find a contractible piece that's infinite and contains all the 1s.

In the ungraded case, well, you can put in a $\mathbb{Z}$-grading if you want, in the graded case this is true on the nose, in the ungraded case you may need to go to the completion, may need to take a power series.

We call $Y * X$ the *fusion* of Y and X, you get something infinite, but isomorphic to something finite. This is the fusion.

The subject of these lectures is the question of how to find $Y * X$, how do we compute this fusion? The first two pairs of names, just tell you that it exists. Actually being able to describe it is a different thing, the subject of these lectures.

Obviously there's theory that goes into answering that question, but if you put in a matrix factorization of $V - W$ and a matrix factorization of W the code will compute the fusion. I wrote this with N. Carqueville to compute knot homologies, do a lot of defect fusions. The code is on github, you'll find this all on my website I gave the link to before. You need to install singular, which is good at Groebner bases. Then you download all the code you need as a zip file.

[examples on the computer]

2. Kwokwai Chan: Scattering diagrams and deformation of complex structures

I'd like to thank the organizers for inviting me to speak here. What I'm going to talk about is joint with Conan Leung and Ziming Nikolas Ma. I'm not sure if all of you have seen scattering diagrams, so let me start with a brief review of scattering diagrams. There are many ways to do this.

Let me look at A, the automorphism group of $\mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]]$ over $\mathbb{C}[[t]]$, and I want to look at certain elements in this group. For any lattice point (a,b), nonzero, in $\mathbb{Z}^2$, let $f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}][[t]]$ be a function of the form

$$1 + tx^a y^b g(x^a y^b, t)$$

where $g(z,t) \in \mathbb{C}[z][[t]]$.

Then we can define a certain element in the group A, which is denoted $\Theta_{(a,b),f} \in A$ by

$$\Theta_{(a,b),f} = \begin{cases} x \mapsto x \cdot f^{-b} \\ y \mapsto y \cdot f^{a} \end{cases}$$

Definition 2.1. (Gross–Pandharipande–Siebert) The *tropical vertex group* $H \subset A$ is (the completion with respect to t of) the subgroup generated by $\Theta_{(a,b),f}$ for all (a,b) in $\mathbb{Z}^2 \backslash \{0\}$ and f of the above type.

Remark 2.1.

$$\Theta^* \omega_0 = \omega_i$$

where $\omega_0 = \frac{dx}{x} \wedge \frac{dy}{y}$ on $(\mathbb{C}^\times)^2$.

What we want is to find factorization formulas for commutators of generators of H. So let me give you an example. There are very special elements S_{ℓ_1} in the sets of generators, usually the first ones to consider, $\Theta_{(1,0)(1+tx)^{\ell_1}}$ and $T_{\ell_2} = \Theta_{(0,1)(1+ty)^{\ell_2}}$, and you want to compute

$$T_{\ell_2}^{-1} \circ S_{\ell_1} \circ T_{\ell_2} \circ S_{\ell_1}^{-1}$$

this is like a circle going around the origin.

You want to express this as a product of Θs.

Lemma 2.1. *(Kontsevich–Soibelman) This works for any two generators but I'll state it for this special case:*

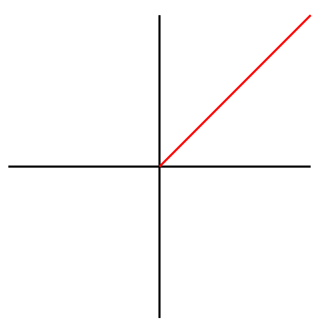
$$T_{\ell_2}^{-1} \circ S_{\ell_1} \circ T_{\ell_2} \circ S_{\ell_1}^{-1} = \vec{\prod} \Theta_{(a,b),f_{a,b}}$$

where the (a,b) are all chosen in the first quadrant, this is the product over primitive vectors (a,b) in the first quadrant.

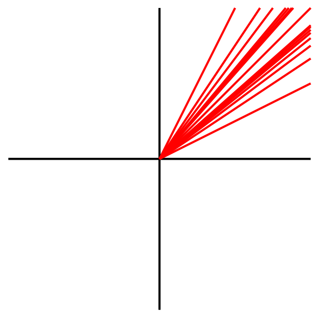
So you draw some rays, and the rays you pass through between them, that's enough, that's what we call a scattering diagram.

Let me give you some concrete example, for $\ell_1 = \ell_2 = 1$. Then

$$T_1^{-1} \circ S_1 \circ T_1 \circ S_1^{-1} = \Theta_{(1,1),1+t^2xy}$$



For another example, $\ell_1 = \ell_2 = 2$, you need to add infinitely many rays already, passing through $(k, k+1)$ and $(k+1, k)$ as well.



For $\ell_1 = \ell_2 = 3$, you get a region where all rational slopes appear, a dense set of rays.

For Gross–Pandharipande–Siebert, these scattering diagrams are related to, for instance,

- Euler characteristics of quiver moduli
- Gromov–Witten invariants of toric surfaces
- cluster algebras,
- wall-crossing formulas for [unintelligible]invariants,

and so on. For me it will be because of the relation to SYZ mirror symmetry.

2.1. **SYZ mirror symmetry.** The SYZ, by the way, stands for Strominger–Yau–Zaslow (1996).

Roughly speaking, we want to understand mirror symmetry. The conjecture says that a pair of mirror Calabi–Yau manifolds should be related geomtrically, admit special Lagrangian fibrations over the same base with dual fibers. The toy example (the only one we will consider in this talk), you look at an affine manifold, a manifold B_0 with integral affine structure, there are charts with transition functions affine linear maps. Then you can look at two manifolds associated to this B_0. So you can look at $X_0 = T^*B_0/\Lambda^*$, the quotient by a lattice (a local system of lattices). There is a lattice structure from the affine structure. Everyone knows that the cotangent bundle has a canonical symplectic structure.

On the other hand, you have a complex manifold $\vee X_0 = TB_0/\Lambda$. If you are taking the tangent bundle of a real manifold you usually don't get a complex manifold but here you do because of the affine change of coordinates.

It makes sense to say that these are a mirror pair.

Now, let's see. Let me briefly talk about a proposal of Fukaya. This is a nice example of mirror symmetry, but it's not interesting at all because it's not what you get in the general case of compact Calabi–Yau manifolds. This is too restrictive. If the base is an affine manifold without any singular fibers, then X_0 and $\vee X_0$ are just tori. To look at more interesting cases of compact (or non-compact) Calabi–Yau manifolds, you need to allow singular fibers.

In general we consider Lagrangian torus fibrations with singular fibers. The general picture, you look at a Calabi–Yau, and if you're lucky enough you find a Lagrangian torus fibration, and you look at B_0 inside B, the locus of smooth fibers, and over the smooth locus you look at T^*B_0/Λ^*, and put that inside X, and define $\vee X_0 = TB_0/\Lambda$, the *semi-flat mirror*. You need to fill in the singular fibers. The problem is that the complex structure J_0 on $\vee X_0$ cannot be extended to any (partial) compactification of $\vee X_0$ because the affine structure around the discriminant locus cannot be extended.

The most important idea of SYZ is to deform J_0 by so-called *instanton corrections* from holomorphic disks bounded by fibers of μ.

Now here comes Fukaya's idea. We say that the correct structure should be related to holomorphic disks. How do we see that? You look at a Maurer–Cartan equation, you search for a solution, and then you Fourier expand the solutions and compare this to holomorphic disk data.

Fukaya's idea is that both things will be related to Witten–Morse theory of multi-valued functions on the base B_0. Fukaya's program is to relate these things.

this is related to work by Floer and later generalized by Fukaya and Oh. You can see that the counting of holomorphic disks is related to the counting of gradient flow trees on the base. On the other side it's Fukaya's conjecture. The paper, I think is in 2005, he proposed to understand this you should go to Witten–Morse theory on the base, but he didn't have any proofs.

How is this related to scattering diagrams? These are exactly tropical images of holomorphic disks. My work with Conan and Ziming is to look at the relationship on the complex side, by taking Fourier series and semi-classical limits.

So take the Kodaira–Spencer differential graded Lie algebra

$$(\Omega^{0,*}(\wedge^* T^{1,0}_{\vee X_0}), \bar\partial, [,])$$

So one question is, what is the corresponding Lie algebra on the A-side? Motivated by Fukaya's proposal, we have $(L_{X_0}, d_W, \{,\})$, where $L_{X_0} = \Omega^*(M, TB_0^{\mathbb{C}})$. You take $M = \sqcup_{x\in B_0}\pi_1(\mu^{-1}(x))$ where μ is the projection of X_0 to B_0.

Then d_W is $e^{-\frac{f}{\hbar}} d e^{\frac{f}{\hbar}}$, where $f : \tilde M \to \mathbb{R}$ is, you look at x in the base a relative homotopy class φ and map it to $\int_{D^2}\varphi^*\omega$, where $\tilde M = \sqcup \pi_2(B, , \mu^{-1}(x))$ over M. Let me skip the bracket, which is the natural bracket on $TB_0^{\mathbb{C}}$ twisted by $e^{\frac{f}{\hbar}}$

The construction for the moment is ad hoc, you just write this down so that the Fourier series interchanges them.

Proposition 2.1. *The Fourier transform F gives an isomorphism of DGLAs $(L_{X_0}, d_W, \{,\}) \cong \Omega^{0,*}(\wedge^* T^{1,0}_{\vee X_0}), \bar\partial, [,])$.*

2.2. **Solving the Maurer–Cartan equation.** We directly solve this equation on the A-side,

$$d_W\Phi + \frac{1}{2}\{\Phi, \Phi\} = 0.$$

How do you relate this to scattering diagrams? For simplicity, let me describe this just for one wall. Then you don't get scattering, but remember that the vector (a, b) you looked at, and then $\Theta_{(a,b),f_{a,b}}$. You can cook up an ansatz giving a solution to the Maurer–Cartan equation, $\Phi \in \Omega^1(M, TB_0^{\mathbb{C}})[[t]]$.

This solution only depends on $f_{a,b}$, whenever you write down f you can write down a solution, and this solution is gauge equivalent to zero, because we're looking at the local case, I'm assuming that $B_0 = \mathbb{R}^2$, no singular fibers. So $\vee X_0$ is $(\mathbb{C}^\times)^2$ and has no nontrivial complex structure. Then this solution is gauge equivalent to zero. So then there exists $\varphi \in \Omega^0(M, TB_0^{\mathbb{C}})[[t]]$ such that $e^\varphi * 0 = \Phi$.

What I want to point out is when you do the semiclassical limit of this gauge,

Proposition 2.2. *(C.–Leung–Ma) The semiclassical limit of φ gives you back $f_{a,b}$.*

Now it comes to the interesting part, about the two-wall case. Then the thing is, you look at two walls, you have an $\vec m_1 = (a_1, b_1)$ and $\vec m_2 = (a_2, b_2)$

Now the problem is, when you look at this $\Phi = \Phi_1 + \Phi_2$, where Φ_i is given by the ansatz, this doesn't solve the equation. You have to modify it, in fact a solution $\tilde{\Phi}$ is given by

$$\tilde{\Phi} = \sum_{k\geq 1} \ell_k(\Phi_, \ldots, \Phi)$$

the sum over trivalent trees, with brackets at the vertices and on internal edges you put a so-called propagator or homotopy, this is the inverse of $\bar{d}$, you have to use the Green's operator. You look at the sum of all these and get a solution.

I am already over time, but let me state the main theorem.

Theorem 2.1. *(C.–Leung–Ma)*

(1) *The nontrivial solution can be decomposed*

$$\tilde{\Phi} = \sum_{(a,b)\in\mathbb{Z}^2_{\geq 0}\setminus\{0\}} \Phi^{(a,b)}$$

(2) *Each* $\Phi^{(a,b)}$ *is a Maurer–Cartan solution*

(3) *there is* $\varphi_{(a,b)}$ *with* $e^{\varphi_{a,b}} * 0 = \Phi^{(a,b)}$ *and* $\varphi_{(a,b)} = \underbrace{\varphi_{(a,b),0}}_{\textit{semiclassical limit}} + O(\hbar^{\frac{1}{2}})$

(4) *Each* $\varphi_{(a,b)}$ *recovers* $f_{a,b}$*, if you look at the ordered product* $\prod e^{\varphi_{(a,b),0}}$ *you get the identity, so this is a* monodromy-free *scattering diagram.*

So solving the Maurer–Cartan equation and including the semiclassical limit gives you scattering diagrams.

3. Kyoji Saito: An introduction to primitive form theory

Thank you very much for the introduction and for inviting me to give some course here.

The talk is some brief intro to primitive form theory. I did bad preparation, so I'll talk from old notes, but the origin is really some hundreds of years old. So using ten year old notes isn't actually that old.

I said this goes to the 1937 or 1938, but recent primitive forms were introduced around the beginning of the 1980s. Let me give some background. I'll start in the first part with a classical story. This is elementary and some people can think of it as sleeping time. I'll talk about elliptic integrals (revisited) or simple Lie algebra theory from the viewpoint of the coadjoint orbit space. In the second part, I'll come to a more technical explanation of primitive forms. In the third part, I'll give some possible applications. For the moment, the whole picture is not yet seen, but I'll try to describe some possible applications. One is mirror symmetry, but as we have seen this morning, this story belongs to Landau–Ginzburg models in the B-model. This could be dual to several different objects, it could be FJRW theory or the compact Calabi–Yau case. The first case is done, by Si Li and his collaborators. The second is only partially done, by Gromov–Witten theory.

A second application is the relation with topological conformal field theory. This is only partially understood, some examples were done by Dijkgraaf, Verlinde, and others. A third application is to primitive automorphic forms.

Some of you like Si Li and Changzheng Li understand much more than me about primitive forms, or professor Losev from the physics side. What I can do is a formal technical description and some of you can understand it better.

3.1. **Elliptic integrals.** You consider integration on a curve, and you want to find a point of distance $a+b$, on a curve and this was studied for cubics and quartics. Euler finally found an equation, and people developed a theory of elliptic integrals. This goes to the integral theory on Riemann surfaces. These are some origins of modern Hodge theory. In my impression, modern Hodge theory is a big machine but within it several fine details are skipped or need to be looked at again.

After Lie, Jacobi, or Abel, people tried to formalize these integrals. There are three that I'll talk about.

- Weierstraß, doing $F_{A_2}(x,y,g_2,g_3) = y^2 - (4x^3 - g_2x - g_3)$.
- Jacobi (Euler), $F_{B_2}(x,y,g_2,g_4) = y^2 - (x^4 + g_2x^2 + g_4 + \frac{g_2^2}{8})$.
- Hesse, $F_{G_2}(x,y,\underline{g}) = x(x+y)(x-y) + g_2(3x^2+y^2) + g_6 - 2g_3^3$.

These correspond to the three rank two root systems. This is a subfamily of some bigger family, so I want to embed B_2 in A_3, $F_{A_3}(x,y,g_2,g_3,g_4) = y^2 - (x^4 + g_2x^2 + g_3x + g_4 + g_2^2/8)$. Similarly, I can put G_2 inside D_4 with $F_{D_4}(x,y,g) = x(x+y)(x-y) + g_2(3x^2+y^2) + g_4x + g_4'y + g_0 - 2g_2^4 - g_2(g_4+g_4')$.

I wanted to describe these loci, but in the Weierstraß case you have two parameters g_2 and g_3 and if you look at it carefully, this curve is singular when the discriminant of the polynomial $(4x^3 - g_2x - g_3)$ is zero, and up to a coefficient, this is $\Delta = g_2^3 - * g_3^7$.

(many pictures).

I want to study integrals, I want to study some tautological differential form on the total space with parameters, nad then I want ot integrate this over these cycles. That was the original study of Abel and Jacobi. I want to talk now about Abel's idea, which was very clever.

So he was studying these, and what happened?

This is simple, consider a polynomial $P(x,y,\underline{g})dxdy$, consider this on the space of parameters x, y, and g. If the parameter is zero, then you get these singular fibers. [pictures]

What he did was $\mathrm{Res}[\frac{P(x,y,g)dxdy}{F(x,y,g)}]$, this is some Poincaré residue, and you can do this by dividing both sides, this is $\frac{P()dx}{\frac{\partial F}{\partial y}}|E_g = -\frac{P()dy}{\frac{\partial F}{\partial x}}|E_g$.

So in the cases of our examples, the rank of the invariant lattice and the [unintelligible]are the same.

Euler didn't say explicitly, but

Lemma 3.1. *Let* $\zeta = \mathrm{Res}\left[\frac{dxdy}{F(x,y,g)}\right]$

For all cases, $\{\nabla_{\frac{\partial}{\partial_i g_i}}(\zeta^{(0)}\}$ gives a basis of the cohomology groups of the $E_{0,g}$.

There's some standard formula that you can prove, that

$$\frac{\partial}{\partial g_i}\int_{[unintelligible]} S^{(0)} = \int_{[unintelligible]} \nabla_{\frac{\partial}{\partial g_i}}\zeta^{(0)}$$

[Too tired to continue.]

4. Liang Kong: Local observable algebras for $2+1D$ topological phases of matter

[I do not take notes on slide talks]

5. January 7: Daniel Murfet: Fusion of defects in Landau–Ginzburg models I

Recall that the first thing I talked about yesterday was $2D$ defect TFTs. This had phases and defect conditions. The objects were decorated circles and the morphisms decorated surfaces. There are labels on the regions and the decorations. There are compatibility conditions using the source and target maps from the defects to the phases.

I'm going to start by defining the homotopy category of matrix factorizations.

Definition 5.1. Given two matrix factorizations X and Y of W, assume they're finite rank, look at $Hom_{\mathbb{C}[x]}(X,Y)$, which is a $\mathbb{Z}_2$-graded free $\mathbb{C}[x]$ module. Then $\delta(\alpha) = d_Y\alpha - (-1)^{|\alpha|}\alpha d_X$.

This still squares to zero even though d_X and d_Y do not. So this is a complex.

Definition 5.2. There's a dg category whose objects are matrix factorizations of W, call it $mf(W)$, between two objects X and Y I take this complex. Composition is composition of matrices. This is a $\mathbb{Z}_2$-graded category. This category doesn't have good properties unless X and Y are assumed to be finite dimensional.

Anyway, the homotopy category is $H^0(mf(W))$. This is the triangulated category with objects finite rank matrix factorizations and morphisms closed degree zero elements in this complex mod exact elements.

I always meant homotopy equivalences by isomorphisms yesterday. This was introduced by Eisenbd in the 80s.

Okay, that was a piece of last lecture present in today's lecture. Today's lecture is going to be about bicategories. Let me begin by repeating and elaborating the connection between $2D$ TFTs and bicategories. If you sit down and look at the definition, you get that, the rigorous and clear direction, from a $2D$ defect TFT Z you can cook up a bicategory where all 1-morphisms have left and right adjoints. It'll be *pivoted* and satisfy some extra conditions. The 2-category satisfies some conditions, and I don't know exactly which conditions are necessary to define Z in the other direction. I'll define bicategories more carefully later. Just a sketch, a bicategory has objects and 1-morphisms. The set of objects is D_2 the set of phases, and the 1-morphisms are the defect conditions. Today I'll talk about a particular bicategory, the bicategory of Landau Ginzburg models. This is another way to look at physicists' LG models. The phases are potentials $W(x)$ in various sets of variables. The defect conditions X with $s(X) = W(x)$ and $t(X) = V(y)$ is a matrix factorization of $V(y) - W(x)$. Then the corresponding bicategory had better have potentials as objects and 1-morphisms matrix factorizations.

The real novelty is that before I learned about this, the defect perspective gives you this idea of one-morphisms between potentials, and it suggests nontrivial properties between the arrows.

Before I make the definitions in higher resolution, let me convince you that there are nontrivial arrows between different singularities, geometrically interesting, you might say.

The first example is generalized orbifolding (Carqueville–Runkel). So professor Saito was talking about ADE singularities. So A_{11} the polynomial is $x_1^{12} + x_2^2$. There's also a polynomial $W_{E_6} = y_1^3 + y_2^4$, and there's an interesting map X from the first to the other, a matrix factorization of rank 2 of $y_1^3 + y_2^4 - x_1^{12} - x_2^2$. You can write down the "dual" and then composition is fusion, and you get a loop, writing down the fusion, at $W_{A_{11}}$, call it A.

