INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS SYMPLECTIC TUESDAY

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1. October 13: Yong-geun Oh: Topological extension of Calabi invariants and its application

Okay, so let me remind you of some background.

1.1. **Background.** Let me start with some generalities. Let (M, ω) be a closed symplectic manifold and let $\mathrm{Symp}(M, \omega)$ be the group of symplectic diffeomorphisms, automorphisms of the symplectic structure.

When M is not simply connected, then the canonical subgroup $\mathrm{Ham}(M, \omega)$, the group of Hamiltonian diffeomorphisms, is a proper normal subgroup.

I should maybe remind you of this Ham. The way how this diffeomorphism group is defined is very strange. The symplectomorphism group is natural, it's the automorphisms of the symplectic structure. The Hamiltonian group is unnatural and it's a historical accident that we consider it. Hamiltonian dynamics studies this object but there's no a priori mathematical reason to look at it.

Look at time-dependent functions $H(t, x)$, time dependent functions on M, and if these are at least C^2 or $C^{1,1}$ functions, well, let me talk about that later. Then X_H is a time-dependent vector field, defined by

$$X_H(t, x) \lrcorner \omega_x = dH_t(x)$$

This uniquely determines X_H by nondegeneracy of ω_x if H is differentiable. If H is C^2 or $C^{1,1}$ (the first derivative is Lipschitz), then $x = X_H(t, x)$ defines a global flow ϕ_H^t, and then $Ham(M, \omega)$ is defined as the subset of symplectic diffeomorphisms which are the time $t = 1$ image of Hamiltonian flow, with H in $C^\infty([0, 1] \times M, \mathbb{R})$.

A lemma is that this forms a subgroup. A theorem of Banyaga is that this is a *simple* group and $[\mathrm{Symp}(M, \omega), \mathrm{Symp}(M, \omega)] = \mathrm{Ham}(M, \omega)$.

Remark 1.1. Say M is connected. At the Lie algebra level, you have a map $0 \to \mathbb{R} \to C^\infty(M) \to \mathcal{X}^{\mathrm{Symp}}(M) \to H^1(M, \mathbb{R}) \to \cdots$ where these take f to X_f and X to $X \lrcorner \omega$.

The ordinary differential equation has global flow. So let's lower the regularity of the Hamiltonian. If the regularity of H is under $C^{1,1}$, then the flow does not exist but we can still think about Hamiltonian functions. Physicists even use non-continuous potentials. It's very tempting to ask what happens, is there any way of completing this group?

I want to, here is Eliashberg–Gromov's C^0-rigidity theorem:

Theorem 1.2. *Let me denote* $\mathrm{Sympeo}(M, \omega) = \overline{\mathrm{Symp}(M, \omega)} \subset \mathrm{Homeo}(M)$*, the* C^0*-closure of* $\mathrm{Symp}(M, \omega)$*. The amazing theorem is that if you look at* $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}(M)$*, and the theorem says that this is* $\mathrm{Symp}(M, \omega)$*.*

This means that a sequence ψ_i of symplectic diffeomorphisms converges in C^0 and its limit is differentiable, then its limit is a symplectic diffeomorphism, preserves the symplectic form as well.

The C^0-closure does not control anything about the behavior of derivatives, but that happens to symplectic diffeomorphisms.

Gromov noticed here that there is something deep in the study of symplectic manifolds.

So a natural question is:

Question 1.3. What is the analog of $\mathrm{Ham}(M, \omega)$ inside $\mathrm{Sympeo}(M, \omega)$.

There is, now, this analog of Ham inside Sympeo and that's what we're talking about.

The most natural way to define Ham is by two stages. First look at the Hamiltonian paths, and then evaluate them at time 1. So we'll define $\mathcal{P}^{\langle am}(\mathrm{Symp}(M, \omega), \mathrm{id})$, which I'll abbreviate $\mathcal{P}^{\langle am}_{\mathrm{id}}$. There's a natural evaluation $\mathrm{ev}_1 : \mathcal{P}^{\langle am}_{\mathrm{id}} \to \mathrm{Symp}(M, \omega)$. Then $Ham(M, \omega) = \mathrm{ev}_1(\mathcal{P}^{\langle am}_{\mathrm{id}})$.

So my first step is to define $\mathcal{P}^{\langle am}(\mathrm{Sympeo}(M, \omega), \mathrm{id})$.