Proposition 5.1. *A is a Frobenius algebra. The modules over A is equivalent to $hmf(\mathbb{C}[y], W_{E_6})$.*

This is a consequence of some general yoga they set up and this is quite surprising. If this seems obvious to you that A_{11} and E_6 are related, please tell me why. So 1-morphisms from $W_{A_{11}}$ to itself is a monoidal category. In this monoidal category I can talk about Frobenius algebras. So A is a Frobenius algebra in that category. There are many more examples in this line that I'll talk about today.

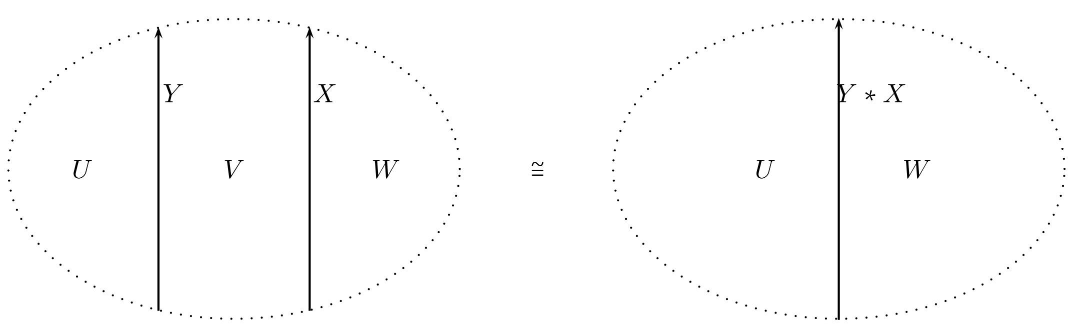
The second example with nontrivial arrows comes from topological Fukaya categories (Dyckerhoff–Kapranov). The starting point for this construction is a cocyclic object $\{\{Z^i\}$, where you have the n simplices the potential z^{n+2}. These matrix factorizations were first considered by Brunner–Roggenkamp. These examples hopefully demonstrate that there are interesting 1-morphisms between different potentials.

So I only talked about fusion in a special case last time, now let me define general fusion and then get into bicategories.

Theorem 5.1. *Given potentials $W(x)$, $V(y)$, and $U(z)$ and finite rank matrix factorizations $Y \in hmf(\mathbb{C}[y,z], U(z) - V(y))$ and $X \in hmf(\mathbb{C}[x,y], V(y) - W(x))$. So we think of this as $W \xrightarrow{X} V \xrightarrow{Y} U$. Then if I tensor these modules over the intermediate variable Y, I get $(Y \otimes_{\mathbb{C}[Y]} X, d_Y \otimes 1 + 1 \otimes d_X)$ which is an infinite ranke matrix factorization of $U - W$. There is a finite rank matrix factorization $Y * X$ of $U - W$ over $\mathbb{C}[x,z]$ which is homotopy equivalent to $Y \otimes X$.*

Definition 5.3. Call this object the *fusion* of Y and X..

I'll give an example of nontrivial fusions, but first let me draw a picture:



Let me give an example, let $V(y) = y^d$ and $W(x) = x^d$. Then $X \in hmf(\mathbb{C}[x,y], y^d - x^d)$. Let $\eta = e^{2\pi i/d}$. Then $y^d - x^d = \prod_{i=0}^{d-1}(y - \eta^i x)$. Choose $S \subset \{0, \dots, d-1\}$. Then let

$$X = P_S = \begin{bmatrix} 0 & \prod_{i \in S}(y - \eta^i x) \\ \prod_{i \notin S}(y - \eta^i x) & 0 \end{bmatrix}$$

which is called a *permutation defect*. You might think that's simple and can't be interesting, but the fusion is nontrivial. I can take $P_{S'}$ which is in $hmf(z^d - y^d)$, and I've got two arrows, so to speak,

$$x^d \xrightarrow{P_S} y^d \xrightarrow{P_{S'}} z^d$$

and I can ask about the fusion, and this is a finite rank matrix factorization of $z^d - x^d$. This turns out to be some sum of P_S things. Let

$$P_{m:\lambda} = P_{\{m,\dots,m+\lambda\}}$$

and then

$$\underbrace{P_{m:\lambda}}_{P_{S'}} * \underbrace{P_{n:\mu}}_{P_S} = \bigoplus_{v=|\lambda-\mu|,\text{ step }2} P_{m+n-\frac{1}{2}(\mu+\lambda-v):v}$$

and this is related to $\widehat{su}(2)_{d-2}$. This was proven by Brunner–Roggenkamp and reproved later by Davydov–Ros Camacho–Runkel.

Let's move to bicategories.

Definition 5.4. A *bicategory* $\mathcal{B}$ has

- *objects* $a, b, c, \dots$;
- for every pair a category $\mathcal{B}(a,b)$, objects are called 1-morphisms $X : a \to b$. Morphisms of $\mathcal{B}(a,b)$ are called 2-morphisms;
- for every triple a, b, c, a functor
$$\mathcal{B}(b,c) \times \mathcal{B}(a,b) \to \mathcal{B}(a,c)$$
which is horizontal composition,
$$(Y,X) \mapsto Y \circ X$$
- for each a a unit $\Delta_a : a \to a$.
- an associator, a natural two-isomorphism $Z \circ (Y \circ X) \cong (Z \circ Y) \circ X$
- unitors for $X : a \to b$ which say that $\Delta_b \circ X \cong X \cong X \circ \Delta_a$ naturally, plus
- coherence conditions, literally the same as for a monoidal category.

A bicategory with one object is literally a monoidal category, so this is a generalization of that. If you change these natural isomorphisms in the associator and unitor, this is a two-category.

So if you take categories, functors, and natural transformations, this is a 2-category, where these two things are strict. Many interesting things are not strict. If I take rings, bimodules, and bimodule maps, for example, that's a bicategory, because the tensor product is not associative on the nose. I can take smooth projective varieties, Fourier–Mukai kernels, and morphisms in the derived category of coherent sheaves, and this is also a bicategory, more interesting than the others if you're a geometer.

Definition 5.5. The bicategory $\mathcal{LG}$ (for Landau–Ginzburg) has objects potentials $(\mathbb{C}[x], W(x))$, and next I should give a category $\mathcal{LG}(W(x), V(y))$, and that's the category of matrix factorizations $hmf(\mathbb{C}[x,y], V - W)$, but I should formally split idempotents, take the idempotent completion. I need to tell you how to compose 1-morphisms, and that's fusion, which we've already defined. If I have two 1-morphisms Y and X,

$$W \xrightarrow{X} V \xrightarrow{Y} U$$

then the fusion by definition is a finite rank matrix factorization of $U - W$, just an isomorphism class but there are two ways to get around that. For lack of time I won't talk about units.

You need to check coherence

Proposition 5.2. *(Lazariou–McNamee; Carqueville–Runkel) $\mathcal{LG}$ is a bicategory.*

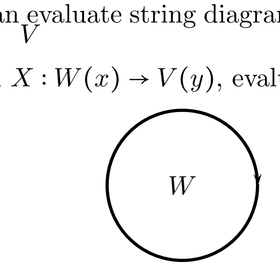
We have ourselves a bicategory which we think of as encapsulating the data of a $2D$ defect TFT. What can we say about the bicategory? I want to get to generalized orbifolding.

Theorem 5.2. *(Carqueville–Runkel; Carqueville–Murfet) Every* 1*-morphism* $X : W \to V$ *in* $\mathcal{LG}$ *has a left and right adjoint and* $\mathcal{LG}$ *is* pivoted.

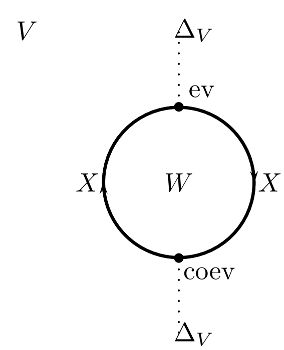
I won't define left and right adjoints in a general bicategory. I'll tell you the adjoint. Assume $|x|$ and $|y|$ are even for ease. $X^\vee = Hom_{\mathbb{C}[x,y]}(X, \mathbb{C}[x,y])$ with $d(\alpha) = -(-1)^{|\alpha|}\alpha d_X$, and $X^\vee$ is both left and right adjoint to X. If you're familiar with Grothiendieck duality this isn't surprising. There are some suspensions here that I'm supressing. You can approach this in a number of ways, but the difficulty is writing this down in a way that you can get explicit units and counits.

This means that you can evaluate string diagrams in the bicategory and actually compute the answers.

So given a 1-morphism $X : W(x) \to V(y)$, evaluating



I put in a Δ_V at the top and bottom, and then this is something in $Hom(\Delta V, \Delta V)$, which is the Jacobi alegbra of V, J_V.



If you plug into the formulas, this is the *right quantum dimension of* X, $\dim_r X \in J_V$. The point is that everything in this Landau–Ginzburg setting is computable, you can sit down and show that

$$\dim_r X = (-1)^{\binom{m+1}{2}} \operatorname{Res}_{\mathbb{C}[x,y]/\mathcal{C}[y]}\left(\frac{str(\partial_{x_1} d_X \cdots \partial_{x_n} d_X \partial_{y_1} d_X \cdots \partial_{y_m}(d_X)) dx_1 \cdots dx_n}{\partial_{x_1} W \cdots \partial_{x_n} W}\right)$$

in $J_V = \mathbb{C}[y]/(\partial V)$ and if $\dim_r X$ is a unit then V is a generalized orbifolding of W, this is a strong relation, and this is what is used in showing the relation I talked about earlier between $W_{A_{11}}$ and W_{E_6}.

6. Matt Young:Cohomological Donaldson–Thomas theory with orientifolds I

The goal of the lectures is to extend Donaldson–Thomas theory to deal with orientifolds. If Donaldson–Thomas is about counting bundles on a Calabi–Yau threefold, then this extends this to G-bundles, for G a classical group. Today I'd like to explain what I mean by cohomological Donaldson–Thomas theory. This is other peoples' work, mainly Kontsevich and Soibelman, and in the other lectures I'll talk about orientifolds and how to extend this theory to them.

Before defining what this is, I want to start with some motivation from physics that was also Kontsevich–Soibelman's motivation. It's an old idea, old by string theory standards, that given a string theory or quantum field theory with extended supersymmetry, more than the minimal nonzero amount of supersymmetry, there is a distinguished subspace, I'll call it H_{BPS} inside H, the Hilbert space of states of your theory graded by some charge lattice Λ with finite dimensional graded pieces. This subspace is interesting from both a physical and mathematical point of view. The motivation is all a huge conjecture, probably not right as stated, but that won't stop me.

What's one caricature of what H_{BPS} could be? In type IIA string theory compactified on X, a smooth projective Calabi–Yau three-fold, in this case, the classical BPS field configurations, these are the fields that obey some partial differential equation of lower order than the equations of motion, these are complex vector bundles on X with a Hermitian Yang–Mills connection. This is a first order partial differential equation you need to solve, this PDE is the BPS condition. You can argue that these connections are the same as semi-stable vector bundles on X. This is a purely classical thing, that the moduli of BPS configurations is $\mathcal{M}_d^{st}(X)$ with Chern character $d \in \Lambda = H^{2*}(X, \mathbb{Q})$. This is an interesting moduli space.

So the naive expectation is that this subspace H_{BPS} is the singular cohomology of this moduli space, $H_{BPS,d} = H^*(\mathcal{M}_d^{st}(X))$, and this is even more problematic than the original conjecture, equality should be "=" here. We have an extra $\mathbb{Z}$-grading from Hodge or cohomological degree, so that H_{BPS} is a $\Lambda \times \mathbb{Z}$-graded vector space.

At increasing levels of sophistication, we could want to compute the following quantities.

(1) the dimensions of the components of H_{BPS}, say the Euler characteristic, so something in $\mathbb{Z}[[\Lambda]] = \mathbb{Z}[[x^d]]$ where $d \in \Lambda$, call this *numerical BPS invariants.*
(2) the Serre polynomial, using the extra $\mathbb{Z}$-grading $[H_{BPS}] = \sum_{d,k} \dim\, H_{BPS(d,k)}(-q^{\frac{1}{2}})^k x^d$, the *refined BPS invariant* which recovers the numerical one when $q^{\frac{1}{2}} = 1$.
(3) Just calculate H_{BPS} itself, call this the *cohomological BPS invariant.*

Physicists can calculate these, maybe with less detail as we go down the list.

A proposal of Harvey–Moore says that there's extra structure, namely that H_{BPS} is a Λ or $\Lambda \times \mathbb{Z}$-graded associative algebra. The definition, the definition that they give is a purely physical definition, so it's not clear at all how you'd make this mathematically precise. They have a more precise definition for type IIA but it turns out to not quite be correct.

This leads to at least two questions that we could try to answer.

(1) The first is whether we can define in some mathematically precise way the BPS invariants, whether numerical, refined, or cohomological.

(2) The second is whether we can actually construct this BPS algebra. Can we come up with a definition so that H_{BPS} has a natural multiplication.

You can extract these two questions from this setup.

I want to explain Kontsevich–Soibelman's approach to these two problems. They take as input a 3-dimensional Calabi–Yau category, and they should spit out a BPS algebra and the invariants that play the role of BPS invariants are Donaldson Thomas invariants.

(1) the first example, from type IIA, is $D^b(Coh(X))$, the derived category of coherent sheaves on X a smooth Calabi–Yau three-fold.
(2) [unintelligible]$D^b(J_{Q,W})$, the Ginzberg dg-algebra for a quiver with potential.

The proposal of Konstevich–Soibelman is rigorous for the second example. Another reason to study this case, is that the quiver case is in some sense the formal local version of the general case.

For the rest of today I want to explain what this approach is in the quiver case.

A *quiver* is a pair (Q_0, Q_1) where Q_0 are *nodes* and Q_1 are *arrows*, so you have a directed multigraph. Attached to Q is the Abelian category $\mathrm{Rep}_{\mathbb{C}}(\mathbb{Q})$ of representations of Q. Such a representation U has for each node i, U_i along with maps $U_i \to U_j$ for each arrow $i \to j$. We can always write these, choosing a basis, as $U_i = \mathbb{C}^{d_i}$, so we call $(d_i)_{i \in Q_0}$ the dimension $\underline{dim}(U)$ and $\Lambda_Q^+ = \mathbb{Z}_{\geq 0} Q_0$.

The *Euler form* of this 1-dimensional category is $\chi(U,V)$ which is $\dim \mathrm{Hom}(U,V) - \dim \mathrm{Ext}^1(U,V)$ and there's a Riemann–Roch type thing, wher this is

$$\sum_{i \in Q_0} d_i e_i - \sum_{i \xrightarrow{\alpha} j} d_i e_j$$

where $\underline{\dim}(U) = d$ and $\underline{\dim}(V) = e$.

Definition 6.1. We call a quiver Q *symmetric* if χ is a *symmetric* bilinear form.

This means there are the same number of arrows in one direction as in the other,

$$\#\{i \to j\} = \#\{j \to i\}.$$

Today we'll always assume that Q is symmetric and from the example, I'll take $W = 0$. Everything can be done without these assumptions but the statements are much more complicated. There's a completely linear description of a representation of a quiver, so we can use this to write down moduli spaces of quiver representations.

For $d \in \Lambda_Q^+$, let

$$R_d = \prod_{i \xrightarrow{\alpha} j} Hom_{\mathbb{C}}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$$

and

$$GL_d = \prod_{i \in Q_0} GL_{d_i}(\mathbb{C}).$$

There's a natural action of GL_d on R_d and the orbits are isomorphism classes of representations so then $M_d = R_d / GL_d$ is the stack or representations of Q with $\underline{\dim} d$.

Definition 6.2. Let $\mathcal{H}_Q = \bigoplus_{d\in\Lambda_Q^+} H^{*-\overbrace{\lambda(d,d)}^{\dim M_d}}(M_d)$, which I can think of as just the sum of $H^*_{GL_d}(R_d)$, the sums of the equivariant cohomology. This is a $\Lambda_Q^+\times\mathbb{Z}$-graded vector space.

So we can take the Serre series $[\mathcal{H}_Q] \in \mathbb{Q}(q^{\frac{1}{2}})[[\Lambda_Q^+]]$, this can be computed explicitly.

We have this Serre polynomial but it's not what we want, we have no stability condition. These could have huge isomorphisms and this isn't clearly related to the moduli of *stable* vector bundles. So $\mathcal{H}_Q$ is in an Abelian monoidal category of $\Lambda_Q^+\times\mathbb{Z}$-graded vector spaces. Then $[\mathcal{H}_Q]$ sits in a subring of the Grothiendieck group of this category $K_0(Vect_{\Lambda_Q^+\times\mathbb{Z}})$. This is a λ-ring, where we have a bunch of additional operations from taking symmetric powers in this ring.

As an aside, let Π_Q be $\mathbb{Q}(q^{\frac{1}{2}})[[\Lambda_Q^+]]$. Its λ-ring structure is determined by a map $\sigma_t : \Pi_Q \to 1+\Pi_Q[[t]]^+$. Suppose I have a graded vector space $[V]$ then I map it to $\sum_{n\geq 0}[\mathrm{Sym}^n V]t^n$, which defines the operations necessary to define a λ-ring.

This λ ring is a complete graded λ-ring, so we can define the plythestic exponential, basically the symmetric power operation, so this maps Π^+ (no constant terms) bijectively on $1+\Pi^+$, taking $[V]\to\sum_{n\geq 0} Sym^n V$. Then there's no convergence problems and you can really take this convergent sum.

What is this exponential explicitly in our example? Let's take the quiver Q consisting of a point. In this case the space of dimension vectors is just $\mathbb{Z}_{\geq 0}$. Then

$$\mathrm{Exp}(f(q^{\frac{1}{2}},x)) = \exp\Big(\sum_{n\geq 1}\frac{f(q^{\frac{1}{2}},x^n)}{n}\Big),$$

the Adams operations.

This definition is motivated by a bunch of physics calculations.

Definition 6.3. (Kontsevich–Soibelman) The *refined Donaldson–Thomas invariant* of Q is the element

$$\Omega_Q \in Q(q^{\frac{1}{2}})[[\Lambda_Q^+]]$$

defined by

$$\mathrm{Exp}\Big(\frac{\Omega_Q}{1-q}\Big) = [\mathcal{H}_Q]$$

in Π_Q.

This is safe because Exp is a bijection.

The first question you might ask is why you'd be interested in an element defined in this way. There's a useful heuristic for why you might come up with this definition.

A basic fact is that every representation of Q has a finite filtration, a *Jordan–Hölder filtration*, so that the subquotients U_i/U_{i-1} are all simple or irreducible representations (no nontrivial subrepresentations). As an aside, the simple representations are exactly the stable representations from geometric invariant theory.

There will be many different filtrations of this form, but the set of subquotients of this kind is well-determined up to isomorphism.

assume that

(1) the Jordan–Hölder filtration is canonical and
(2) splits canonically

This is saying that every representation of the quiver is isomorphic to a direct sum of simple representations, so V_i is stable for $v_i \geq 0$. This only holds true when Q is a point but let's see what we can do.

What does this say at the level of cohomology? Let $\mathcal{M}_d^{st}$ be the moduli space of stable representations of $\underline{\dim} = d$. Then I want to take $\mathcal{M}^{st} = \amalg_{d\geq 1}\mathcal{M}_d^{st}$. I'm taking the trivial stability condition, from the GIT point of view [unintelligible].

At the level of cohomology,

$$[\mathcal{H}_Q] = [H^*(M)] = \sum_{n\geq 0}[H^*(Sym^n\mathcal{M}^{st})]$$

which is not quite true because we need to take care of automorphisms. We can remember that the stable representations have automorphisms, a $\mathbb{C}^*$-worth. So if we mod out by that,

$$[\mathcal{H}_Q] = [H^*(M)] = \sum_{n\geq 0}[H^*(Sym^n\mathcal{M}^{st})/\mathbb{C}^*]$$

then this is actually true.

So how do we compute this? We take the S^n coinvariants of the nth power, this is

$$\sum_{n\geq 0}[\mathrm{Sym}^n H^*(\mathcal{M}^{st}/\mathbb{C}^*)] = \mathrm{Exp}([H^*(\mathcal{M}^{st}/cC^*)]) = \mathrm{Exp}(\frac{[H^*(\mathcal{M}^{st})]}{1-q}).$$

I'm almost done. Here we've used that $H^*(B\mathbb{C}^*) = \mathbb{Q}[u]$ where u has degree $(0,2)$. This suggests that the Donaldson–Thomas invariant Ω_Q is something like $[H^*(\mathcal{M}^{st})]$.

Motivated by this, there are two natural conjectures you could make, due to Kontsevich–Soibelman and Joyce–Song

Conjecture 6.1.
(1) Integrality: if I fix d, I'm looking at the cohomology of a single part, and that should be a finite dimensional graded vector space. That is, $\Omega_{Q,d}$ is a function of $q^{\frac{1}{2}}$, and this should sit in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$, polynomials and not rational functions.
(2) Positivity: If we think of this as an Euler characteristic, then when we substitute in $-q^{\frac{1}{2}}$, then we should get a polynomial with nonnegative coefficients, in $\mathbb{Z}_{\geq 0}[q^{\pm\frac{1}{2}}]$, motivated by the heuristic belief that there is some connection to the homology.

The first conjecture was proven by Kontsevich–Soibelman, using some complicated techniques. I'll describe a conceptual proof of both that is easy, using some additional conditions.

7. Kyoji Saito: An introduction to primitive form theory II

Thank you very much for the introduction. Yesterday I described the classical theory, and how to find a differential form, elliptic integral of the first kind, and I'll discuss a completely different subject today, a Lie theoretic approach. Later tomorrow I'll generalize both properties. Today I'll explain just the Lie theoretic story.

Start with $\mathfrak{g}$ a simple Lie algebra over $\mathbb{C}$. If you don't know this, a typical example is $sl(\mathbb{C}, \ell+1)$. Then you consider the adjoint gorup G, that means in this case $SL(\mathbb{C}, \ell+1)$, and the Cartan algebra $\mathfrak{H}$, the maximal Abelian subalgabra, in this case diagonal matrices in $sl(\mathbb{C}, \ell+1)$, and the Weyl group maybe I'll skip. So you consider this very classical setting. Then G acts on $\mathfrak{g}$ adjointly. In the matrix case it's by conjugation on the trace-free Lie algebra. In general this action has fixed points. Nevertheless you consider the quotient map $\mathfrak{g} \to \mathfrak{g}//\operatorname{ad} G$, where this is $\operatorname{Spec}(\{s(\mathfrak{g}^*)\}^{\operatorname{ad} G})$. If you follow the Cartan subalgebra $\mathfrak{H}$, there is a finite Weyl group W, the normalizer of $\mathfrak{H}$ divided by $\mathfrak{H}$ itself, and this becomes $\mathfrak{H}/W$ in the quotient. The Weyl group acts on $\mathfrak{H}$ as a finite reflection group, and this quotient is naturally identified with $\mathfrak{g}//\operatorname{ad} G$. There are singular orbits, and to detect them, well, in $\mathfrak{H}$, there are hyperplanes H_α which are fixed by reflections, and then the image in the quotient is called the *discriminant* D and is a divisor in $\mathfrak{H}/W$. Then an orbit is singular if its orbit is in D. So $\pi^{-1}(t)$ is smooth if $t \notin D$. If $t \in D$, the fiber in general decomposes as a finite number of orbits, and the most generic open orbit, there is a single generic nonsingular orbit and all the others are singular.

In the case that G is the special linear group so that $\mathfrak{H}$ is diagonal matrices, then to x in $\mathfrak{g}$, the space is $\det(\lambda I - x)$. It's a theorem of Chevalley that I hope many of you know that says that if you are going to look at this process, then you are looking at the function ring fixed by W, $S(\mathfrak{H}^*)^W \cong \mathbb{C}[p_1, \dots, p_\ell]$ where $p_1, \dots, p_\ell$ are homogeneous polynomials in the polynomial ring $S(\mathfrak{g}^*)$ of degree $d_1, \dots, d_\ell$, where d_i is $q_i + 1$ and q_i is called the *exponent*. Here $1 = q_1 < \cdots < q_\ell = h - 1$ and these are somehow symmetric.

You associate the variable $(p_1, \dots, p_\ell)$ in $\mathbb{C}^\ell$ and get the image in $\mathfrak{H}/W$. This is not a linear isomorphism, but this quotient space has a very common structure, a linear structure, and this later becomes what people call a Frobenius structure. I want to talk about where the flat structure comes from in this setting.

This projection map π on $\mathfrak{g}$ has a very specific structure. People may know that the cohomology ring is isomorphic to the flag variety, but I'll show that if you consider the direct image, then from the viewpoint of D-modules, there is a single generator, which is what I call the primitive form.

Before going further, maybe I should say something about singularities, go back to $\mathfrak{g} \to \mathfrak{g}//\operatorname{ad} G$. In order to adjust to the picture, let $S := \mathfrak{g}//\operatorname{ad} G$. I've already discussed the fiber. For generic t this is smooth. How about the most singular point? Then you see that this quotient space is the space of [unintelligible]. So you should look at all matrices in this space with eigenvalues 0. So $\mathfrak{g}_0$, the preimage of 0, consists of nilpotent elements. This is the space where there are as many orbits as possible. There is some resolution of this singularity using the cotangent space of the Grassman variety, $T^*(G/B)$. This is often called the Grothiendieck or Springer resolution.

In some sense, in order to describe this resolution, all other fibers, once you take the finite cover of the base space, somehow you can be convinced that this gives a generic fiber for $t \in S \backslash D$, this gives $\mathfrak{g}_t$. From this description, your fiber becomes a symplectic manifold, identified as it is with the cotangent space, and the cohomology ring of the fiber, which is the same as the cohomology ring of the cotangent bundle, which is the same as the cohomology ring of the base, and this is a Lie group modulo its maximal torus. This is generated by its lowest degree, degree two elements, this is a famous theorem of Borel. In the degree 2 part, there are still some classes. Then the homology group is identified with the root lattice of a certain Lie algebra. Today I'll show further that every cohomology class is obtained by [unintelligible]two-forms. Let us see how this thing works.

Maybe I'll just make the following remark, made by Brieskom, and clarified by [unintelligible]. This part also I won't use today, but, well, this is another remark, here we see that the fiber over $\mathfrak{g}_0$ is the nilpotent variety N, so consists of several singular orbits, but generically you have somethnig smooth. The codimension two part is called subregular. Then if you look at a singular point, and consider, since I don't use it today, consider some subvariety transversal to the subregular orbit. Then if you restrict the quotient map π to this transversal slice, you obtain a family of varieties whose fiber over 0 is the intersection of the slice and the nilpotent orbit, so a 2-dimensional singularity which is actually an ADE singularity (or BCF), depending on $\mathfrak{g}$.

The fiber over zero is the simplest singularity and all of $X \to S$ is the universal deformation of X_0. Then $H^2(X_t, \mathbb{C})$ for $t \in S \backslash D$ naturally embeds into $H^2(\mathfrak{g})$, and this is an isomorphism, so the generic fiber was the flag variety, but the second cohomology is identified with [unintelligible].

So my goal is to look at this part of the origin here.

In order to describe the cohomology ring $H^2(X_t, \mathbb{C})$ over $t \in S$, you need to get some vector bundle or coherent sheaf in the phase space. Then we should look at the de Rham cohomology group. So I'll switch now to that story.

Then for some people who are not used to it, I'll define notation. If there exists a map $\mathfrak{g} \to S$, then there's a notion of relative de Rham Kähler form. So let $\Omega^p_{\mathfrak{g}/S}$ is the sheaf of holomorphic relative p-forms on $\mathfrak{g}/S$. You're going to look at $\Omega^p_{\mathfrak{g}}/$ (the holomorphic p-forms on $\mathfrak{g}$ divided by $\Omega^{p-1}_{\mathfrak{g}} \wedge \pi^* \Omega^1_S$). This can also be defined where π is singular.

Then you are going to consider the hypercohomology group $\mathbb{R}^2 \pi_*(\Omega^{\cdot}_{\mathfrak{g}/S})$, a coherent sheaf on S of finite rank. Then this is an $\mathcal{O}_S$-free module of rank ℓ. This is the object we want to study now. This has a kind of D-module structure. In such setting, there is a concept if you have a family of varieties and look at the direct image of a D-module, then you get one on the base space. That is the so-called Gauss–Manin connection. One should be a little bit careful in the following sense. First let us denote by Der_S the sheaf of holomorphic vector fields on S, which some people denote by T_S.

Then what I was saying, if you have a D-module, then on the quotient space you again have a D-module structuer, and you can get

$$\mathrm{Der}_S \times \mathbb{R}^2 \pi_*(\Omega^*_{\mathfrak{g}/S}) \to \mathbb{R}^2 \pi_*(\Omega^*_{\mathfrak{g}/S})$$

which works outside the discriminant, $v \times \zeta \mapsto \nabla_v \zeta$, the Gauss–Manin connection.

Let Δ be defined in $S \backslash D$ by $\Delta^2 = 0$. Introduce $\mathcal{O}_S(\frac{1}{\Delta})$ and then this is correct.

Let me concentrate on the rank two part, although there are aspects of this I did not explain clearly.

The adjoint map has a very particular property, that the fibers of the type $\pi_{\mathfrak{g}} \to S$ are symplectic manifolds, and the symplectic form, called the Kostant-Kirillov form, since I have a bad memory I might say this badly, but let me describe it the following way. Let X be in an orbit of $\mathfrak{g}_t$. What is the tangent space $T_{g_t X}$? This is $\ker \mathrm{ad}\, X$. Then the symplectic form we will define $T_{\mathfrak{g}_X} \times T_{\mathfrak{g}_X} \to \mathbb{C}$, this is $A, B \mapsto tr(\mathrm{ad}(X)\, \mathrm{ad}[A, B])$, and this ω gives as symplectic structure. This is only for the singular fiber, but you can do this together for all fibers simultaneously. Then ω naturally extends to a global form $\Omega^2_{\mathfrak{g}/S}$, a closed form. This is more of a standard well-known story. Let's call this the Kostant–Kirillov form.

Now coming back to the Gauss–Manin connection, now for some technical reason it looks, actually, the base space is just given by the coordinate system $P_1, \ldots, P_n$, so this is just $\sum \mathcal{O}_S \frac{\partial}{\partial P_i}$, since [unintelligible], the space is somehwo graded, and with respect to this grading, the partial derivative $\frac{\partial}{\partial P_\ell}$ is degree $-d_\ell$. Up to the unique lowest degree element.

So let us call for the moment, D and sometimes δ this operator, and call this the primitive vector field. Then what I wanted to say was the following.

Usually with the Gauss–Manin connection you consider [unintelligible],

$$\mathrm{Der}_S : \mathbb{R}\pi_A(\Omega^{\cdot}_{\mathfrak{g}/S}) \otimes \mathcal{O}[D, D^{-1}],$$

(the primitive vector field) and there is something I should explain. The first theorem I want to state is the following.

Theorem 7.1. *Let $Der_S \otimes_{\mathbb{C}} \mathbb{C}[D^{-1}] \to \mathbb{R}^2\pi_*\Omega^*_{\mathfrak{g}/S}[D^{-1}]$, then for v in the domain, let it act naturally, $\nabla_v\omega$, this is an isomorphism. Today I couldn't find my old notes, maybe there is some degree shift necessary, I forgot, so to be safe, say up to a shift in degree.*

In this way, my first program is satisfied by this theorem. The second theorem is more important for the next program, the flat metric and Frobenius manifold. Let me say that we can understand the right hand side above as $\mathrm{Der}_S \oplus \mathrm{Der}_S D^{-1} \oplus \mathrm{Der}_S D^{-2} + \cdots$, and how does the connection look in these terms? This is a highly nontrivial phenomenon. For convenience, $D^k\omega$ I'll denote $\omega^{(k)}$.

Theorem 7.2. *There exists a commutative associative product $* : \mathrm{Der}_S \times \mathrm{Der}_S \to \mathrm{Der}_S S$. where $D = 1$ and an integrable connection (torsion free) $\nabla : \mathrm{Der}_S \times \mathrm{Der}_S \to \mathrm{Der}_S$, along with an Euler vector field E and an $\mathcal{O}_S$ endomorphism N from Der_S to itself such that*

(1)
$$\nabla_U \nabla_V \omega^{(-2)} = \nabla_{U*V}\omega^{(-1)} + \nabla_{\nabla_U V}\omega^{(-2)}$$
and all higher terms vanish.

(2)
$$\nabla_{\frac{\partial}{\partial D}} \nabla_U \omega^{(-1)} = \nabla_{E*U}\omega^{(-1)} - \nabla_{NU}\omega^{(-2)}.$$

So we get the Gauss–Manin connection but with more delicate structure. Here ∇ is a Levi–Civita connection of a metric $J : \mathrm{Der}_S \times \mathrm{Der}_S \to \mathcal{O}_S$; let me define $J^* : \Omega^1_S \times \Omega^1_S \to \mathcal{O}_S$, well $J^*(dP_i, dP_j)$, well, let I be the Killing form of the Lie algebra, usually defined on the space $\mathfrak{H} \times \mathfrak{H} \to \mathbb{C}$, a bilinear form, let $X_1, \ldots, X_n$ be the coordinates of $\mathfrak{H}$, a basis of the dual space. Then the Killing form defines $I^*(X_i, X_j) \to \mathbb{C}$. Then I define

$$J^*(dP_i, dP_j) = I^*(dP_i, dP_j) = \sum \frac{\partial P_i}{\partial X_k}\frac{\partial P_j}{\partial X_\ell} I^*(X_k, X_\ell)$$

and this would be invariant but you introduce the primitive vector field and then it's not degenerate:

$$J^*(dP_i, dP_j) = \underbrace{D}_{\frac{\partial}{\partial P_\ell}} I^*(dP_i, dP_j) = D \sum_{k,\ell} \frac{\partial P_i}{\partial X_k}\frac{\partial P_j}{\partial X_\ell} I^*(X_k, X_\ell)$$

so then $S(\mathfrak{H}^*)^W = \mathbb{C}[P_1, \dots, P_\ell]$ so there exist Q_1 through Q_ℓ so that $J^*(dQ_i, dQ_j)$ is constant, more exactly, its an antidiagonal matrix

$$\begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$$

so $(\nabla, *)$ is a flat structure.

So then twenty years after I introduced this, Dubrovin came to the same structure in fluid dynamics and he called it a Frobenius manifold structure. I'm sorry, I'll stop here.

8. Young-Hoon Kiem: Categorification of Donaldson–Thomas invariants I

I'd like to thank the organizers for the invitation. Honestly, before yesterday, I had no idea what the title of the conference means. I still don't have a clear idea, but I learned a lot yesterday and today. I learned a lot so far. I'm going to talk about categorification of Donaldson–Thomas invariants. I learned today that I'm actually doing cohomological BPS invariants. I'll give three lectures. Today I'll talk about some structure, which I'll call critical virtual manifolds, then sheaves on Calabi–Yau three-folds, and then applications. Everything I'll talk about is joint with Jun Li.

So I'm going to talk about critical virtual manifolds and perverse sheaves.

I'll mostly talk about the former and a little bit about the latter. To motivate the notion of critical virtual manifolds, let me start very basic with a complex manifold. A complex manifold, we all know, is a 2nd countable paracompact Hausdorff topological space equipped with an open covering $X = \cup_\alpha X_\alpha$ and homeomorphisms $\varphi_\alpha : X_\alpha \to \varphi_\alpha(X_\alpha) \subset \mathbb{C}^n$, a homeomorphism with open image so that the transition functions $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is holomorphic.

What about a *Kähler* manifold? It's a complex manifold with a Hermitian metric h on TX, where $\omega = \mathrm{im} h$ and $d\omega = 0$.

Suppose X is a compact Kähler manifold, $H^*(X, \mathbb{C})$ comes with Poincaré duality, the Lefschetz hyperplane theorem, the hard Lefschetz theorem, the Hodge decomposition and the Hodge index theorem. These I'll collectively call the Kähler package and the comprise important tools in complex algebraic geometry.

But if X is singular, none of these properties survive. What can we do? The way, in order to get these nice properties even when X is singular, it turns out we shouldn't use the ordinary cohomology but instead a different cohomology theory. I'll talk about this today for a special type of singular space.

So I'll talk about this in the category of *analytic spaces*.

or me an *analytic space* is a locally ringed space $(X, \mathcal{O}_X)$ that comes with an open covering $\cup_\alpha X_\alpha$ where $X_\alpha \subset V_\alpha \subset \mathbb{C}^n$, where X_α is defined as the zeros of a finite number of functions $f_{\alpha,i}$ and $\mathcal{O}_{X_\alpha} = \mathcal{O}_{V_\alpha}/(f_{\alpha,i})$. I believe these analytic spaces form a nice category.

So what's known is that we still have the Kähler package if we use the hypercohomology $\mathbb{H}^*(X, P)$ for P a perverse sheaf on X which underlies a polarizable Hodge module. This is a result of M. Saito. For Poincaré duality this was first studied by Goresky–MacPherson with intersection homology. I should also mention Kashiwara, Beilinson–Bernstein–Deligne, and so on. It's this amazing theorem that for the hypercohomology in this setting, we have all these nice properties and we only had the hard Lefschetz theorem for the absolute version, but now we get a relative version, so it's even better in that case.

Maybe it's wise not to explaine what a polarizable Hodge module or a perverse sheaf is (in one sentence a complex of sheaves, cohomology of global sections of an injective resolution is the hypercohomology).

Let's do examples.

(1) So Let P be the intersection cohomology. You only allow chains that intersect nicely with the singular locus. It's a theorem of Saito that these underly a polarizable Hodge module. So if I take $IH^*(X)$ that's $\mathbb{H}^*(X, IC_*)$.

(2) The second example, more relevant for us, is the perverse sheaf of vanishing cycles. We have a complex manifold V and a holomorphic function $f : V \to \mathbb{C}$. Then you have $f^{-1}(0)$ inside V. So what do you do? Let me remind you of the definition. V^* denotes $V - f^{-1}(0)$ and f gives a morphism $V^* \to \mathbb{C}^*$, and then the fiber product with the universal cover $\widetilde{\mathbb{C}^*}$ is $\widetilde{V}^*$. Then we can inject V^* by i in V and then pick $f^{-1}(0)$ out by j, and $\psi_f\mathbb{Q}$ is $j^*i_*\pi_*\mathbb{Q}_{\widetilde{V^*}}$.

Let me talk about the *Milnor fiber*, for x in $f^{-1}(0)$, well $MF_x = f^{-1}(\delta) \cap B_\epsilon(x)$ for $0 < \delta \ll \epsilon$

1. So you look at the Milnor fiber at x, mapping to S^1, and the cover is the real line, and when you pull back, you get something very much like MF_x, and it has the same cohomology. So $\psi_f\mathbb{Q}$ keeps track of nearby cycles.

So you have a natural morphism $\mathbb{Q} \to \psi_f\mathbb{Q}$, and then the mapping cone $\phi_f\mathbb{Q}$ takes a map from $\psi_f\mathbb{Q}$. So this is the reduced cohomology of the Milnor fiber. The perverse sheaf of vanishing cycles is $\phi_f\mathbb{Q}[\dim V - 1]$. This has the following property, that $\chi(P_f|_x)$, I get the Euler number of the Milnor fiber with degree shifted, $(-1)^{\dim V - 1}(\chi(MF_x) - 1)$.

This is an object in the derived category of constructible sheaves on the critical locus of f.

This is something, in this case, in the case where we have a holomorphic function on a complex manifold, this is the Behrend function for the critical locus of f. This is a theorem proved in Behren's paper. There's a constructible function on any scheme. The Behrend function should coincide with this, with the Euler number of pervese sheaves [unintelligible].

So for our purpose, the perverse sheaf of vanishing cycles is more relevant, for Donaldson–Thomas theory. Tomorrow I'll show you that the moduli of stable sheaves on a Calabi–Yau three-fold, because locally it's always the critical locus of a holomorphic function on a complex manifold.