Definition 1.4. We say a path $\lambda : [0, 1] \to \mathrm{Sympeo}(M, \omega)$ is a topological hamiltonian path if there is a sequence $H_i = H_i(t, x)$ such that

(1) the flow ϕ_{H_i} C^0 converges to λ uniformly in time, and
(2) H_i converges in $L^{1,\infty}$ (you can assume C^0 if this is uncomfortable. This means L^1 in time and L^∞ in space and is much morp natural)

Suppose you are given that the Hamiltonian flow converges. Suppose ϕ_{H_i} and Φ_{F_i} converge to λ and both H_i and F_i converge. Then do the limit of H_i and F_i agree?

Theorem 1.5. *(Viterbo (C^0), Buhovksy–Seyfaddini ($L^{1,\infty}$)) They do agree*

Theorem 1.6. *(Oh, earlier, easier but still nontrivial) If H_i and F_i converge to the same function and ϕ_{H_i} and ϕ_{F_i} converges, then the limiting flows are equal*

$$\lim \phi_{H_i} = \lim \phi_{F_i}$$

This enables us to define a topological Hamiltonian and its flow. A topological Hamiltonian means the C^0 limit of Hamiltonians that arises in this way.

Remark 1.7. Let me extend to open manifolds. Look at $\mathbb{R}^{2n}$. Look at compactly supported symplectic diffeomorphisms and compare compactly supported Hamiltonian diffeomorphisms. Then their C^0 closures coincide:

$$\overline{\mathrm{Symp}_c(\mathbb{R}^{2n})} = \overline{\mathrm{Ham}_c(\mathbb{R}^{2n})}.$$

So you get nothing interesting.

I will freely use ϕ_H for topological Hamiltonian flows.

Definition 1.8. By $\mathcal{P}^{\langle am}(\mathrm{Sympeo}(M, \omega), \mathrm{id})$, we mean the set of topological Hamiltonian paths from the identity.

Definition 1.9. The group $\mathrm{Hameo}(M, \omega)$ is $ev_1 \mathcal{P}^{\langle am}_{\mathrm{id}}(\mathrm{Sympeo})$.

It's an interesting exercise that

Proposition 1.10. *The group* $\mathrm{Hameo}(M, \omega)$ *is always a normal subgroup of* $\mathrm{Sympeo}(M, \omega)$.

The main question you ask is

Question 1.11. Is Hameo(M, ω) proper in Sympeo(M, ω)

That's the general question.

1.2. 2 **dimensions.** In two dimensions, what does Sympeo mean? In two dimensions, symplectomorphisms are just area-preserving. Write $M = \Sigma$. So Symp(Σ, ω) is the group Diff$^\omega(\Sigma)$ of area-preserving diffeomorphisms of Σ.

Theorem 1.12. *(Smoothing theorem) Any area-preserving homeomorphism can be smoothly approximated. That is,* Homeo$^\omega(\Sigma)$ *is the same as* $\overline{\text{Diff}^\omega(\Sigma)}$ *in* Homeo(Σ)

This is not very nontrivial, but that's the theorem.
That basically implies that Homeo$^\omega(\Sigma) =$ Sympeo(Σ, ω).

Corollary 1.13. Hameo(Σ, ω) *is a normal subgroup of* Homeo$^\omega(\Sigma)$*.*

Now this connects to a well-known open problem in dynamical systems.

Question 1.14. Is the group Homeo$^\omega(D^2, \partial D^2)$ or Homeo$^\omega(S^2)$ simple?

This is the only dimension that is not understood. This kind of simpleness question, all the other dimensions are understood and this case, just the disk and the sphere have been open.

If that question, that Hameo is proper in Sympeo, would imply that these groups are not really simple.

I should say that all of this can be done with boundary.

Theorem 1.15. *(Oh)*Hameo$(D^2, \partial D^2)$ *is a proper subgroup of* Sympeo$(D^2, \partial D^2)$*.*

I think I can prove this in any dimension but this is the most interesting case.

My student Müller and myself introduced Hameo around 2004 and Fathi proposed some "wild" area-preserving homeomorphisms. Let me describe this. It's very simple, well, not very simple actually.

Look at the radius of dyadic integers, decompose the disk into an infinite sequence of annuli of radius $\frac{1}{2^k}$. On each annuli, consider the diffeomorphism given by $(r, \theta) \mapsto (r, \theta + \rho_k(r))$. Our r lives in $[\frac{1}{2^k}, \frac{1}{2^{k-1}}]$. So ρ is a rotation in the middle, with something like $\rho_k(r) = 2^k \rho_{k-1}(r)$. The infinite product $\phi = \prod_{k=1} \phi_k$ is well-defined and differentiable everywhere except at the center, where it is still continuous.