I will not give you the definition of pervese sheaf, it's not a sheaf, it's a complex of sheaves. It comes from a sheaf of D-modules, and they behave like sheaves. It's a well-known fact that perverse sheaves glue! What do I mean? Suppose you have an open covering of an analytic space and perverse sheaves on each X_α, and suppose you have an isomorphism $\lambda_{\alpha\beta} : P_\alpha|_{X_{\alpha\beta}} \to P_\beta|_{X_{\alpha\beta}}$ satisfying the cocycle condition $\lambda_{\alpha\beta\gamma} = \lambda_{\gamma\alpha}\lambda_{\beta\gamma}\lambda_{\alpha\beta} = 1$, then there is a perverse sheaf P on X such that $P|_{X_\alpha} \cong P_\alpha$ in the category of bounded constructible complexes of $\mathbb{Q}$-sheaves, where perverse sheaves form an Abelian subcategory.

Okay, so now I'm ready to introduce the notion of critical virtual manifold. I should talk about the local model first.

I had to cook up a name, I hope you like it,

Definition 8.1. (1) An *LG pair* is a pair of complex manifold V and a holomorphic function f on V which has only one critical value, namely 0. I don't assume that it is nondegenerate or anything like that. The critical locus is not necessarily isolated.
(2) Two LG pairs (V_1, f_1) and (V_2, f_2) are equivalent if there is a biholomorphic map $\varphi : V_1 \to V_2$ such that $f_2 \circ \varphi = f_1$.
(3) $\mathrm{Crit}(f)$ is the zero locus of $df : V \to \Omega_V$.
(4) X_f is the reduced part of $\mathrm{Crit}(f)$

A very simple lemma says that if you have equivalent LG pairs then you can think about perverse sheaves of vanishing cycles for f_1 and f_2, and then you have a map $\bar{\varphi}$ from X_{f_1} to X_{f_2}, and then $P_{f_1} \cong \bar{\varphi}^* P_{f_2}$. So the perverse sheaves are essentially the same, this comes from the definition.

Now I'm ready to define critical virtual manifolds.

Definition 8.2. A *critical virtual manifold* is an analytic space $X = \cup_\alpha X_\alpha$ where $X_\alpha \cong \mathrm{Crit}\, f_\alpha$ inside $V_\alpha \xrightarrow{f_\alpha} \mathbb{C}$ where (V_α, f_α) is an LG pair.

On $X_{\alpha\beta} = X_\alpha \cap X_\beta$ you have an open neighborhood $V_{\alpha\beta}$ of $X_{\alpha\beta}$ inside V_α and a biholomorphic map between $V_{\alpha\beta}$ and $V_{\beta\alpha}$ that makes the diagram commute:

$$\begin{tikzcd}
& V_{\alpha\beta} \arrow[r, hook] \arrow[dd] & V_\alpha \arrow[dr, "f_\alpha"] & \\
X_{\alpha\beta} \arrow[ur, hook, "\text{closed}"] \arrow[dr, hook, "\text{closed}"'] & & & \mathbb{C} \\
& V_{\beta\alpha} \arrow[r, hook] & V_\beta \arrow[ur, "f_\beta"'] &
\end{tikzcd}$$

The only thing I assume is that $\varphi_{\alpha\alpha} = \mathrm{id}$ and $\varphi_{\beta\alpha} = \varphi_{\alpha\beta}^{-1}$.

Remark 8.1. The notion of critical virtual manifolds is equivalent to Joyce's notion of d-critical locus.

The first thing I should mention about these critical virtual manifolds is the notion of orientation. Think about T_{V_α} restricted to X_α, and the determinant of this, call this $K_\alpha^\vee$, and this is in $\mathrm{Pic}(X_\alpha)$. We have $\xi_{\alpha\beta} : K_\alpha^\vee|_{X_\alpha\beta} = \det T_{V_\alpha\beta}|_{X_{\alpha\beta}} \xrightarrow{\det(d\varphi_{\alpha\beta})} \det T_{V_{\beta\alpha}}|_{X_{\alpha\beta}} = K_\beta^\vee|_{X_{\alpha\beta}}$.

Then $(\xi_{\alpha\beta\gamma}) = (\xi_{\gamma\alpha}\xi_{\beta\gamma}\xi_{\alpha\beta}) \in H^2(X, \mathbb{Z}_2)$

We call a critical virtual manifold *orientable* if ξ is 0 in $H^2(X, \mathbb{Z}_2)$, that is, if $\{K_\alpha^\vee\}$ glue to a line bundle on X.

So just a few examples. First of all, all complex manifolds are orientable critical virtual manifolds. I can take V_α to be X_α and f_α to be 0. Then $K_X^\vee$ is $\det T_X$. Tomorrow I'll show that all moduli of simple sheaves on a Calabi–Yau three-fold are critical virtual manifolds, orientable if there is a universal family.

There are many nice structures on critical virtual manifolds. I don't know if I have enough time to talk about them.

- it comes with a virtual fundamental class $[X]$ of degree 0. There's a theory of Chang–Li called semi-perfect obstruction theory. So X_α is in V_α as the critical locus of f_α, so this is the zero locus of df_α. There's a canonical obstruction theory $[T_{V_\alpha}|_{X_\alpha} \xrightarrow{d(df_\alpha)} \Omega_{V_\alpha}|_{X_\alpha}]$. You might not be able to glue the constituent parts, but obstruction sheaves and obstruction assignments glue. Locally you have a perfect obstruction theory, and so if these two things happen, then you get a global virtual fundamental class. The obstruction sheaves obviously glue to Ω_X, and it's easy to check that assignments glue. So the degree of the virtual fundamental class is the Donaldson–Thomas type invariant, in $\mathbb{Z}$.

- Moreover, there is a theory of Behrend which applies microlocal analysis, saying that if you have a symmetric obstruction theory, then $DT(X) = \chi_v(X) = \sum_{n\in\mathbb{Z}} n\chi v^{-1}(n)$. I explained what v is when locally it's a critical locus of a function, and this holds for all critical virtual manifold.

I think the categorification problem was raised for Donaldson–Thomas invariants for a Calabi–Yau three-fold, but naturally generalizes to critical virtual manifolds. So find a cohomology theory $\mathcal{H}^*(X)$ for critical virtual manifolds so that $\chi(\mathcal{H}^*(X))$ is $DT(X) = \chi_v(X)$. For instance if X is smooth, I could take $\mathcal{H}^*(X) = H^{*+\dim X}(X)$.

The main theorem for critical virtual manifolds that I'll talk about is that the answer is yes.

Theorem 8.1. *(K.–Jun Li) Let X be an orientable critical virtual manifold. So $X = \cup X_\alpha$ and $X_\alpha = \mathrm{Crit}(f_\alpha) \subset V_\alpha \xrightarrow{f_\alpha} \mathbb{C}$ and there is a perverse sheaf P on X such that $P|_{X_\alpha} \cong \phi_{f_\alpha}\mathbb{Q}[\dim V_\alpha - 1]$. Then $\chi(\mathbb{H}^*(X,P)) = \chi_v(X) = DT(X)$ because this is the Euler characteristic of the perverse sheaf of vanishing cycles.*

We obtain P by gluing perverse sheaves of vanishing cycles. The two-cocycle obstruction for gluing is the same as the 2-cocycle obstruction for gluing $K_\alpha^\vee$. That's the main part.

Tomorrow I'll prove that moduli spaces of simple sheaves on Calabi–Yau threefolds have the structure of a critical virtual manifold, and then on Saturday applications.

9. January 8: Daniel Murfet: Fusion of defects in Landau–Ginzburg models III

While I'm fiddling with this, let me remind you what happened last time. I defined the bicategory $\mathcal{LG}$. The objects are potentials, the 1-morphisms are matrix factorizations of differences of potentials, and the composition of 1-morphisms is fusion, within the tensor product is a finite model and that's the composition. I talked about some applications of units and counits. I defined the quantum dimension and you can compute it in the Jacobi ring of the potential V. There are also some special defects P_S for S a subset of $\{0,\dots,d-1\}$ that come back in this lecture.

Take two potentials where the number of variables are even $W(x)$ and $V(y)$.

Theorem 9.1. *(Carqueville–Runkel) Let $X : W(X) \to V(Y)$ have this property that I mentioned, that the quantum dimension is invertible. Then I can draw the diagram in the bicategory*

$$W \underset{X^\vee}{\overset{X}{\rightleftarrows}} V$$

Composing $X^\vee$ with X to get a loop at W¡ then A is a separable symmetric Frobenius algebra in $\mathcal{LG}(W,W)$.

Moreover there is an equivalence

$$mod_{\mathcal{LG}(W,W)}(A) \cong hmf(V)$$

and one says that V is a generalized orbifold *of W.*

That's a theorem. The second theorem is that there are interesting examples. And anything you can say about V can be said in terms of W and the algebra A.

Theorem 9.2. *(Carqueville– Ros Camacho–Runkel) Let d be even. Then $W_{A_{d-1}} \xrightarrow{GO} W_{D_{\frac{d}{2}+1}}$ where the potentials are $x_1^d + x_2^2$ and, respectively, $x_1^{\frac{d}{2}} + x_1x_2^2$.*

The same is true for $A_{11} \xrightarrow{GO} E_6$, for $A_{17} \xrightarrow{GO} E_7$, and for $A_{29} \xrightarrow{GO} E_8$.

I've only defined the right quantum dimension, but the left quantum dimension (which is the right quantum dimension of the dual) is also invertible, then $V \sim W$, then they are orbifold equivalent. So this theorem is saying that if two singularities have the same Coxeter number then they are orbifold equivalent.

So the Frobenius algebra of A_{11}, if you compute the fusion, it's isomorphic to $\Delta \oplus P_{-3,\ldots,3}$ (for $d = 12$). All the alegbras for the ADE cases are of this form, direct sums of P_S type defects.

Some open questions, there's some groups working on this.

Question 9.1. What is the geometric or quiver or Lie theory explanation of these statements? Many of you know this theorem of Kajiura–Saito–Takahashi that says that hmf for ADE singularities are the same as derived categories of quivers, so there must be a version that doesn't mention matrix factorizations.

The next simplest class of singularities is unimodular singularities. It's natural to extend this to a question about those.

Conjecture 9.1. Unimodular singularities with the same *central charge* are orbifold equivalent

The central charge is $\sum_i (1 - |x_i|)$ if I choose a rational grading for the variables, with $|W| = 2$. This could be false, there's one example known, which is that $Q_{10} \sim K_{14}$. This is just one example.

The technology levels are low. The way to do this is to find a matrix that does the job.

Why do you need the same Coxeter number? The quantum dimension needs to be invertible. There's a shift of c_W, and it will land in a graded piece above degree 0 unless $c_V = c_W$. '' Let me add one more open question. I mentioned the cocyclic object of Dyckerhoff–Kapranov. I mentioned it in connection with the topological Fukaya category. It's interesting to try to orbifold it using these ideas. If you want something of D or E type, you can do it in A type and make it equivariant appropriately, so you might try something like that here.

Okay, so the rest of the lecture will be about how the code works that I've been mentioning. So the singular code for computing these fusions is central to this story. You end up doing a lot of computations with this code. For the rest of the lecture, let me try to explain how the code computes fusions.

Okay, so let me restate the problem. $\mathcal{LG}(V,U) \times \mathcal{LG}(W,V) \to \mathcal{LG}(W,U)$, composition in the bicategory, is defined as a finite rank representative $Y * X$ over $\mathbb{C}[x,z]$

in the tensor product $Y \otimes X$. The code uses an intermediate object, of possibly independent interest, $Y|X$, the *cut* of Y and X, a finite rank matrix factorization of $U - W$ together with an action of a Clifford algebra. It's a Morita trivial Clifford algebra, since we're over $\mathbb{C}$, so this is a spinor representation, and you can split an idempotent and the other stuff is $Y * X$.

That states it as an algorithm to compute the fusion. The cut can be promoted to a composition rule for another kind of bicategory. Then there's an equivalence of that new bicategory with $\mathcal{LG}$.

So let me define the cut. I have to define $Y|X$ and the action of a Clifford algebra. Let me be clearer about the setup, $W(x)$, $V(y)$, and $U(z)$ are potentials. Let $m = |y|$. Let Y be a matrix factorization of $U - V$ and X a factorization of $V - W$. The Jacobi algebra J_V is $\mathbb{C}[y]/\partial V$, finite dimensional.

Definition 9.1. The cut $Y|X$ is $Y \otimes_{\mathbb{C}[y]} J \otimes_{\mathbb{C}[y]} X$, since $\mathbb{C}[y,z] \otimes_{\mathbb{C}} J_v \otimes_{\mathbb{C}} \mathbb{C}[x]$ which has rank the dimension of J_v.

Now I need a Clifford action, I need closed odd $\mathbb{C}[x,z]$-linear actions γ_i and $\gamma_i^\dagger$ on $Y|X$ satisfying Clifford relations up to homotopy. What are the Clifford relations? They are that $\gamma_i\gamma_j + \gamma_j\gamma_i \cong 0$, that $\gamma_i^\dagger\gamma_j^\dagger + \gamma_j^\dagger\gamma_i^\dagger \cong 0$, and that $\gamma_i\gamma_j^\dagger + \gamma_j^\dagger\gamma_i \cong \delta_{ij}$, and the ingredient here is the Atiyah class, which is related to curvature of superconnections (let me acknowledge a debt to Calin).

Definition 9.2. Let $t_i = \partial_{y_i} V$

there is a $\mathbb{C}$-linear flat connection— If you give me a power series in y variables, I can always find a connection that differentiates in the t direction $\nabla : \mathbb{C}[[y]] \to \mathbb{C}[[y]] \otimes_{\mathbb{C}[t]} \Omega^1_{\mathbb{C}[t]/\mathbb{C}}$. Think about taking a power series and writing it sort of, I'll resist the temptation to give an example because I'll just run out oftime.

Lemma 9.1. *The operator* $[d_{Y\otimes X}, \partial_{t_i}]$ *on* $Y \otimes X$ *is* $\mathbb{C}[t]$*-linear.*

I should complete $Y \otimes X$ to power series in y, and anyway so this operator passes to an operator on the quotient $(Y \otimes X) \otimes_{\mathbb{C}[t]} \mathbb{C}[t]/t$. I've killed the partial derivatives, and that's $Y|X$.

So let me prove it's $\mathbb{C}[t]$-linear. Well, $[[d_{Y\otimes X}, \partial_{t_i}]t_j]$ by Jacobi is

$$-[[\partial_{t_i}, t_j], d_{Y\otimes X}] - [[t_j, d_{Y\otimes X}], \partial_{t_i}]$$

The second term vanishes and because t_i are flat, the first one is just δ_{ij}, so this is

$$[\delta_{ij}, d_{Y\otimes X}]$$

which is zero.

Definition 9.3. The *Atiyah class* At_i is $[d_{Y\otimes X}, \partial_{t_i}]$ acting on $Y|X$.

The Clifford operators are $\gamma_i = \mathrm{At}_i$ and $\gamma_i^\dagger = 1 \otimes \partial_{y_i}(dx) - \frac{1}{2}\sum_q \partial_{y_i y_q}(V)\,\mathrm{At}_q$.

This term might look weird but it comes out of homological perturbation so it's kind of a miracle it's even this simple.

I claim they satisfy the Clifford relations, you can do it by hand. You sit down and compute the commutators, you play around, it's not difficult. You can see that those things satisfy the Clifford relations up to homotopy.

Definition 9.4. The *cut* is this matrix factorization $(Y|X)$ together with this Clifford action $\{\gamma_i, \gamma_i^\dagger\}$.

Now I can state the theorem.

Theorem 9.3. *(Dyckerhoff–M., M.) The idempotent* $e = \gamma_1 \cdots \gamma_m \gamma_m^\dagger \cdots \gamma_1^\dagger$ *acting on* $Y|X$ *splits to give me* $Y \otimes X$*. To be more precise, there's a diagram in* $HMF(U - W)$*, the category of infinite rank matrix factorizations*

$$Y|X f\curlywedge \qquad Y \otimes X g\curlywedge$$

with $fg \sim 1$ and $gf \sim e$.

So the code computes e as a matrix. It strictifies $ee \sim e$ to $EE = E$. In the graded case it definitely terminates, in practice this works in the ungraded case too although its not guaranteed.

Then you compute $\mathrm{im}(E)$, this is a Gröbner basis calculation. That's a matrix calculation, this is a finite rank splitting of the same idempotent that gives the tensor product, so it must be homotopy equivalent to the tensor product. So $Y * X$ is $\mathrm{im}(E)$.

I'm out of time so let me state the souped up version of the theorem which explains, these formulas are maybe mysterious looking.

Let $S_m = \wedge(k\mathcal{O}_1 \oplus \cdots \oplus k\mathcal{O}_n)$ with $|\mathcal{O}_i| = 1$. Let $C_m = End_k(S_m)$, this is a Clifford algebra, with $\gamma_i = \mathcal{O}_i^* \lrcorner$ and $\gamma_i^\dagger = \mathcal{O}_i \wedge$.

Theorem 9.4. *There is an isomorphism of* C_m*-representations between* $Y|X$ *and* $S_m \otimes_{\mathbb{C}} (Y \otimes X)$*.*

This is by homotopy perturbation. The exterior algebra is really a Koszul complex on these partial derivatives $(\partial_{y_i} V)_i$. The connection that differentiates in these directions, this is the standard way to get a contracting homotopy on the Koszul complex. You have this initial strong deformation retract. Then use the perturbation lemma to put in the differential on both sides. There's an extra differential you absorb in some way and then this is what you get.

Let me summarize and then stop. I explained that $2D$ defect TFTs are like certain bicategories. I gave the example of $\mathcal{LG}$ and described generalized orbifolding, and then advertised that it was computable.

10. Matt Young:Cohomological Donaldson–Thomas theory with orientifolds II

Today the subject will be cohomological Hall algebras and representations. That's the subject of today's talk. Let's remind ourselves where we were at the end of the last lecture. We were given a quiver Q which we assumed symmetric to make things easier. We formed the graded vector space $\mathcal{H}_Q$, and the components were $H^{\cdot - \dim M_d}(M_d)$ where $M_d = R_d / GL_d$, this is a $\Lambda_Q^+ \times \mathbb{Z}$-graded vector space, and then we gave a definition of the refined Donaldson–Thomas invariant, this is $\Omega_Q \in \mathbb{Q}(q^{\frac{1}{2}})[[\Lambda_Q^+]]$ defined by $\mathrm{Exp}(\frac{\Omega_Q}{1-q}) = [\mathcal{H}_Q]$. We want to give some structure on this vector space that gives more evidence that this is the right definition. What I'll explain today gives a much clearer explanation (to me) of this formula.

The idea is Kontsevich–Soibelman, following ideas of others, is to consider $\mathcal{H}_Q$ as an algebra, using a Hall algebra type construction.

The category $\mathrm{Rep}_{\mathbb{C}} Q$ is an Abelian category, so in particular we know what it means to have short exact sequences in this category, and we can form the following diagram, with $M_{d,d'}$, the stack of flags $U \subset V$ with $\underline{\dim}(U) = d$ and $\underline{\dim}(V) = d + d'$. We can think of this flag as being a short exact sequence $0 \to U \to V \to V/U \to 0$.

So we can do

$$\begin{tikzcd} & M_{d,d'} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M_d \times M_{d'} & & M_{d+d'} \end{tikzcd}$$

where the two projections take the short exact sequence to $(U, X/U)$ and V, respectively.

The map π_2 is a smooth proper morphism of stacks. What are the fibers? They are Grassmannians of representations. We can take this diagram and then look at the structure it induces on the cohomology of stacks.

Theorem 10.1. *(Kontsevich–Soibelman) The maps $\mathcal{H}_{Q,d} \otimes \mathcal{H}_{Q,d'} \xrightarrow{(\pi_2)_! \circ \pi_1^*} \mathcal{H}_Q(d+d')$ given $\mathcal{H}_Q$ the structure of an associative $\Lambda_Q^+ \times \mathbb{Z}$-graded algebra, the* cohomological Hall algebra*.*

So from yesterday, recall that $\mathcal{H}_Q$ is the model for the BPS algebra. Somehow, this is the definition, we want to think of this algebra structure, you think of this representation, smash them together and get a bigger representation. Consider a short exact sequence $0 \to U \to V \to W \to 0$. Then in this sequence V is never stable. These are the simple representations. We can say that anything that lies in the image of the product will not be stable. We should compute some minimal generators for the algebra.

Let's give an example. These cohomology groups can be computed explicitly using localization and equivariant cohomology. Let Q bt then m-loop quiver, which has one vertex and m loops. So what is the Hall algebra $\mathcal{H}_{L_m}$, the dimension vectors are nonnegative integers, and so

$$\mathcal{H}_{L_m} = \bigoplus_{d \geq 0} H^*(gl_d^{\oplus m}/GL_d)$$

this is equivariant cohomology of m-tuples of matrices, which is contractible, so this is $\bigoplus H^*_{GL_d}(\mathrm{pt})$ so this is $\bigoplus \mathbb{Q}[x_1, \ldots, x_d]^{S_d}$, so independent of m. The product is where we see m. Take $f_1(x_1, \ldots x_{d'}) f_2(x_1, \ldots, x_{d''})$, and so we take

$$\sum_{\pi \in S_{d',d''}} \pi(f_1(x'_1, \ldots x'_{d'}) f_2(x''_1, \ldots, x''_{d''}) \times \prod_{i=1}^{d''} \prod_{j=1}^{d'} (x''_i - x'_j)^{m-1})$$

For ease, if $m = 0$, I think $x^i \cdot x^j = x_1^i x_2^j (x_2 - x_1)^{-1} + x_2^i x_1^j (x_1 - x_2)^{-1}$, which is the Schur polynomial $\frac{x_1^i x_2^j - x_1^j x_2^i}{x_2 - x_1}$. You can use this to prove that powers of x generate the Hall algebra freely in this example.

The first thing to notice is that if $i = j$ then this is 0, this is a Clifford algebra, so this turns out to be $\mathcal{H}_{L_0} = \wedge^*[x^0, x^1, \ldots]$.

We can generalize this example. In the case where Q is symmetric, so the m-loop quiver is symmetric, then the Hall algebra is actually a supercommutative algebra with $\mathbb{Z}_2$-grading induced by the $\mathbb{Z}$-grading. In the case of the 0-loop quiver, there's a shift I omitted.

Theorem 10.2. *(Efimov) There exists a graded subspace $V_Q^{\mathrm{prim}} \otimes \mathbb{Q}[u]$, where u has degree $(0,2)$, such that the map $\mathrm{Sym}(V^{\mathrm{prim}_Q} \otimes \mathbb{Q}[u]) \to \mathcal{H}_Q$ is an isomorphism of algebras and the dimension of $V_{Q,d}^{\mathrm{prim}}$ is non-infinite for all $d \in \Lambda_Q^+$.*

What is the corollary? If we take the Serre polynomials of the isomorphism, we see that $[\mathcal{H}_Q] = [\mathrm{Sym}(V_Q^{\mathrm{prim}} \otimes \mathbb{Q}[u])]$, but this is exactly the plethystic exponential, $\mathrm{Exp}(\frac{[V_Q^{\mathrm{prim}}]}{1-q})$, so V_Q^{prim} is the cohomological Donaldson–Thomas invariant.

Secondly, the integrality conjecture holds, and finally we get positivity, since this is the Serre polynomial of a single vector space.

Explicitly, what is $V_Q^{\mathrm{prim}} \otimes \mathbb{Q}[u]$? This is a quotient of $\mathcal{H}_Q$ by the maximal ideal of $\mathcal{H}_Q$ times itself. So splitting that and thinking of it as a subspace, that's what you'd do.

So let me give a geometric interpretation due to Chen. Let Q be the double of a quiver. This is a simplifying assumption. For example L_{2m} is the double of a quiver but L_{2m+1} is not.

Theorem 10.3. *(Chen) $\mathcal{H}_{Q,d} = H^*(M_d)$, and inside here you have $V_{Q,d}^{\mathrm{prim}} \otimes \mathbb{Q}[u]$, and this maps to $H^*(\mathcal{M}_d^{\mathrm{st}} \otimes \mathbb{Q}[u])$, and the image here takes the u factor to the u factor, and has as image $PH^*(\mathcal{M}_d^{\mathrm{st}}) \otimes Q[u]$, so we can identify $V_{Q,d}^{\mathrm{prim}}$ as $PH^*(\mathcal{M}_d^{\mathrm{st}})$, the pure homology.*

I now want to move on and talk about representations of $\mathcal{H}_Q$. To motivate this, I want to return to the string theory picture we talked about yesterday. If we add additional structure to our string theory we should get additional structure on our BPS states and that will let us get a representation.

So my physics caricature, in type IIA string theory, we have maps of Riemann surfaces (allowing boundaries) into X a Calabi–Yau three-fold. Whatever this theory is, you can extract some parts of this theory. You can ask about the boundary conditions that make such maps of Riemann surfaces well-defined. So at some level, this should be the bounded derived category of coherent sheaves on X. So the BPS number should count stable objects of this category in some sense.

What is an orientifold? You start with a string theory, apply the construction, and get a new string theory. You apply it to an oriented theory and get an unoriented theory. You have a closed Riemann surface with an orientation reversing involution. You map into X which has an isometric involution σ, and now you want $\mathbb{Z}_2$-equivariant maps. So what happens to the boundary conditions in the original theory? You get orientifold data on the category $D^b(\mathrm{Coh}(X))$.

So what is orientifold data? It's a contravariant involution of the D-brane category. This is a contravariant triangulated functor $S : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(X))$. This should be an involution, so we should give $\theta : 1_D \to S^2$, with coherence conditions.

in this particular example, if you have, well, the functor $D^b(X)$, the functor S is the derived pullback composed with the derived dual, $S = \sigma^* \circ (\quad)^\vee$, and here we can take θ to be $\pm$ the canonical thing.

In orientifold string theory. We've been trying to capture stable objects, and now we want stable self-dual objects. A self-dual object is a D-brane that descends to the orientifold theory. This is N along with an isomorphism $\psi_N : N \to S(N)$ which is symmetric, $S(\psi_N) \circ \theta_N = \psi_N$.

Suppose σ is the identity. Then this S is just derived dual. Then any orthogonal or symplectic vector bundle gives an example, because an orthogonal or symplectic bundle is a vector bundle with an isomorphism with its dual with appropriate compatibility with its symmetry. So this passes from vector bundles to G-bundles and this is an explanation for why mathematicians might be interested.

So what are some examples on quiver categories? We need to fix some data, an analogue of the involution. The first part is a contravariant involution $\sigma : Q \to Q$ of the quiver. So then $i \xrightarrow{\alpha} j$ and under the involution $\sigma(j) \xrightarrow{\sigma(\alpha)} \sigma(i)$ and we also fix combinatorial data $s : Q_0 \to \pm 1$ and $\tau : Q_1 \to \pm 1$. There's a compatibility, $s_i = s_{\sigma(i)}$ and $\tau_a \tau_{\sigma(\alpha)} = s_i s_j$ if $i \xrightarrow{\alpha} j$.

Now we can define $S : \mathrm{Rep}_{\mathbb{C}}(Q) \to \mathrm{Rep}_{\mathbb{C}}(Q)$ where $(U, u) \mapsto (S(U), S(u))$ where $S(U)_i = U^\vee_{\sigma(i)}$ and $S(u)_\alpha = \tau_\alpha u^\vee_{\sigma(\alpha)}$. Then $\theta_u = \bigoplus s_i ev_{u_i}$.

Let's look at examples, let's take the quiver with a single loop and single node. This is the adjoint quotient map, $R_d/GL_d = gl_d/GL_d$. There's only one involution on this quiver, fixing both the node and the edge. We have four choices, $s = \pm 1$ and $\tau = \pm 1$. So if $s = +1$ and $\tau = -1$ then $\mathfrak{R}^\sigma/[\textit{unintelligible}]$ and this is so_e/O_e, for $\tau = +1$ this is $\mathrm{Sym}^2 \mathbb{C}^e/O_e$. If we take $\sigma = -1$ we get the symplectic group.

In the last few minutes, what can we do with this? We should emulate the ending point and forget the starting point. It's hard to compute numerically in the G-case, and this is a technique to do so.

So consider the vector space which takes the role of the Hall algebra. Take

$$\mathcal{M}_Q = \bigoplus_{e \in \Lambda_Q^{\sigma,+}} H^{\cdot - \dim M_e^\sigma}(M_e^\sigma).$$

This is a $\Lambda_Q^{\sigma,+} \wedge \mathbb{Z}$-graded vector space. But there's no interesting Abelian structure for self-dual objects. The Hall object construction was entirely based on having interesting extensions.

Instead we use a slightly different construction. Suppose U is a self-dual representation, $U \hookrightarrow (N, \psi_N)$ is isotropic, that is, so that $U \hookrightarrow N \to S(N) \to S(U)$ is zero. Then $N//U = U^\perp/U$ has a canonical self-dual structure. So we can think of this as saying we don't have short exact sequences, but we have mixed short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r] & N \arrow[r, dotted] & P \arrow[r] & 0 \end{tikzcd}$$

where N and P are self dual objects.

From this we can correct our naive guess that we want an algebra and get a module. We modify $M^\sigma_{d,e}$, we get stacks of flags of isotropic representations, $U \subset N$ isotropic, with fixed dimension vectors $\underline{\dim}(U) = d$ and $\underline{\dim}(N) = d + \sigma(d)$, and we again get a diagram

$$\begin{tikzcd} & M^\sigma_{d,e} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M_d \times M^\sigma_e & & M^\sigma_{d+\sigma(d)+e} \end{tikzcd}$$

and this gives $\mathcal{M}_Q$ the structure of a $\Lambda_Q^{-1} \times \mathbb{Z}$-graded $\mathcal{H}_Q$ module, and the proposal is that $\mathcal{M}_Q$ is the "BPS module" for orientifold string theory. So I'll give some evidence for this tomorrow.

11. Kyoji Saito: An introduction to primitive form theory III

Okay, thank you very much. This is the third part of my lecture. In the first part, I explained somehow, revisited elliptic integrals, found elliptic integrals of the first kind. In the second lecture, we explained some Lie theory, where the Kostant–Kirillov form played some primitive role, and now we'll further generalize these and call what we get a primitive form.

There are many options but I'll stick to isolated critical points for [unintelligible]. There are some notes by A. Takahashi and myself, an introduction on this and its connection to Frobenius manifold structure.

If some of you who know this are in the audience, I apologize, but I think that many of you don't know. I'll skip the applications because of time, but what are some applications? One is to use the integrals of the periods of the integral form to construct certain vertex operator algebras. Then you can get an integrable hierarchy for all kinds of singularities. This is going on these days, let me mention Milanov.

There are a lot of dualities related to primitive forms. One is Landau–Ginzburg-Landau–Ginzburg mirror symmetry, related to FJRW, done by Si Li and his collaborators. I'll explain partition functions today, and this will be related to some FJRW partition function.

There's also Landau–Ginzburg Calabi–Yau duality, and then the other side is Gromov–Witten theory. The third duality is to topological conformal field theory, that's Dijkgraaf and Verlinde in the early 1990s. The Russians are trying to clarify this for much further cases.

Another application is that integration of primitive forms satisfies some Picard–Fuchs equation, and this gives some period map. Inverting [unintelligible]should lead us to automorphic forms but this is not well-studied except in the versions I discussed in the first talk. I don't think I can talk about any of this today, and I'll focus on what are primitive forms.

Today is rather, in some sense boring, monotonic, I'll just explain, monotonically, the story.

The starting point is the following object. You'll look at $f : \mathbb{C}^{n+1}, 0 \to \mathbb{C}, 0$, holomorphic, and assume that 0 is an isolated critical point of f. This means that if you solve the equation $\frac{\partial f}{\partial x_0} = \cdots = \frac{\partial f}{\partial x_n} = 0$ the solution locus has 0 as an isolated point.

This uses some standard technique in complex geometry, by using Hilbert nullenstatz, then $\dim \mathcal{O}_{\mathbb{C}^{n+1},0} / (\frac{\partial f}{\partial x_0}, \dots \frac{\partial f}{\partial x_n}) < \infty$.

I don't have time to discuss details, but the topology, this is also complex geometry notation, we only care about a neighborhood of the origin but don't care which one. If we choose representatives X in $\mathbb{C}^{n+1}$ and S in $\mathbb{C}$, then $f : X - f^{-1}(0) \to S \backslash \{0\}$ is a locally trivial fibration so that only the middle homology group exists, $H_n(f^{-1}(t) = X_t, \mathbb{Z}) = \mathbb{Z}^\mu$ where $\mu = \dim_{\mathbb{C}} J_f$.

Let's go to some conceptual universal unfolding due to Thom. He was studying some singularities in the 60s, and he saw that you can understand more carefully if you unfold things, so study $F(\underline{x}, \underline{t}) : \mathbb{C}^{n+1} \times \mathbb{C}^m \to \mathbb{C}$ so that $F(x, 0) = f(x)$. This is called an *unfolding*. He called an unfolding *universal* if the following holds. It's a little bit technical, but for later use, let me, I called these X and S. Let me call this total space X and S the space $\mathbb{C}^m, 0$. I also introduce C_F to be the relative critical set, the common critical locus of $\frac{\partial F}{\partial x_0} = \cdots = \frac{\partial F}{\partial x_n} = 0$. The dimension of C_F is, this is an m-dimensional space. Then $C_F^m \to \mathbb{C}^m$, this is a finite cover.

Consider $\mathcal{O}_C$, the ring of holomorphic functions on C_F. Then $\pi_* \mathcal{O}_C$ is well-defined as a sheaf, a coherent sheaf on S.

Now I want to introduce a map $\mathrm{Der}_S \to \pi_* \mathcal{O}_C$. Since we have X a product space, you can easily lift vector fields by having the other part be arbitrary. So $v \mapsto \hat{v}|_C$, and Thom's definition, from this point of view, if this is an $\mathcal{O}_S$ isomorphism, then this is universal.

You can identify universal ones, they're all isomorphic. I'll introduce much simpler ones that most people use. Consider $F(\underline{x}, \underline{t}) = f(x) + \sum t_i \phi_i(x)$ where ϕ_i represents a $\mathbb{C}$-basis of the Jacobi ring J_f. It's not hard to see that this is a universal unfolding in this sense.

From now on I'll fix just one universal unfolding. Then you see clearly that your tangent space, the module of sections of the tangent bundle is identified with the ring of functions on the critical set. The tangent space then has a ring structure. In that way, the universal unfolding has this structure, the tangent space has a ring structure, the so-called Frobenius structure.

That's the first step. Before going further, by this identification, there is a particular element in $\pi_* \mathcal{O}_C$, this contains 1, this should go to something in Der_S. Let me (unfortunately) call this δ_W, the *primitive vector field*. This is not so nice. In yesterday's talk, this was denoted by D. Another element in this ring is $F|_C$. This should correspond to something in the derivations, we'll call this E, the *Euler vector field*; this showed up yesterday. Why is the primitive field called primitive? It will later be identified with primitive forms. Maybe this is too complicated.

Next let us introduce, since the lack of time, I may not have time to discuss the topology. I at least mentioned the Milnor fibration. If I look at the family of unfoldings, the setting for $t = 0$, let us consider $X|_{t=0}$, the function at some isolated critical point, moving t to something nonzero, you get X_t may have a decomposed critical point, and some critical values may split. Anyhow, what you want to say is, at $t = 0$ you have the Milnor fiber with vanishing cycles, they survive away from $t = 0$ and you want to study these by the de Rham cohomology group.

I don't want to go in historical order. So let us consider Ω_{X_S}, relative differential forms, so $\Omega^p_{X_S} = \Omega^p_X / \sum dt_i \wedge \Omega^{p-1}_X$, now I don't want to justify it but yesterday we saw that we're tensoring with a variable D and this appears again in this story now. We have two differential operators. One is the regular de Rham differential. The other one is the wedge product with dF. Then you'll consider

$$\Omega^{\cdot}_{X/S}[[\delta_W^{-1}]][\delta_W], \delta_w^{-1} d_{X/S} + dF.$$

I should have said [unintelligible]from the beginning.

But let me introduce the module

$$\mathcal{H}_f = \mathbb{R}^{n+1}\pi_*(\Omega^{\cdot}_{X/S}[[\delta_W^{-1}]][\delta_W], \delta_W^{-1} d_{X/S} + dF)$$

You want to prove that it contains $\mathcal{H}^{(k)} = \mathbb{R}^{n+1}\pi_*(\Omega^{\cdot}_{X/S}[[\delta_W^{-1}]]\delta_W^k, \delta_W^{-1} d_{X/S} + dF)$, so you get $0 \subset \cdots \subset \mathcal{H}_F^{(0)} \subset \mathcal{H}_F^{(1)} \subset \cdots \subset \mathcal{H}_F$.

If you write down the module explicitly, it's not hard to check that $0 \to \mathcal{H}^{(-1)} \to \mathcal{H}_F^{(0)} \to \Omega_F \to 0$ is the same as Ω_F, well Ω^{n+1}_X, the ring of holomorphic functions on C with volume 1, so $\Omega_F = \pi_* \Omega^{n+1}_{X/S}$. This module has strong structure, a *residue pairing*; namely we have the following thing, $J : \Omega_F \times \Omega_F \to \mathcal{O}_S$ where if you have $\omega_i = \varphi_i() dx_0 \cdots dx_n$, then you take the residue of

$$\left(\begin{array}{c} \varphi_1 \varphi_2 dx_0 \wedge \cdots \wedge dx \\ \frac{\partial F}{\partial x_0} \cdots \frac{\partial F}{\partial x_n} \end{array} \right)$$

and this is in $\mathcal{O}_S$. This is $\mathcal{O}_S$-bilinear, symmetric, and nondegenerate. This is another important ingredient used to produce the structure. We'll also consider the Gauß–Manin connection. There's some process to get, you'll consider holomorphic vector fields acting on the base space. Let $v \in \mathrm{Der}_S$. Define $\nabla_v : \mathcal{H}_F \to \mathcal{H}_F$ where if you have $\zeta = \phi dx_0 \wedge \cdots \wedge dx_n = [(F\phi + \delta_W v(F)\phi) dx_0 \wedge \cdots \wedge dx_n]$. So

$$\nabla_{\frac{d}{d\delta_W}} \zeta = [(F\phi + \frac{\partial \phi}{\partial \delta_W}) dx_0 \wedge \cdots \wedge dx_n].$$

So ∇_v, *Griffiths transversality*, goes $\mathcal{H}_F^{-k} \to \mathcal{H}_F^{-k+1}$ while $\nabla_{\frac{d}{d\delta_W}}$ preserves the filtration.

Another important ingredient is the *higher residue pairing*, which gives a kind of *polarization*. Let me just write down the result.

Theorem 11.1. *There exists a unique $\mathcal{O}_S$-bilinear map $K_F : \mathcal{H}_F \times \mathcal{H}_F \to \mathcal{O}_S[[\delta_W^{-1}]][\delta_W]$ satisfying the following five axioms:*

(1) *Symmetry. For all ω_1 and ω_2,*

$$K_F(\omega_1, \omega_2) = K_F(\omega_2, \omega_1)^*$$

where $*$ is the involution which takes δ_W to $-\delta_W$.

(2) *for any $P \in \mathcal{O}_S[[\delta_W^{-1}]][\delta_W]$,*

$$PK_F(\omega_1, \omega_2) = K_F(\omega_1, P\omega_2)$$

(3) *Consider $K_F : \mathcal{H}_F^0 \otimes \mathcal{H}_F^{(0)} \to \delta_W^{-n-1}\mathcal{O}_S[[\delta_W^{-1}]]$, we have commutativity of*

$$\begin{CD} \mathcal{H}_F^{(0)} \times \mathcal{H}_F^{(0)} @>{K_F}>> \delta_W^{-n-1}\mathcal{O}_S[[\delta_W^{-1}]] \\ @VVV \\ \Omega_F \times \Omega_F @>>{J_F}> \mathcal{O}_S, \end{CD}$$

that is, we recover the classical pairing.

(4) *Compatibility with Gauß–Manin 1.*

$$\frac{d}{d\delta_W} K_F(\omega_1, \omega_2) = K_F(\Delta_{\frac{d}{d\delta_W}} \omega_1, \omega_2) + K_F(\omega_1, \Delta_{\frac{d}{d\delta_W}} \omega_2)$$

(5) *Compatibility with Gauß–Manin 2. For v in* Der_S*, we have* $vK_F(\omega_1, \omega_2) = K_F(\nabla_v \omega_1, \omega_2) + K_F(\omega_1, \Delta_v \omega_2)$

Definition 11.1. $\zeta^{(0)}$ in $\Gamma(S, \mathcal{H}_F^{(0)})$ is called a *primitive form* if (let $\delta_W^k \zeta^{(0)}$ be $\zeta^{(k)}$ for $k \in \mathbb{Z}$.)