I claim that this is still not contained in Hameo. I'll use the "Calabi invariant." This kind of construction exists only in the continuous and not in the smooth category. Freedman uses this to prove the 4-dimensional Poincaré.

This invariant is on Diff$^\omega(D^2, \partial D^2)$, this is area-preserving diffeomorphisms supported on the interior of the disk. In a neighborhood of the boundary this is the identity. Then the Calabi invariant can be defined in two different ways.

One way is using the Hamiltonian. I should say that this group is contractible, Diff$^\omega(D^2, \partial D^2)$ is the same as $Ham(D^2, \partial D^2)$. So for $\phi \in Diff^\omega(\partial D^2)$, choose H that gives rise to ϕ and then take $\int_0^1 \int_{D^2} H(t, x) dx dt$. One proposition is the following.

Proposition 1.16. *This integral depends only on* ϕ_H^1*.*

The proof is by Stokes' formula, using that $\mathrm{Diff}^\omega(D^2, \partial D^2)$ is contractible.

It looks like the integral is well-defined for each Hamiltonian path, but there's no way to prove this independence for the continous case.

The second definition is the following. Let ω be $d\alpha$, since we're on the disk. Then $\phi^*\omega = \omega$ means that $\phi^*\alpha - \alpha$ is closed. This is a compactly supported closed one-form. But since $H^1(D^2) = 0$, there is a compactly supported function $h_\phi : D^2 \to \mathbb{R}$ such that $\phi^*\alpha - \alpha = dh_\phi$. Then $\mathrm{Cal}(\phi) = \frac{1}{2}\int_{D^2} h_\phi dx$.

This involves taking a derivative, so it doesn't work to extend to hameomorphisms. So there's something nontrival that one has to do.

What I want to do, my goal, is to define or extend $\mathrm{Cal} : \mathrm{Diff}^\omega(D^2, \partial D^2) \to \mathbb{R}$ to $\overline{\mathrm{Cal}} : \mathrm{Hameo}(D^2, \partial D^2) \to \mathbb{R}$.

We'll use the first definition, starting by defining $\overline{\mathrm{Cal}}\surd^{ath} : \mathcal{P}^{\langle am}_{\mathrm{id}}(\mathrm{Sympeo}(D^2, \partial D^2)) \to \mathbb{R}$. But that's obvious, you just take the integral. I'll need the developing map $\mathrm{Dev}(\lambda) = H$ when $\lambda = \phi_H$. So you say

$$\overline{\mathrm{Cal}}\surd^{ath}(\lambda) = \int_0^1 \int_{D^2} \mathrm{Dev}(\lambda) dx dt.$$

So the main task is to prove that $\overline{\mathrm{Cal}}\surd^{ath}(\lambda) = \overline{\mathrm{Cal}}\surd^{ath}(\mu)$ when $\lambda(1) = \mu(1)$ for λ and μ in $\mathcal{P}^{\langle am}_{\mathrm{id}}(\mathrm{Sympeo})$. So for a given topological Hamiltonian loop based at the identity, is $\overline{Cal}\surd^{ath}(\lambda) = 0$?

This uses a couple of things

(1) The proof of this involves an extension of Alexander isotopy in D^2 which exists in the topological but not the diffeomorphism category. The extension will exist in the topological Hamiltonian category.
(2) In differential geometry, a Lagrangian can be thought of as a graph. There's a correspondence between Hamiltonian isotopy and the so-called Lagrangian suspension.
(3) We'll use a C^0 intersection, well, that doesn't quite work, so we'll use an $L^{1,\infty}$ version of Lagrangian intersection theorem on the cotangent bundle.
(4) Then we have, I asked about this in lunchtime seminar, this is about re-arrangement of "Hamiltonian mass."

Maybe I'll just answer Jae-Suk's question. Before answering Jae-Suk's question, I want to make an important remark.

Remark 1.17. Suppose you have a topological Hamiltonian loop. By definition, it's the C^0-limit of ϕ_{H_i} and the H_i are smooth Hamiltonians converging in the C^0 topology. If your path happens to be a loop, the smoothing sequence starts from the identity, and you can ask whether the time 1 thing can be chosen to be the identity.

In general ϕ_{H_i} may not be a loop, so only, we know that $\phi^1_{H_i} \to \mathrm{id}$. This is the source of all kind of difficulty.