(1) (primitivity) $\zeta^{(0)}$ induces an $\mathcal{O}_S$-isomorphism

$$\mathrm{Der}_S \otimes_{\mathcal{O}_S} \mathcal{O}_S[[\delta_W^{-1}]] \cong \mathcal{H}_F^0$$

where

$$\sum v_\ell \delta_W^\ell \mapsto \sum \nabla_{v_\ell} \zeta^{(-1)} \delta_W^{-\ell}$$

(2) For $k \geq 1$ and v and v' in Der_S,

$$K^{(k)}(\nabla_v \zeta^{(-1)}, \nabla_{v'} \zeta(-1)) = 0$$

(3) There is a constant r, some homogeneity condition, so that

$$\nabla_{\frac{d}{d\delta_W}} \zeta^{(0)} = \nabla_E \zeta^{(-1)} - r\zeta^{(-1)}$$

(4) For $k \geq 2$ and u, v, and w in Der_S, we have $K^{(k)}(\nabla_u \nabla_v \zeta^{(-2)}, \nabla_w \zeta^{(-1)}) = 0$
(5) For $k \geq 2$ and u and v, in Der_S,

$$K^{(k)}(\nabla_{\frac{\partial}{\partial \delta_W}} \nabla_u \zeta^{(-1)}, \nabla_v \zeta^{(-1)}) = 0$$

This looks technical but if you remember your elliptic integrals and the Kostant–Kirillov forms, you can see that all of these conditions are satisfied. Then let us call this object a primitive form.

The definition is okay, but whether there exist other examples, well, that's another question. So some examples are

(1) elliptic integrals and Kostant–Kirillov form.
(2) $f = x^3 + y^3 + z^3 + \lambda xyz$ (called $E_6^{(1,1)}$) or $x^4 + y^4 + z^2 + \lambda xyz$ (called $E_7^{1,1}$) or $x^6 + y^3 + z^2 + \lambda xyz$ (called $E_8^{(1,1)}$), in these examples,

$$\zeta^{(0)} = \frac{dxdydz}{\int \frac{dx}{\sqrt{\text{something}}}}$$

and this contains some highly transcendental structure inside.
(3) M. Saito showed the existence for all isolated hypersurface singularities. But this is just a local existence theorem, and we described on the first and second day, the description yesterday is global. We don't know how far primitive forms can go globally. But in general these are not unique, but in the elliptic singularity case, you have a choice of cycle to integrate over, the ambiguity is one parameter. So in the second example this depends on one parameter. With Si Li and Changzheng Li, we studied [unintelligible], and this distinction I skip. Unfortunately I have no more time. Only I will say three things.

One is that a consequence of this structure, you get the Frobenius (or flat) structure, $(S, J, *, \delta_W, E)$, and this leads to the potential, there exists a function, S has some coordinates, flat coordinates $t_1, \ldots t_\mu$, and with respect to them, let u, v, w be in $\{\frac{\partial}{\partial t_0}, \ldots, \frac{\partial}{\partial t_\mu}\}$, then there is a function $\mathcal{F}$, called the *prepotential* so $uvw\mathcal{F} = J(u * v, w)$. Then another is that if you write down the covariant differentiation of ζ by u and v, it terminates in the second stage. We saw yesterday in the Kostant–Kirillov example, this structure. Then the period satisfies a second order equation. Your period map is completely controlled by a holonomic system, not yet done but this should lead to a quite rich period map theory.

12. Young-Hoon Kiem: Categorification of Donaldson-Thomas invariants II

So last time, I introduced the notion of a critical virtual manifold, an analytic space X with an open covering X_α, and each set in the covering is the critical set of a function f_α from V_α to $\mathbb{C}$. There is a compatibility

$$\begin{tikzcd}
& V_{\alpha\beta} \arrow[r, hook] \arrow[dd, "\varphi_{\alpha\beta}"] & V_\alpha \arrow[dr, "f_\alpha"] & \\
X_{\alpha\beta} \arrow[ur, hook, "\text{closed}"] \arrow[dr, hook, "\text{closed}"'] & & & \mathbb{C} \\
& V_{\beta\alpha} \arrow[r, hook] & V_\beta \arrow[ur, "f_\beta"'] &
\end{tikzcd}$$

and this is orientable if $\xi \in H^2(X, \mathbb{Z}_2)$ is zero. I gave you a theorem that if X is orientable, you can glue perverse sheaves and mixed Hodge modules. By M. Saito's theorem, if you have a perverse sheaf underlying a mixed Hodge module, then the hypercohomology satisfies many nice properties like the relative hard Lefschetz property and others.

Tomorrow I'll use these nice properties to give an application of the existence of the perverse sheaf P, the gluing of the perverse sheaves of vanishing cycles for X_α. Then $\chi(\mathbb{H}^*(X, P)) = \chi_v(X)$, weighted by the Behrend function, because restricting to a point you get [unintelligible]the Behrend function. This is the Donaldson–Thomas invariant of X when X is compact.

Today I want to think about moduli spaces of sheaves on Calabi–Yau three-folds. The goal is to show that the moduli of simple sheaves are always critical virtual manifolds.

How do we show this? We use gauge theory. Not physical but mathematical gauge theory.

Let me give you a background on gauge theory on Calabi–Yau three-folds. Now Y is a smooth projective variety over $\mathbb{C}$, it's Calabi–Yau, so that $K_Y = \Omega^3_Y \cong \mathcal{O}_Y$, and I fix Ω a nonzero holomorphic $(3,0)$-form in $H^0(Y, \Omega^3_Y)$.

I have the stack of simple sheaves $\mathrm{Sh}^c_{\mathrm{si}}$, where simple means that $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) = \mathbb{C}\mathrm{id}$. Then I have the stack of simple vector bundles $\mathcal{V}^c_{\mathrm{si}}$ sitting inside tihs open, and I'll think mainly about open analytic subspaces, within that.

The only thing I want from X is that the dimension of the tangent space is bounded above. So first of all, since we fixed the topological type, the underlying complex vector bundle E on Y is always fixed. Then $\mathcal{A}^{0,q}(E)$ denotes smooth sections of $E \otimes \wedge^{0,q} T^*_y$, these are E-valued $(0,q)$-forms. This is an infinite dimensional vector space, sections of this vector bundle.

The gauge group $\mathcal{G}$ is $C^\infty(\mathrm{Aut}(E))$, a principal bundle over Y, and let me introduce the semi-connection, a $\mathbb{C}$-linear map $\bar\partial : \mathcal{A}^{0,0}E \to \mathcal{A}^{0,1}E$, and this should satisfy the Leibniz rule $\bar\partial(s \otimes f) = f\bar\partial s + s \otimes \bar\partial f$. Of course this extends to $\mathcal{A}^{0,q}(E) \to \mathcal{A}^{0,q+1}(E)$ in the obvious way, $\bar\partial(s \otimes \alpha) = \bar\partial s \otimes \alpha + s \otimes \bar\partial\alpha$.

Next let me talk about curvature. So $F^{0,2}_{\bar\partial} = \bar\partial^2 : \mathcal{A}^{0,0}(E) \to \mathcal{A}^{0,2}(E)$, and $F^{0,2}_{\bar\partial} \in \mathcal{A}^{0,2}(\mathrm{End}\, E)$. It's a standard exercise (the Bianchi identity) that $\bar\partial F^{0,2}_{\bar\partial} = 0$. We say $\bar\partial$ is *integrable* if $F^{0,2}_{\bar\partial} = 0$.

Then $\mathcal{E}(U) = \{s \in \mathbb{C}^\infty(E) | \bar\partial s = 0\}$. this turns out to be a locally free sheaf of $\mathcal{O}_X$-modules, a holomorphic vector bundle on Y.

Then two integrable connections $\bar\partial$ and $\bar\partial'$ define isomorphic holomorphic vector bundles if and only if they are related by, if $\bar\partial' = \bar\partial \cdot g$ for some $g \in \mathcal{G}$.

So we can give a gauge theoretic description of the moduli space of vector bundles. Let $\mathcal{A}$ be the space of semi-connections on E, and we take the Sobolev completion, we'll eventually apply elliptic differential operator theory. Let's not talk about this because we're not analysts, but trust me, that can be handled. In that we look at $\mathcal{A}_{\text{si}}$, the simple semiconnections, which have the minimal possible stabilizer, $\mathcal{G}_{\bar\partial} = \mathbb{C}^*\text{id}$. Inside there we consider $\mathcal{A}_{\text{si}}^{\text{int}}$, the space of simple and integrable semiconnections. Then $\mathcal{A}_{\text{si}}$ maps to $\mathcal{B}_{\text{si}} = \mathcal{A}_{\text{si}}/\mathcal{G}$, a Banach manifold. Inside there, the quotient of $\mathcal{A}_{\text{si}}^{\text{int}}$ is $\mathcal{V}_{\text{si}}$, the space of simple holomorphic vector bundles, and this is a finite dimensional space.

To make things explicit, if we fix $\bar\partial$, then we can write $\mathcal{A} = \bar\partial + \mathcal{A}^{0,1}(\operatorname{End} E)$, so if I pick something like $\bar\partial + a$, I'll write this sometimes as $\bar\partial_a$. With this notation, the curvature of $\bar\partial a$ is

$$F^{0,1}_{\bar\partial a} = (\bar\partial + a)(\bar\partial + a) = \bar\partial a + a \wedge a,$$

this is the curvature.

What is the gauge group action?

$$(\bar\partial + a)g = g^{-1}(\bar\partial + a)g = \bar\partial + (g^{-1}dg + g^{-1}ag)$$

There's also the Laplacian,

$$\Delta^{0,q}_{\bar\partial} = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial : \mathcal{A}^{0,q}(\operatorname{End} E) \to \mathcal{A}^{0,q}(\operatorname{End} E)$$

and

$$\Delta^{-1}_{\bar\partial}(0)^{0,q} = H^q(\operatorname{End}\mathcal{E}) = \operatorname{Ext}^q(\mathcal{E},\mathcal{E})$$

the harmonic forms, where $\mathcal{E}$ is the holomorphic vector bundle defined by $\bar\partial$.

Now the holomorphic Chern–Simons function is

$$CS : \mathcal{A}_{\text{si}} \to \mathbb{C}$$

with

$$CS(\bar\partial + a) = \frac{1}{8\pi^2}\int_Y \operatorname{tr}(\bar\partial a \wedge a + \frac{2}{3} a \wedge a \wedge a) \wedge \Omega$$

and $\delta CS(\bar\partial + a)(b) = CS(\bar\partial + a + b) - CS(\bar\partial + a)|_{b=0}$ which is

$$\frac{1}{8\pi^2}\int \operatorname{tr}(F^{0,2}_{\bar\partial a} \wedge b) \wedge \Omega = 0$$

which is true if and only if $F^{0,2}_{\bar\partial a} = 0$

So we have

$$\begin{tikzcd}
\operatorname{Crit}(CS) = \mathcal{A}_{\text{si}}^{\text{int}} \arrow[r, hook] \arrow[d] & \mathcal{A}_{\text{si}} \arrow[r, "CS"] \arrow[d, "\pi"] & \mathbb{C} \\
\operatorname{Crit}(\overline{CS}) = \mathcal{V}_{\text{si}} \arrow[r, hook] & \mathcal{B}_{\text{si}} \arrow[ur, "\overline{CS}"'] &
\end{tikzcd}$$

and if we're okay with infinite dimensional complex manifolds like $\mathcal{B}_{\text{si}}$ then we're fine. But maybe we're not.

Definition 12.1. Let $r \ge \dim T_x\mathcal{V}_{\text{si}} = \dim \operatorname{Ext}^1_Y(\mathcal{E},\mathcal{E})$ with $x = \bar\partial \in \mathcal{A}_{\text{si}}^{\text{int}}$. A *Chern–Simons chart* of dimension r at x is an r-dimensional submanifold V of $\mathcal{A}_{\text{si}}$ such that $x \in V$ and letting $f = CS|_V$, we have $\operatorname{Crit}(f)$ an open neighborhood of $\pi(x)$ in $\mathcal{V}_{\text{si}}$.

Now we have to show the existence of a Chern–Simons chart. This is just a definition. There is a well-known choice due to Miyajima–Joyce–Song. We have $\bar\partial \in \mathcal{A}_{\mathrm{si}}^{\mathrm{int}}$, with $\epsilon > 0$ small. Then let $V_0 = \{\bar\partial + a \| \|a\| < \epsilon, \bar\partial^* a = 0, \text{ with } \bar\partial^*(\bar\partial a + a \wedge a) = 0\}$. I want the curvature to be zero eventually, so I put $\bar\partial^*$ on the curvature. It turns out that this is a Chern–Simons chart of minimal dimension, meaning that $V_0 = \dim T_x \mathcal{V}_{\mathrm{si}}$. We are not happy with this, we cannot get a critical virtual manifold structure here because the dimension varies from point to point, this is not good, for a critical virtual manifold, this should be fixed. To get uniform dimension, you cannot decrease the dimension, there's a definite lower bound. You can increase the dimension of the smaller chart, increase the number of coordinates. Suppose we have a function f of $x_1, \dots, x_n$, then we can just add x_{n+1}^2, and call that $\tilde f(x_1, \dots, x_{n+1})$, and the critical ponits of f are the same as the critical points of $\tilde f$.

The perverse sheaf of vanishing cycles also doesn't change, this is the Sebastiani–Thom isomorphism. There's also a mixed Hodge module version.

You can't do this in a stupid way, you have to do this in a very careful way. The way to increase the dimension that doesn't cause any trouble, you have to increase it in a controlled fashion:

Definition 12.2.

$$Q(a,b) = \frac{1}{8\pi^2} \int_Y \mathrm{tr}(\bar\partial a \wedge b) \wedge \Omega.$$

An *r-dimensional Chern–Simons framing* at $x = \bar\partial \in \mathcal{A}_{\mathrm{si}}^{\mathrm{int}}$ is an r-dimensional subspace Ξ in $T_x \mathcal{A}_{\mathrm{si}}$ such that

(1) $\Xi \supset \Delta_{\bar\partial}^{-1}(0)$
(2) $Q|_{\Xi/\Delta_x^{-1}(0)}$ is nondegenerate.

Lemma 12.1. *For any $r \geq \dim T_x \mathcal{V}_{\mathrm{si}}$, there is an r-dimensional Chern–Simons framing.*

This is because we have a nondegenerate quadratic form, we just extend.

Theorem 12.1. *Let $r \geq \dim T_x \mathcal{V}_{\mathrm{si}}$, let Ξ be an r-dimensional Chern–Simons framing at x. Let $V = V(x, h, \Xi)$ be the space*

$$\{\bar\partial + a \| \|a\| < \epsilon, \bar\partial^* a = 0, \bar\partial^*(\bar\partial a + a \wedge a) \in \Delta_{\bar\partial}(\Xi)\}.$$

This is a Chern–Simons chart of dimension r.

So I can always find a chart that is locally the critical locus of a function on a complex manifold. But I need to get the compatibility condition on $\varphi_{\alpha\beta}$. But that can be guaranteed.

Proposition 12.1. *This Chern–Simons chart doesn't really depend on the choices of x, h, and Ξ so $V(x,h,\Xi) = V(x', h', \Xi')$ locally at x if $\pi(x) = \pi(x')$, we needed to specify the Hermitian metrics h or h' to get the adjoint of $\bar\partial$, and Ξ and Ξ' are Chern–Simons framings.*

Proposition 12.2. *If you have a Chern–Simons chart $V = V(x, h, \Xi)$, and inside you have* Crit(f)*, then for every point y in that critical locus there exists a Chern–Simons framing Ξ' so that $V(y, h', \Xi')$ is locally equivalent to $V(x, h, \Xi)$ locally at x.*

Then after refining your covering a few times, it's like an analysis exercise to see that these glue together.

Theorem 12.2. *For $X \subset \mathcal{V}_{\mathrm{si}}$ open, this is a critical virtual manifold.*

So far I assumed X is the moduli of vector bundles.

Remark 12.1. By the Seidel–Thomas twist, for $X \subset \mathrm{Sh}^{c}_{\mathrm{si}}$, open, you can always find some $\mathcal{V}^{\tilde{c}}_{\mathrm{si}}$, maybe a bigger rank or something

Then I should talk about the orientation issue here.

Theorem 12.3. *If you can find $\sqrt{\det R\pi_* R\operatorname{Hom}(\mathcal{E},\mathcal{E})} \in \operatorname{Pic}(X)$ then the critical virtual manifold is orientable.*

Then there is a nice argument of Okounkov that says

Proposition 12.3. *(Okounkov) If there is a universal family on $X \times Y$ then there is a square root*

Then there is a critical virtual manifold, and thus a perverse sheaf, and a Hodge module, and then all these nice properties.

13. January 9: Qin Li: 1D Chern–Simons theory and algebraic index theorem

I would like to thank the organizers for the invitation. I was going to talk about 2D B-model theory, and I thought that this is similar, this is a $1D$ theory but it has many of the same features and you can still see some interesting results.

This is joint work with Ryan Grady and Si Li.

I'll say later why I call it $1D$ Chern–Simons theory. It's a $1D$ sigma model with target a symplectic manifold (M,ω) and the interesting thing is that, I'll explain the BV quantization of this theory, and how the deformation quantization of symplectic manifolds and the algebraic index theorem is encoded in the BV quantization of this theory.

So let me start from some classical, twenty years ago, a construction of Fedosov. Given this symplectic manifold, Fedosov considered the following geometric object, called the Weyl bundle,

$$\mathcal{W}(M) = \prod_{k\geq 0} \operatorname{Sym}^k(TM^\vee)[[\hbar]]$$

and there's actually an interesting product, the Weyl product, that comes from the fiberwise Moyal product, so then $\mathcal{W}(M)$ is an alegbar bundle. Locally

$$a \circ b = \sum_{k=0}^{\infty} (\frac{\hbar}{2})^k \frac{1}{k!} \omega^{i_1,j_1}(x) \cdots \omega^{i_k,j_k}(x) \frac{\partial^k a}{\partial y^{i_1}\cdots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1}\cdots \partial y^{j_k}}$$

So the way he constructed his famous [unintelligible]is, you choose a symplectic connection, and modify it to be a flat connection by adding a bracket with γ, the bracket being associated to the Weyl product,

$$\nabla + \frac{1}{\hbar}[\gamma, \quad]_*$$

and this is called Abelian if it squares to zero. Here $\gamma \in \mathcal{A}^1(W)$, it's a one-form valued in the Weyl bundle.

The theorem of Fedosov is

Theorem 13.1. *(Fedosov) Given a sequence $\{\omega_k\}_{k\geq 1}$ of closed 2-forms on M theree exists a unique Abelian connection $\nabla + + \frac{1}{\hbar}[\gamma, \quad]_*$ such that*

(1)
$$\gamma = \sum_{i,j} \omega_{ij} y^i dx^j + r$$
for some $r \in \mathcal{A}^1(\mathcal{W})$ and such that $\delta^{-1}(\gamma) = 0$

(2)
$$\nabla\gamma + \frac{1}{2}[\gamma,\gamma]_* + R_\nabla = -\omega + \sum \hbar^k \omega_k.$$
Here R is defined in terms of the curvature of the symplectic connection.

(Fedosov)

Theorem 13.2. *Flat sections of $\mathcal{W}$ is isomorphic to $C^\infty(M)[[\hbar]]$*

The way Fedosov defines this deformation quantization is as follows. Suppose you are given f and g, then you can make them flat sections of this Weyl bundle, then the Weyl product is compatible with the Abelian connection, and this is still a flat section, so
$$\sigma^{-1}(\sigma(f) \circ \sigma(g)) \in C^\infty(M)[[\hbar]]$$
and we can let this define a star product $f * g$, so this is, if we expand in $\hbar$,
$$f * g = fg + \frac{\hbar}{2}\{f,g\} + O(\hbar^2)$$
So something which does not appear in Fedosov's work is the BV bundle, and this is motivated from quantum field theory or quantum mechanics.

Definition 13.1. The BV bundle of (M,ω) is
$$\Omega^{-*}_{\mathcal{W}} = \widehat{\mathrm{Sym}}(TM^\vee) \otimes \wedge^{-*}(TM^\vee)[[\hbar]]$$

and later after explaining $1D$ field theory and BV quantization I'll explain how this can be seen.