Let me answer about Lagrangian suspension. Suppose you have a Lagrangian submanifold in (M, ω) and Hamiltonian flow ϕ_H. Then look at the isotopy $\phi^t_H(L)$. This can be embedded, there is a natural Lagrangian embedding $[0,1] \times L \hookrightarrow T^*[0,1] \times M$. This is defined in the following way. More generally, let me start with a Lagrangian embedding $\ell : L \to (M, \omega)$ which I denote $\iota_{(L,H)}$ where $\iota_{(L,H)}(t, y) = (t, -H(t, \phi^t_H(y)), \ell(y))$.

Now if you apply $\phi_F : (D^2, \partial D^2) \to (D^2 \partial D^2)$, this is a compactly supported flow, and you want to consider the double. You extend ϕ_F by the identity.

Now, this may not close up, there is a simple way of closing this, by taking the "odd double Lagrangian suspension," I want to double it $\iota^{\mathrm{od}}_{\ell,H}(t,y)$ is $(t, -H(t, \phi^t_H(y)), \ell(y))$ for $0 \le t \le 1$ and then

$$2-t, H(2-t, \phi^{2-t}_H(y)), \ell(y)$$

for $1 \le t \le 2$. [something about this making a loop]

2. October 20: Cheol-Hyun Cho: Introduction to Fukaya category and mirror symmetry I

I was wondering how to proceed for these lectures, and for my lectures I plan to give four or five lectures, and so, the first lecture and maybe the second will be about the basic setup of all these things, J-holomorphic curves and Lagrangian Floer theory and maybe the Fukaya category. The first two lectures will be very elementary. Actually I read a sentence in some preprint, "abbreviations do injury to knowledge," or in Korean [[unintelligible]]. The setup is full of subtleties and to give all of them takes maybe a year.

The third lecture will be either, well, I want to talk about $\mathbb{CP}^1$, then in four an elliptic curve, and then in the fifth $K_{\mathbb{P}^1}$ which is $\mathcal{O}_{\mathbb{P}^1}(-2)$. These generalize to the toric case, to Calabi–Yau hypersurfaces, and to toric Calabi–Yaus. The mirrors are like $z + \frac{q}{z}$. To a hypersurface you have a mirror hypersurface.

Mirror symmetry has many different phenomena, and Lagrangian Floer theory seems to explain how this mirror symmetry work. I'll try to show how this machinery shows how mirror symmetry works in these three examples. If you're a symplectic geometer it will be very boring, please feel free to leave.

Basically, we're doing symplectic geometry so we're working with a symplectic manifold. We have this closed non-degenerate 2-form and the way to use the 2-form is whenever we have a covector you can turn it into a vector, isomorphically $T^*M \xrightarrow{\omega} TM$. So if you have a function, you can take df and then turn it into a vector field X_f, which gives you a flow, Hamiltonian flow (or diffeomorphism). Symplectic geometry is about symplectic diffeomorphisms that preserve the form, or about Hamiltonian flows. If you haven't seen this setup before then maybe try, for real starters, look at $\mathbb{R}^2$ with variables x and y. Look at $f = \frac{y^2}{2} + U(x)$. This is, y is momentum and U is potential. Here ω is $dx \wedge dy$.

The relation between X_f and df is that $df(V) = \omega(X_f, V)$. The $\mathbb{R}^2$ is like a phase space, you have position and momentum and then the flow tells you how you move around in the phase space.

The J, we want to relax a condition, is an "almost-complex structure," an endomorphism from TM to TM which squares to $-\mathrm{Id}$. We want it to be compatible with ω in the sense that $\omega(JV, JW) = \omega(V, W)$ and $-\omega(V, JV) > 0$. When I see this, I try to think of $dxdy(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$, Then $g(v,w) = \omega(v, Jw)$ is a Riemannian metric. This implies that the space of compatible Js is contractible. This is a good thing. Somehow symplectic geometry is about global phenomena. Later we'll change the almost-complex structures and find that something is invariant. We want the invariant to not depend on J.

We consider J-holomorphic curves from a sphere to M or a disk to M, and you can think of the sphere as $\mathbb{C} \cup \{\infty\}$, and the disk as, well, then in either case you have

$$\frac{\partial}{u}\partial s + J_u \frac{\partial u}{\partial t} = 0.$$

This is almost the Cauchy–Riemann equation, but J can look different in different places. Analysis people would say that this dependence on u makes this equation non-linear so that the analysis is difficult.

We want to say that this kind of J-holomorphic curve knows something about mirror symmetry.