There are certain operators defined on this BV bundle, for instance $d_\omega : \Omega^*_{\mathcal{W}} \to \Omega^{-*-1}_{\mathcal{W}}$. There is also the BV operator $\Omega^*_{\mathcal{W}} \to \Omega^{-(*-1)}_{\mathcal{W}}$ which I will explain later.

Let me turn to one dimensional Chern–Simons theory. This theory describes, it's not the whole sigma model, but the sigma model from S^1 to a symplectic manifold in a formal neighborhood of a constant map. There's a supersymmetric localization, which is what lets us restrict to this neighborhood.

The space of fields is $\mathcal{E} = \mathcal{A}_{S^1} \otimes_{\mathbb{C}} (\mathcal{A}_M \otimes T_M)$, and this is actually the way we'll describe a formal neighborhood of constant maps from S^1 to M. We'll sometimes write the second term in the tensor product as $g_M[1]$, which is an L_∞ algebra (I won't get into this). This is where my "one dimensional" comes from.

We will let $\mathcal{E}^\vee$ denote the $\mathcal{A}_m$-linear dual of $\mathcal{E}$, so if we spell this out it's
$$\mathrm{Hom}_{\mathcal{A}_m}(\mathcal{E}, \mathcal{A}_m)$$
We can also consider functionals on this vector space
$$\mathcal{O}(\mathcal{E}) = \prod_{k\geq 0} \mathrm{Sym}^k(\mathcal{E}^\vee)$$
and the action functional, there's a natural map $\rho : \mathcal{A}_M(\mathcal{W}) \to \mathcal{O}_{\mathrm{loc}}(\mathcal{E})$, where *local* functionals, this is related to S^1, my spacetime. This is defined in the following

way. Let I_k be a section of $\mathrm{Sym}^k)T_M^\vee$, the degree k-component of the Weyl bundle, and this map is

$$\rho(I_k) = \alpha \mapsto \int_{S^1} I_k(\alpha, \dots, \alpha).$$

The action functional we will consider is

$$S(\alpha) = \int_{S^1} \omega(d_{S^1}\alpha + \nabla\alpha, \alpha) + \rho(I)(\alpha)$$

for some $I \in \mathcal{A}_M^1(\mathcal{W})$.

This is a functional, we're describing a quantum field theory, the next thing we want to talk about is BV quantization.

We first fix a flat metric on S^1 and there is an Hodge Laplacian $D = [d_{S^1}, d_{S^1}^*]$, and we will let $\mathbb{K}_t$ in $\mathrm{Sym}^2 \mathcal{E}$ denote the kernel of the operator $e^{-tD} : \mathcal{E} \to \mathcal{E}$

Definition 13.2. The scale L BV Laplacian on $\mathcal{O}(\mathcal{E})$, $\Delta_L : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$, is the second order differential operator given by the contraction with $\mathbb{K}_L$, which lives in $\mathrm{Sym}^2(\mathcal{E})$.

We can also define the effective propagator as

$$\mathbb{P}_\epsilon^L = \int_\epsilon^L (d_{S^1}^* \otimes 1)(\mathbb{K}_t) dt$$

and then we can define the renormalization group flow, and the renormalization operator is $W(\mathbb{P}_\epsilon^L, \quad) : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E})$, and we can write this as a sum over connected graphs

$$\sum_\gamma \frac{\hbar^{g(\gamma)}}{\mathrm{Aut}\,\gamma} W_\gamma(\mathbb{P}_\epsilon^L, \quad)$$

and the Feynman weight associated to the graph is as follows. Suppose you have a graph with internal edges, vertices, and external edges. The external edges are labeled by the inputs, and on the vertices you put the functionals and on the internal edges the effective propagator. You can contract the propagators with the functionals I to get the Feynman weights.

This is the renormalization group flow. The reason we want to consider these, we want to consider what is a quantization of our theory.

Definition 13.3. A family of functionals $I[L] \in \mathcal{O}(\mathcal{E})$ parameterized by $L > 0$ is said to be a perturbative *quantization* of $\lim_{L\to 0} I[L] \pmod{\hbar}$ if

(1) $I[L] = W(\mathbb{P}_\epsilon^L, I[E])$ and
(2) the quantum master equation is satisfied:

$$(Q + \hbar\Delta_L)e^{I[L]/\hbar} = 0$$

Let $L = \infty$. Then we can restrict to harmonic fields in $\mathcal{E}$. The functionals on the harmonic fields motivate the *BV* bundle. There is the symmetric part $\widehat{\mathrm{Sym}}(T_M^\vee)$ and the wedge part $\wedge^*(T_M^\vee)$, you have the symmetric part on the 0-forms and the part on the one-forms.

Let me say something after the quantization procedure. You can consider local quantum observables. The picture is as follows, the spacetime is S^1, consider U an open interval in S^1. Then you have a structure called the factorization algebra of quantum observables, and if we compute $H^0(\mathrm{Obs}^q(U))$, this is isomorphic to flat sections of the Weyl bundle.

I should mention actually here, one thing I didn't mention, when we quantize if we want to find something satisfying the quantum master equation that's the same as saying Fedosov's equation is satisfied. After computation, we can say that quantum observables supported on U are flat sections of the Weyl bundle.

There's this factorization structure, if you have U and U' within V, you have this factorization product $H(\mathrm{Obs}^q(U))\otimes H(\mathrm{Obs}^q(U')) \to H^{\mathrm{Obs}^q(V)}$, and after identifying with flat sections of the Weyl bundle, this gives rise to Fedosov deformation quantization.

For global observables, and after the BV quantization, you can consider the partition function, and the partition function of 1 gives rise to the algebraic index theorem.

$$\int_M e^{\frac{\omega_\hbar}{\hbar}} \hat{A}(M)$$

To compute this we need to consider the S^1-equivariant localization, we're looking at a neighborhood of constant maps in the loop space of M. This genus shows up in [unintelligible].

14. Matt Young:Cohomological Donaldson–Thomas theory with orientifolds III

So the last lecture is going to be on orientifold Donaldson–Thomas theory. Now that we've set up this Hall algebra structure and the representation of the Hall algebra it will be easy to write down a candidate for the orientifold Donaldson–Thomas invariant.

We've been working with a quiver Q and in the orientifold setting we added σ a contravariant involution of the quiver, and some combinatorial signs s and τ, to talk about self-dual representations. Associated to this data we cooked up a contravariant functor S and wrote down an isomorphism from the identity functor to the square S^2. The objects we're interested in, the cohomological Hall algebra of Kontsevich–Soibelman is $\bigoplus H^{-\chi(d,d)}(M_d)$ and the cohomological Hall module $\mathcal{M}_Q = \bigoplus H^{-\mathcal{E}(e)}(M_e^\sigma)$.

We need to know a bit more about the self-dual objects. I need just two basic facts about these representations. We've been studying symmetric quivers. This was to avoid some technicalities and make things easier but it's not strictly necessary. I'll restrict even further now to make things even easier.

Definition 14.1. Given a representation $U \in \mathrm{Rep}_{\mathbb{C}} Q$, le $\mathcal{E}(U) = \dim \mathrm{Hom}(S(U),U)^{-S} - \dim \mathrm{Ext}^1(S(U),U)^S$.

This has the following properties.

Lemma 14.1.
(1) *$\mathcal{E}(U)$ depends only on* $\underline{\dim} U$
(2) $\dim M_e^\sigma = -\mathcal{E}(e)$, *which is similar to* $\dim M_d = -\chi(d,d)$. *There's a stronger statement that says these vector spaces are actually the tangent spaces of the appropriate stacks*
(3) *the dimensions of the fibres of the map from isotropic flags mapping to the first and last terms of the flag has dimension $\chi(e,d) - \mathcal{E}(d)$. This has a stronger statement, again, at the deformation theory level. If e is 0, then this is a flag of Lagrangian extensions, the deformation theory of Lagrangian extensions is also controlled by $\mathcal{E}$.*

So these are some ways to interpret this number.

Definition 14.2. A quiver with all this data is called σ-symmetric if

(1) it's symmetric, and
(2) $\sigma^*\mathcal{E} = \mathcal{E}$ under this involution.

For example, take an m-loop quiver. A non-trivial example, take the affine A^1 quiver. the conditions to be σ-symmetric are that τ and s are constant. If the signs of the arrows are different it's not σ-symmetric.

We'll restrict only to such quivers today for simplicity.

Definition 14.3. A self-dual representation is called σ-stable if it has no nontrivial isotropic subrepresentations.

This differs from the usual case by adding "isotropic." This should maybe be considered a theorem, because there's a definition of stability from GIT. Then you get $\mathcal{M}_e^{\sigma,\mathrm{st}}$. This is a non-projective orbifold. You don't have a smooth moduli space passing from the general linear group to other groups.

Our goal is that we want to compute $\mathcal{H}^*(\mathcal{M}_e^{\sigma,\mathrm{st}})$. One thing to note immediately is that if you have a short exact sequence in the self-dual setting, $0 \to U \to N \to P \to 0$ then N won't be σ-stable, it has U as a subthing.

So we're not interested in the things that are in the image of the module action, we only want the minimal module generators, that's the algebraic translation.

Lemma 14.2. *Every self-dual representation has a σ-Jordan–Hölder filtration. This is a filtration by isotropic subrepresentations $0 = U_0 \subset \cdots \subset U_r \subset N$ such that*

(1) *U_i/U_{i-1} is stable, and*
(2) *$N//U_r$ is zero or σ-stable.*

Then up to extensions, $N \sim \bigoplus H(U_i/U_{i-1}) \oplus N//U_r$ where $H(V) = V \oplus S(V)$ gives a way to make a representation self-dual.

We might expect, if we look at the stack of all representations, how do we build this up? We need symmetric powers of $\mathcal{M}^{\mathrm{st}}/\mathbb{C}^\times$, we only need to add on a single factor of the moduli space of stable self-dual representations. We need to be a bit more careful, we don't want to count both V and its dual. So we need to quotient by that action. So

$$M^\sigma \sim Sym(\mathcal{M}^{st}/\mathbb{C}^\times)/\mathbb{Z}_2 \times \mathcal{M}^{\sigma,\mathrm{st}}$$

Now we can give a definition.

Definition 14.4. The cohomological orientifold Donaldson–Thomas invariant is defined to be W_Q^{prim}, the space of minimal module generators $\mathcal{M}_Q/\mathcal{H}_{Q,t} * \mathcal{M}_Q$. This is a $\Lambda_Q^{\sigma,+} \times \mathbb{Z}$-graded vector space.

We don't have the extra $\mathbb{Q}[u]$ term here because [unintelligible].

So we can easily define both the refined and numerical invariants from this. The first is that the analogue of integrality and positivity of Kontsevich–Soibelman hold.

Theorem 14.1. *(Y.)*

$$\dim W_{Q,e}^{\mathrm{prim}} < \infty$$

We can now take the Euler characteristic and get a numerical invariant.

This approach via Hall algebras is nice. The main downside is that you've only heuristically related to geometry. The next question is whether this is a geometric object.

There's a partial result in this direction. We can't compute the cohomology of this moduli space because of the mixed Hodge structure but we can compute the pure part.

Theorem 14.2. *There is a canonical surjection*

$$W^{\mathrm{prim}}_{Q,e} \to PH^{\cdot - \mathcal{E}(e)}(\mathcal{M}^{\sigma,\mathrm{st}}_e)$$

The obvious conjecture is that this is an isomorphism.

Chen's proof in the general case uses Nakajima varieties which just don't exist when you leave the general linear group, so you really need a new idea to prove injectivity.

Let's discuss an example. Take Q to be the quiver with one node and one loop. You check directly that $\mathcal{H}_{L_1}$ is $\mathrm{Sym}(\mathbb{Q}(1,0) \otimes Q[u])$, so this is an infinite number of even variables this is generated by. This corresponds to, what are the stable moduli spaces? A representation is a vector space and an endomorphism. In dimension vector 2 or higher, you can choose an eigenvector and that's a subrepresentation. You have a single $\mathbb{C}$ for $d = 1$.

What about, take $s = +1$ and $\tau = -1$. Recall that M^o_e is so_e/O_e. What is the module in this case?

$$\mathcal{M}_{L_1} = \bigoplus_{e \geq 0} H_{O_e}(so_e)$$

which since so_e is contractible, is

$$\bigoplus \mathbb{Q}[z_1^2, \dots z_e^2]^{S_e} \oplus (\mathcal{O}_{2e+1})$$

In this case we can compute directly that this module (let's look at the even dimensional subpiece is a free module over some half-dimensional subalgebra,

$$\mathrm{Sym}(Q_{(1,0)} \otimes \mathbb{Q}[u^2])$$

with basis, generated by $\mathbf{1}^\sigma_o$. We should still look at the geometry. The stable moduli spaces can be computed directly and are all empty (except basically by definition for dimension vector 0 it's a point).

We should think, why are we taking this half-dimensional subspace of the generators, it's exactly the $\mathbb{Z}_2$-quotient because of the self-duality.

Let's look at this more interesting example, with $s = +1, \tau = +1$, where we look at symmetric powers of the fundamental representation. In this case $\mathcal{M}_{L_1}$ is free over $\mathrm{Sym}(\mathbb{Q}_{(1,0)} \otimes u\mathbb{Q}[u^2])$. This has a single generator in each dimension vector. What is the geometry of this? We can again compute the moduli space, and it's $\mathcal{M}^{\sigma,\mathrm{st}}_e = \mathrm{Sym}^e \mathbb{C} \backslash \Delta^{\mathrm{big}}$ and then the homology is $\mathbb{Q}(0)_0 \oplus \mathbb{Q}(-1)_1$, and [unintelligible]. That's the conjecture in this example.

We see the algebra over which this module will be free changes depending on the data.

Can we describe the full algebra structure of the Hall module. How do wechoose the right algebra over which it's free? There are two basic results we can get. What extra structure do we have on the Hall algebra? It's built by smashing two objects together to make a short exact sequence. We have an induced anti-involution $S : \mathcal{H}_Q \to \mathcal{H}_Q$, it reverses the order of the product (since the functor is contravariant and reverses the order of short exact sequences) which induces the *geometric involution* on the primitive part $V^{\mathrm{prim}}_{Q,d} \to V^{\mathrm{prim}}_{Q,\sigma(d)}$. So this extends the geometric involution. This is basically a formal property.

The second part is more interesting. We're strongly using for the second part that Q is σ-symmetric. If we're given $f \in \mathcal{H}_{Q,d}$ and $g \in M_{Q,e}$, then acting by f and its dual we get $(f - (-1)^{\chi(e,d)+\mathcal{E}(d)}S(f)) * g = 0$. The twisting sign is important and depends on what we're acting on. You should think of this as the analogue of supercommutativity in the module case.

We have this property when we restrict to the primitives of the Hall algebra, so consider $V_Q^{\text{prim}} \otimes \mathbb{Q}[u]$ with the e-twisted $\mathbb{Z}_2$ action.

Then from this lemma, we get an induced action of

$$\mathcal{H}_Q(e) = \mathrm{Sym}((V_Q^{\text{prim}} \otimes \mathbb{Q}[u])_{\mathbb{Z}_2,e})$$

on

$$\mathcal{H}_Q * W_{Q,e}^{\text{prim}} \subset \mathcal{M}_Q.$$

The conjecture is the following.

Conjecture 14.1. $\mathcal{H}_Q * W_{Q,e}^{\text{prim}}$ is free over $\mathcal{H}_Q(e)$ with basis $W_{Q,e}^{\text{prim}}$

I should say

Theorem 14.3. *All conjectures hold for the zero loop quiver, the one loop quiver, the two node affine quiver and (suitably modified) for all finite type quivers (this is an infinite family).*

In the last couple of minutes I want to talk about some corollaries of this conjecture.

It's nice we have this definition, but maybe I don't want to compute this, I only want the numerical or refined invariants.

Recall that we defined the refined Donaldson–Thomas invariants by this plethystic exponential

$$Exp(\frac{\Omega_Q}{1-q}) = [\mathcal{H}_Q]$$

and this is not as easy to understand but you can have a computer do it. In general we can't compute this without, well, a corollary of the last conjecture is an equality.

$$[M_Q] = \sum_{e \in \Lambda_Q^{\sigma,+}} [\mathcal{H}_Q(e)][W_{Q,e}^{\text{prim}}]$$

This is an equality in $\mathbb{Q}(q^{\frac{1}{2}})[[\Lambda_Q^{\sigma,t}]]$. We'd like to compute these via computer. Can we do this? The element $[\mathcal{H}_Q(e)]$ is in general not determined by Ω_Q, we need to know it as a $\mathbb{Z}_2$ character.

This is annoying but also interesting. It motivates why you need the cohomological ones. It's not enough to know the graded dimensions. This also tells you that you need at least this $\mathbb{Z}_2$ refinement. However, it's not all bad. In some cases, the $\mathbb{Z}_2$ structure is trivial. For example, the m-loop quiver, the $\mathbb{Z}_2$-equivariant theory has no more information than the usual theory. In particular, we know how to compute these series directly, by work of Reineke, Ω_{L_M} are known. There is an explicit complicated formula. So we can numerically compute the refined invariants $[W_{Q,e}^{\text{prim}}]$ in the orientifold setting.

It seems like, sometimes you're in luck and sometimes not.

One last comment to make, let's go back to the heuristic. We hoped that M^σ would be something ike $(\mathrm{Sym}(\mathcal{M}^{st}/\mathbb{C}^\times)/\mathbb{Z}_2) \times \mathcal{M}^{\sigma,\text{st}}$, so we might hope it's $\mathrm{Sym}(PH(\mathcal{M}^{\text{st}}/\mathbb{C}^\times)/\mathbb{Z}_2) \times [PH(\mathcal{M}^{\sigma,\text{st}})]$, but this says that's just not true. There's something non-geometric going on [missed a little]. Maybe it doesn't make sense to start something new so maybe I'll stop there.

15. Atsushi Takahashi: From Calabi–Yau dg categories to Frobenius manifolds via primitive forms

Thank you for the introduction and to the organizers for the invitation. I'd like to explain how to define a Frobenius structure from a Calabi–Yau dg category. The motivation is from Kontsevich's homological mirror symmetry conjecture that says that $D^b \operatorname{Coh} X$ and $D^b \operatorname{Fuk}(Y)$ for mirror pairs (X, Y). This should imply the so-called classical mirror symmetry conjecture, which is an isomorphism of Frobenius manifolds between $\bigoplus H^q(X, \wedge^p TX) \cong \bigoplus H^q(Y, \Omega^p_Y)$.

The key observation is that the Hochschild cohomology of X is the same as $\bigoplus H^q(X, \wedge^p TX)$ as a vector space. So the problem is to construct a Frobenius structure on $HH^{\cdot+2}(\mathcal{A})$ for a smooth compact Calabi–Yau A_∞-category.

I will also use the theory of primitive forms developed by Kyoji Saito. I will recall classical Saito theory, review the talk yesterday in one or two minutes. The initial data was an isolated singularity $f : \mathbb{C}^{n+1} \to \mathbb{C}$, with an isolated singularity at 0, and then he got a filtered de Rham cohomology $\mathcal{H}^{(0)}_F$ with a Gauß–Manin connection ∇ and a higher residue pairing K_F, let me call this a Saito structure. Then you construct a primitive form for this Saito structure, and once one is given it automatically gives a Frobenius structure (flat structure). Therefore what we should do in order to do this is give an analogue of f starting from a dg category.

So now I'll write a table from an appendix in a paper with Kyoji Saito

classical	categorical
$(f, \mathcal{O}_{\mathbb{C}^{n+1}})$	Calabi–Yau (weak) A_∞ category
$Jac(f)$	$HH^\cdot(A)$
Ω_f	$HH_\cdot(A)$
$\mathcal{H}_f, \mathcal{H}^{(0)}_f$	$HP^\cdot(A)$, $HC^\cdot(A)$

The correspondence is not so precise and to do this I need formality. In order to have the Jacobian ring starting from this data, this is a quotient of some ring by the Jacobian ideal, and this ideal, well, this is not the total cohomology group but the cohomology of some cohomology. So in order to connect them we need some formality assumption. If we assume these, we obtain a Frobenius structure from a dg category. This is what I will explain today.

So let me start with some basic terminology. Today I'll work over a field k which is algebraically closed and characteristic zero. Usually I assume k is a complex number field or the universal Novikov field. For me, a differential graded algebra is $\mathbb{Z}$-graded and d_A is an operator of degree 1 satisfying $d_A^2 = 0$ and $d_A(ab) = d_A(a)b + (-1)^{\bar{a}} a d_A b$, where $\bar{a}$ is the degree of a.

So A is *non-negatively graded* if $A^p = 0$ for $p < 0$ and A is *compact* if $\dim_k H^\cdot(A, d_A)$ is finite. It is *smooth* if A is a perfect $A^e = A^{op} \otimes_k A$-module. Here $\operatorname{per}(A)$ is the smallest triangulated subcategory containing A closed under isomorphism, direct sum, and direct summand. It is *connected* if $H^0(A, d_A) = k[1_A]$. We'll assume these conditions on all algebras.

Next I want to define the Calabi–Yau condition. Let me say $A^! = \mathbb{R}\operatorname{Hom}_{(A^e)^{\mathrm{op}}}(A, A^e)$, this is an inverse dualizing complex.

Remark 15.1. There is a map $A \to \mathbb{R}\operatorname{Hom}_{A^e}(\mathbb{R}\operatorname{Hom}(A, A^e), A^e) = \mathbb{R}\operatorname{Hom}_k(A^!, A^e)$ and this map is an isomorphism.

This means that $A \otimes^{\mathbb{L}}_{A^e} A \cong \mathbb{R}\operatorname{Hom}_{A^e}(A^!, A)$ in $\operatorname{per}(k)$.

Definition 15.1. (Ginzburg) A is *Calabi–Yau* of dimension w if $A^! \cong T^{-w}A$ in $\operatorname{per}(A^e)$.

Remark 15.2. $A^* = \mathbb{R}\operatorname{Hom}_k(A, k)$, this givves the Serre functor or $\operatorname{per}(A)$, and this, $A^* \otimes^{\mathbb{L}}_A A^! \cong A \cong A^! \otimes^{\mathbb{L}}_A A^*$.

By using A^*, this is also $T^w A$, and so from this isomorphism and also the above one, we obtain

$$\mathbb{R}\operatorname{Hom}_{A^e}(A, A) \cong T^{-w}(A \otimes^{\mathbb{L}}_{A^e} A).$$

Now in order to describe the correspondences I should introduce Hochschild (co) homologies. There exist a double complex $(C^{\cdot}(A), \partial = d + \delta)$ which is quasiisomorphic to $\mathbb{R}\operatorname{Hom}_{A^e}(A, A)$. The differentials come from d_A and "the usual" H codifferenetial.

Now the cohomology of this complex is called the Hochschild cohomology. Later I want to use another one, $T_{\mathrm{poly}}(A)$, which is cohomology with respect to the second differential δ, and for safety, we have $d^2 = 0$, $\delta^2 = 0$, and $d\delta + \delta d = 0$.

Now we have some operators to define on Hochschild cochains. First, there is a product

$$\circ : C^{\cdot}(A) \otimes C^{\cdot}(A) \to C^{\cdot}(A)$$

and a Gerstenhaber bracket

$$[\ ,\ ]_G : C^{\cdot+1}(A) \otimes C^{\cdot+1}(A) \to C^{\cdot+1}(A)$$

which satisfy the properties

(1) $\circ$ induces a graded associative *commutative* product on $HH^{\cdot}(A)$ and $T^{\cdot}_{\mathrm{poly}}(A)$.
(2) $[\ ,\ ]$ induces a graded Lie bracket on $HH^{\cdot+1}(A)$ and $T^{\cdot+1}_{\mathrm{poly}}$ which satisfies

$$[X, Y \circ Z]_G = [X, Y]_G \circ Z + (\pm 1) Y \circ [X, Z]_G.$$

Now we can give an analogue of f. There exists m_A in $C^2(A)$, from the dga structure on A. Here $m_A = m_1 + m_2$ where m_1 comes from the differential and m_2 from the algebra structure. Then $[m_A, m_A]_G = 0$, and this comes from knowing that $[m_1, m_1]_G = 0$ from $d_A^2 = 0$, that $[m_1, m_2]_G = 0$ from the Leibniz rule, and $[m_2, m_2]_G = 0$ fro the associativity of the product. Then $dX = [m_1, X]_G$, $\delta X = [m_2, X]_G$, and $\partial X = [m_A, X]_G$.

Definition 15.2.

$$f_A \coloneqq [m_A] \in T^2_{\mathrm{poly}}$$

Proposition 15.1. *(Euler's identity)*

$$f_A[\deg_A, f_A]_G$$

where $\deg_A \in C^1(A)$ *is* $\deg_A(a) = \bar{a}a$. *I can't do this if I don't have a* $\mathbb{Z}$*-grading.*

Okay, Hochschild homology, $(C.(A), d + \delta) \cong A \otimes^{\mathbb{L}}_{A^e} A$.

By considering the total homology of this one you get the Hochschild homology and also here I need $\Omega.(A) = H(C(A), \delta)$. Here there is also the Connes operator $B : C \cdot (A) \to C_{\cdot+1}(A)$, satisfying $B^2 = 0, Bd + dB = 0$, and $B\delta + \delta B = 0$. There are also contraction and Lie derivative operators ι_X and L_X on $C(A)$

Proposition 15.2. *(Daletzki–Gelfand–Tsygan) For X in $T_{\text{poly}}(A)$, thera are ι_X and $L_X : \Omega(A) \to \Omega(A)$ such that $\iota_X \iota_Y = \iota_{X \circ Y} = [L_X, L_Y] = L_{[X,Y]_G}$, such that $L_X \iota_Y + (\pm 1)\iota_Y L_X = L_{X \circ Y}$, so that $[\iota_X, L_Y] = \iota_{[X,Y]_G}$, so that $[B, \iota_X] = -L_X$, so that $[B, L_X] = 0$, and so that $L_{f_A} = -d$.*

This is the *Cartan calculus.*

On $(C^{\cdot}A, C_{\cdot}A)$, we have only these structures "up to homotopy." This is called *homotopy calculus* by Dolgushev–Tamarkin–Tsygan.

Now we can discuss formality.

Conjecture 15.1. $(C^{\cdot}(A), C_{\cdot}(A)) \cong (T_{\text{poly}}(A), \Omega(A))$ as homotopy calculus algebras.

If A is a usual algebra, you have this kind of formality in many situations.

The main statement is that if we assume this conjecture plus something, then we have a primitive form. Today we assume this conjecture and I think the following statement follows from the conjecture, but also that $(T_{\text{poly}}(A), d) \cong (\Omega(A), d)$ as an isomorphism of complexes, so that I can move B to T_{poly}.

Then $HH^{\cdot}(A) \cong H^{\cdot}(T_{\text{poly}}(A), d) = Jac(f_A)$ and $HH_{\cdot}(A) \cong H_{\cdot}(\Omega(A), d) = T^w \Omega_{f_A}$.

Then $H_w(\Omega(A), d) \cong HH_w(A) \cong HH^0(A) \cong k[1_A]$.

Here w is the Calabi–Yau dimension.

So we can define $\mathcal{H}_{f_A} = H_{\cdot}(T^{-w}\Omega(A)((u)), d + uB)$, where $u = \delta_w^{-1}$ of yesterday and similarly

$$\mathcal{H}_{f_A}^{(-p)} = H_{\cdot}(T^{-w}\Omega_{\cdot}(A)[[u]]u^p, d + uB).$$

Then from Kaledin's Hodge to de Rham we have $0 \to \mathcal{H}_{f_A}^{(-p-1)} \to \mathcal{H}_{f_A}^{(-p)} \to \Omega_{f_A} \to 0$, and then nondegeneracy is Hodge to de Rham degeneration. Then $\nabla_{\frac{d}{du} - \frac{1}{u^2}\iota_{f_A}}$, degined on $\Omega(A)((u))$

Proposition 15.3.

$$[\nabla_{\frac{d}{du}}, d + uB] = d + uB$$

and this preserves $\mathcal{H}_{f_A}^{(0)}$.

This propertiy was also explained by Kyoji Saito about primitive forms. In order to calculate this, we use Cartan calculus, and especially we have the following formula,

$$\nabla_{u\frac{d}{du}} = u\frac{d}{du} + L_{\deg_A} + [d + uB, \iota_{\deg_A}]$$

where $L_{\deg_A}$ is N_A and "counts exponents."

Proposition 15.4. $[N_A, d] = d$ *which means that* N_A *defines an element of* $\text{End}(\Omega_{f_A})$.

Now we can define

$$\Omega_{f_A}^{p,q} = \{\omega \in \Omega_{f_A} | \bar{\omega} = w - p + q, N_A \omega = q\omega\}$$

and in this way we get a Hodge filtration. The q is called the *exponent*, and this is not the sum of p and q, just simlpy q.

We choose a homogeneous basis and lift to this to get the primitive form. Let me write down the main statement here.

Theorem 15.1. *(existence of a very good section) We have $0 \to \mathcal{H}_{f_A}^{(-1)} \to \mathcal{H}_{f_A}^{(0)} \xrightarrow{r^{(0)}} \Omega_{f_A} \to 0$ a short exact sequence, and there is a section $S^0 : \Omega_{f_A} \to \mathcal{H}_{f_A}^{(0)}$ such that $\nabla_{u\frac{d}{du}}(S^{(0)}(\Omega_{f_A})) \subset S^{(0)}(\Omega_{f_A})$.*

Corollary 15.1. *Defining S as $S^{(0)} \otimes_k k[u^{-1}]u^{-1} \subset \mathcal{H}_{f_A}$, and $\mathcal{H}_{f_A} = \mathcal{H}^{(0)}_{f_A} \oplus S$, we have $u^{-1}S \subset S$, $\nabla_{u\frac{d}{du}} S \subset S$*

and $K_{f_A}(S,S) \subset k[u^{-1}]u^{[unintelligible]}$.

[some fast details about the higher residue pairing K].

16. Young-Hoon Kiem: Categorification of Donaldson-Thomas invariants III

So last time, X was simple sheaves on a Calabi–Yau three-fold, and [unintelligible]universal family. We saw that X was an orientable critical virtual manifold, and there is this perverse sheaf P and the mixed hodge module $M \in HM^p(X)$ where $\mathrm{rat}(M) = P$. We saw that $\chi(\mathbb{H}^*(X,P)) = DT(X)$ and then $\mathbb{H}^*(X,P)$ satisfies nice properties, in particular the relative hard Lefschetz and decomposition theorems. Today I'll talk about an application of this.

16.1. Curve counting.

Y is a smooth projective 3-fold, $K_Y \cong \mathcal{O}_Y$. There are many way to count them, you can think of them as maps up to parameterization and get Gromov–Witten invariants, you can think of them as defining equations and get Donaldson–Thomas invariants, you can get many other versions as well.

Today I'll talk about a proposal of 1998 of Gopakumar–Vafa. Imagine the following, imagine that you have a space of curves of given topological type, whose class is $\beta \in H_2(Y,\mathbb{Z})$, and suppose you have a moduli space, $X = \{(C,L) | C \in S, L \in \mathrm{Pic}(C)\}$, and say you have a the forgetful projection $h : X \to S$. Imagine everything is nice, we have $h^{-1}(C_g^{\mathrm{sm}}) = \mathrm{Jac}_g$

We have $S = \sqcup S_g$ where S_g are the genus g curves in S, and we want to count S_g. Recall that the cohomology of $H^*(\mathrm{Jac}_g)$ is the same thing as $H^*(\mathrm{Jac}_1)^{\otimes g}$, and the Hodge diamond is

$$1$$

$$2$$

$$1.$$

We use hard Lefschetz, and we have two trivial representations and one 2-dimensional representation, and you take g copies of this, call this representation I_g. By the Clebsch–Gordan rule, you find that any sl_2-representation is an integral linear combination of these I_g.

Now imagine you have a nice cohomology theory $\mathcal{H}^*(X)$ which satisfies a relative hard Lefschetz property. We have $X \xrightarrow{h} S \xrightarrow{c} \mathrm{pt}$. We have relative ample line bundles $\mathcal{O}_h(1)$ and $\mathcal{O}_S(1)$. The Chern classes of these are ω_L and ω_R respectively, and the relative hard Lefschetz theorem give us two sl_2-actions, the left and the right action on $\mathcal{H}^*(X)$, they commute, and so we get an $(sl_2)_L \times (sl_2)_R$-action on $\mathcal{H}^*(X)$.

Let's think only about the left action for the time being. Consider $(sl_2)_L$, write $\mathcal{H}^*(X) = \bigoplus I_g \otimes R_g$, then this is irreducible, there's another sl_2-action, this R_g is an $(sl_2)_R$-space.

Now another imagination, well $I_g = H^*(\mathrm{Jac}_g)$, and my X looks like $\sqcup \mathrm{Jac}_g \times S_g$, let's just imagine this is true. If you have a smooth genus g curve, the fiber is Jacobian, then R_g should be the cohomology of S_g. Okay, so we are counting virtually, so suppose for instance, if R_g is a finite set we count the Euler number, then $\#^{vir}(S_g) = \chi(S_g)$, then you do this by thinking of the trace $\mathrm{Tr}(-1)^{H_R}$ where H_R is the diagonal element $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ in sl_2. So why is this? If I take $(-1)^{H_R}|_{\mathcal{H}^i}$ then I get $(-1)^{i,\mathrm{id}}$.

So then if all these things are true then the *Gopakumar–Vafa invariant* is

$$n_g(\beta) = \mathrm{Tr}_{R_g}(s1)^{H_R}$$

and these numbers are integers.

Conjecture 16.1. (Gopakumar–Vafa)

$$\sum_{g,\beta} N_g(\beta) q^\beta \lambda^{2g-2} = \sum_{g,\beta} n_g(\beta) \frac{1}{k} (2 \sin \frac{k\lambda}{2})^{2g-2} q^{k\beta} \tag{1}$$

where $N_g(\beta)$ is the Gromov–Witten invariants for Y. It's obvious that you can compute $n_g(\beta)$ from $N_g(\beta)$, what's not obvious is that $n_g(\beta)$ is an integer.

Okay, so we want to make it rigorous.

16.2. **Translation.** We need a space X and S. Somehow, many mathematicians seem to agree (let me mention S. Katz in 2000) that X should be the stable sheaves on Y

If $i : C \to Y$ is a curve in Y and $L \in \mathrm{Pic}(C)$, then i^*L, the direct image is a coherent sheaf on Y.

So we want to take X as stable 1-dimensional sheaves E on Y , with $(E) = \beta, \chi(E) = 1$.

Then S is the image of X under h into $\mathrm{Chow}^{1,\beta}(Y)$ and everything in side is projective.

Conjecture 16.2. Hosono–Saito–Takahashi and Katz conjectured that $n_0(\beta)$ should be the Donaldson–Thomas invariant.

There are some cases where X is smooth. Then $\chi(H^*(X))$ is $\chi(X)$, and this guy, the cohomology of X, the Euler characteristic of the Jacobian is 0, so $\chi(R_0)$ survives, this is $n_0(\beta)$ with the sign $(-1)^d$, and this is the Donaldson–Thomas invariant.

The consequence of this conjecture is the following, combining these two conjectures:

Conjecture 16.3.

$$N_0(\beta) = \sum_{k|\beta} \frac{DT(\beta/k)}{k^3}$$

Now Hosono–Saito–Takahashi observed that if you can find a perverse sheaf P which underlies [unintelligible], then $\mathbb{H}^*(X,P)$ satisfies the desired properties. They propose to use intersection cohomology, the most obvious one, for P, so $\mathbb{H}^*(X,P) = IH^*(X)$. Maybe I should explain something about relative hard Lefschetz in this case. So $h : X \to S$ and $\omega = c_1(\mathcal{O}_h(1))$ and $\omega^k : {}^p\mathcal{H}^{-k}(Rh_*IC_X) \to {}^p\mathcal{H}^k(Rh_*IC_X)$, an sl_2-action, and Rh_*IC_X decomposes into simple summands

$$Rh_*IC_X \cong \bigoplus_k {}^p\mathcal{H}^k(Rh_*IC_X)[-k]$$

and this gives an $(sl_2)_L \times (sl_2)_R$-action on $IH^*(X)$ which gives us $n_g(\beta)$. They checked equation 1 for a special $K3$-fibered Calabi–Yau three-fold.

This is probably not the best example.

Now let me summarize what has been expected about Gopakumar–Vafa theory. We expect the following:

(1) $\{n_g(\beta)\}$ should arise from an $(sl_2)\times(sl_2)$-action on some cohomology theory $\mathcal{H}^*(X)$, with X the moduli of stable 1-dimensional sheaves on Y.
(2) The genus zero invariants $n_0(\beta)$ should be the Donaldson–Thomas invariants.
(3) Gopakumar–Vafa should be Gromov–Witten, we want equation 1.

(This is all still joint with Jun Li).

The paper of Hosono–Saito–Takahashi realized the first item perfectly, but we're not sure about the second one, because this does not capture the vanishing cycles. Also, IH^* is not deformation invariant but the Donaldson–Thomas invariants in the second item are.

So now we should look for a different perverse sheaf also underlying a polarized Hodge module.

16.3. **Categorification helps.** We have a nice perverse sheaf underlying a polarized Hodge module. So we propose to use the perverse sheaf P made by gluing the perverse sheaves of vanishing cycles.

Here X, I need it to be the moduli of stable sheaves E on Y with Hilbert polynomial $dm+1$, because the coefficient of the first term and the constant term are coprime, there is a universal family [unintelligible]coscheme descent, and so X is an orientable critical virtual manifold. What's automatic about this perverse sheaf P from last time is that $\chi(\mathbb{H}^*(X,P)) = DT(X)$ and $\mathbb{H}^*(X,P)$ comes with a $sl_2 \times sl_2$-action. Using the recipe of Gopakumar–Vafa and Hosono–Saito–Takahashi we get $n_g(\beta)$ invariants. Then when $g = 0$, $\chi(R_0) = n_0(\beta)$, so condition 2 is automatic and condition 1 is also satisfied by construction.

We're in much better shape. We checked, the remaining issue is the formula 1. By the same argument as in Hosono–Saito–Takahashi, the formula holds for some special $K3$-fibered Calabi–Yau three-folds. What we are trying to prove at this point is the local genus 2 curve case. Here Y is a total space of a general rank 2 stable vector bundle F on a smooth genus 2 curve C with det $F = K_C$ (the Calabi–Yau condition). It turns out the rank 1 case is easy. The important thing is the rank 2 case, so X is stable sheaves E on Y with $\chi(E(m)) = 2m+1$. So this is a smooth five dimensional projective variety. This has a divisor with non-reduced scheme structure, so we have to do some calculations related to the vanishing cycles there. That's what we're working on now.

This is a kind of bonus for the categorification problem. It would be nice, let me finish this lecture, we're all tired,

16.4. **Problems and projects.** Maybe I'll work on some of these but not all of these.

(1) The most important is to prove Equation 1, the Gromov–Witten Gopakumar–Vafa connection, this is probably hard.
(2) Probably easier is to prove that $n_g(\beta)$ is independent of the choice of polarization of Y. This would be true because Gromov–Witten invariants do not depend on polarization. Also we should show that this is deformation invariant (we know this for genus 0 since Donaldson–Thomas is deformation invariant).
(3) So how would you do this? You could use a wall-crossing formula between X^+ and X^-, then $X^+ - X^- = \{0 \to E_1 \to E \to E_2 \to 0 | [\mathit{unintelligible}]\}$ and similarly for $X^- - X^+$. Then $d_+ = \mathrm{ext}^1_Y(E_2, E_1)$ and $d_- = \mathrm{ext}^1_Y(E_1, E_2)$ and $[\mathrm{Supp}\, E_i] = \beta_i$

Conjecture 16.4.

$$n_g(\beta)^+ - n_g(\beta)^- = (-1)^{d_+ - d_- - 1}(d_+ - d_-)\sum_{h=0}^{g} n_h(\beta_1) n_{g-h}(\beta_2)$$

So in the case $d_+ = d_-$ so for instance $c(E_1) = c(E_2)$, this would mean that $n_g(\beta)^+ = n_g(\beta)^-$, which would give you the independence of choice of polarization.