So when you have this curve you can define its energy, the L^2-energy,

$$\iint | \frac{\partial u}{\partial s} |^2 + | \frac{\partial u}{\partial t} |^2$$

You can also consider the symplectic area

$$\iint u^*\omega$$

and actually these are the same notion, if you evaluate on two vectors, you are integrating

$$\iint \omega(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t})$$

and then you use the J-holomorphic condition and this is eventually $||\frac{\partial u}{\partial s}||$ or $||\frac{\partial u}{\partial t}||$. If you put in a half this is the L^2 energy.

These two numbers are analytic and topological respectively, so it's interesting that they coincide.

I should have said that the pseudoholomorphic curve should have Lagrangian boundary. If the dimension of L is half that of M and $\omega(v, w)$ is 0 for any v and w in T^*L. So the condition is for the boundary of the curve to lie on a chosen Lagrangian.

Now let $\tilde{M}(\alpha)$ be the space of all J-holomorphic curves whose homology class is that of α in either $H_2(M)$ or $H_2(M, L)$ as the case may be.

You want to analyze this moduli space, and it's improtant to look at the limiting behavior, the compactification, which is very important. It's called the Gromov compactification. Let's see how it goes.

So we fix α, then the symplectic energy is fixed, it's $\omega(\alpha)$. So this means that the L^2 energy is fixed, and that's somehow $\iint |Du|^2 < \infty$.

If you have a sequence of J-holomorphic curves, assume that u_n is a sphere mapping to M. What happens in the limit? The total integral is finite, but actually the values at indidual points go to ∞. In a picture what happens is [picture]

Let me look at an example. Send D^2 to itself by sendinig z to $\frac{z-\alpha_n}{1-\alpha_n z}$) where $\alpha_n = 1 - \frac{1}{n}$. The Lagrangian is the circle. Then this converges to -1.

Something terrible happened, this converged to a constant map. We need a better limit. You're moving 0 to $-\alpha$. Somehow [picture].

You need a new coordinate, which is $\tilde{z} = n(z - (1 - \frac{1}{n}))$. You magnify near this point and you can check that this map, in this new coordinate, $U_n(\tilde{z}) = \frac{\tilde{z}}{2+\tilde{z}-\frac{1}{n}(1+\tilde{z})}$. As $n \to \infty$ this goes to $\frac{\tilde{z}}{2+\tilde{z}}$

So what happens, roughly, is that z is $\alpha_n + \tilde{z}n$. If you take a derivative with respect to z, I'm confused, hold on.

[pictures]

So that's Gromov compactness. The more difficult part is gluing, which I won't talk about at all. Whenever you have a configuration of curves, you can find something limiting. Is it generally true that two holomorphic curves that touch have a holomorphic curve near their union? No, that's a delicate question. It takes a bunch of machinery to do this. We won't go into that and instead will just assume that the technique works.

You can ask why we care about pseudoholomorphic curves? Gromov used them to prove a surprising non-squeezing theorem. Why is it so powerful? It's related to Lagrangian Floer homology in the following way.

Think of a Lagrangian as a curve on a plane. Suppose you have these two Lagrangians L_0 and L_1. We consider $P(L_0, L_1)$, and γ is an element of it. We want, of course, the path space is too big. Ignoring this problem we want to consider some function and gradient flows on it. We fix some reference path. The most natural function on the path space, choose some bounding surface and consider its symplectic area. Then the action functional $A(\gamma, U)$ is $\iint_{\text{bounding surface}} u^*\omega$. The choice of bounding surface may not be unique. Then the symplectic area will change. So we're really defining this on the universal cover of the path space.

So A is defined from the universal cover to $\mathbb{R}$.

The question is, what is dA? To discuss this we need to know about the tangent space. The tangent space can just be thought of as the space where points move. [picture]

So we have a $u_\tau : [0,1] \times [0,1] \to M$ with the condition $\frac{d}{dt}|_{\tau=0} u_\tau(1,t) = \xi(t)$, along with regular homotopy conditions, and we evaluate $dA(\gamma)(\xi)$ and it's $-\frac{d}{d\tau}|_{\tau=0} \iint u_\tau^*(\omega)$.

You go to evaluate this and see that it's a Lie derivative and so you get

$$-\iint \mathcal{L}_{\frac{\partial U_\tau}{\partial \tau}} = -\iint d(i_{\frac{\partial U_\tau}{\partial \tau}} \omega)$$

and by Stokes you can get

$$\int_0^1 \omega(\gamma', \xi) dt = 1.$$

Then γ' must be 0 which implies that γ is a constant path and $L_0 \cap L_1$.

We'll see what a gradient flow is and that'll be gradient flow.

What is gradient flow? We have $\langle nableA, \xi \rangle = dA(\xi)$. The natural inner product is $\int_0^1 \omega(\eta_1, J\eta_2) dt$. This is a very natural definition. This looks like $\langle \nabla A, \xi \rangle = \int_0^1 \omega(\nabla A, J\xi) dt$. We can also see that $dA(\xi) = \int_0^1 \omega(\gamma', \xi) dt$. In the first case you can see we get $-J(\nabla A) = \gamma'$, and $\nabla A = J\gamma'$.

If you consider negative gradient flow, the family of trajectories, write it as $U(s,t)$. Then $\frac{\partial U}{\partial s}$ should be $-\nabla A(U)$. In the other direction you get $\frac{\partial u}{\partial t} = -J\frac{\partial u}{\partial t}$. To find these negative gradient flows, you should find solutions to this Cauchy–Riemann equation.

There may be infinitely many critical points in the cover of the path space, and to handle them all at once, you use the so-called Novikov ring, so that doing a deck transformation has to do with multiplying by some element of the ring.

In the cotangent bundle, well, think of T^*L, and the fibers are covectors. The zero section is a Lagrangian submanifold. The graph of a 1-form is also a Lagrangian. This should be C^2-small, and a Morse function. What are the intersection points? These are critical points of f. In this kind of setup, we want to look at the intersection points and count trajectories between them. We consider the vector space, $CF(L_0, L_1)$, the vector space generated by $L_0 \cap L_1$, with differential defined by counting the number of rigid J-holomorphic strips with limit p and q at $\pm\infty$ and with boundary on L_0 and L_1 respectively. Rigid means these don't come in families (except modulo the obvious translation).

[Discussion about Morse theory. Pictures.]

Floer's great idea is that there is a correspondence between Morse flow lines and J-holomorphic curves. Let's see how it goes. The basic idea is, you have the function $f : L \to R$, which has a gradient vector field on the Lagrangian. That's in the tangent space of the Lagrangian. Obivously, you have the covector df. In a neighborhood of the zero section you can define an almost complex structure by moving ∇f to df. You want to extend this near the zero section. You consider the projection to the zero section. You can project and then apply f, call that F. This goes $T^*L \to \mathbb{R}$. Then you can give it the Hamiltonian vector field X_F, and then you can consider its flow. So time t flow will be ϕ_F^t. So define $\tilde{J}$ by $\phi_F^t(J\phi_F^{-t})_*$. An easy exercise is to show that this one-parameter family of almost complex structures, well, anyway, I'm over time, if I have a gradient flow line, we'll lift by the Hamiltonian vector field, call this $\gamma(s)$, and this is supposed to be, show that, well, $u(s,t)$, call this $\phi_H^t(\gamma(s))$. You should show that $\frac{\partial u}{\partial s} + \tilde{J}(u)\frac{\partial u}{\partial t} = 0$. A more difficult thing to show is that this is a generic complex structure.

The picture is the following: [picture]

3. October 27: Cheol-Hyun Cho: Introduction to Fukaya category and mirror symmetry II

Last time we discussed about the J-holomorphic example. What we did was consider the path space and find that the kind of holomorphic strip, you can think of it as a Morse flow of paths and the critical points are given by intersection points between Lagrangians.

Let me formally write down the definition of Floer homology of the chain complex. The chain complex $CF(L_0, L_1)$ is generated by intersection points of $L_0 \cap L_1$ and the coefficients, it will be the direct sum $\Lambda\langle p\rangle$ where p is an intersection point. Here Λ is the Novikov ring, which we need because we have a choice of bounding surface. [picture] So instead of working on the covering space, we decided to work with the usual space with Novikov ring coefficients. To make a long story short, the Novikov ring can be written in the following way (this is the universal Novikov ring)

$$\Lambda = \sum_{i=0\infty} a_i T_i^{\lambda} | a_i \in \mathbb{C}, \lambda_i \in rR, \lambda_i \to \infty$$

you can have infinitely many strips with bigger and bigger energy, so you have to allow this kind of coefficient ring. The differential counts J-holmorphic maps u from $\mathbb{R}\times[0,1]$ where $\mathbb{R}\times 0 \to L_0$, $\mathbb{R}\times 1 \to L_1$, $-\infty\times[0,1] \to p$, and $\infty\times[0,1] \to q$. This hsould satisfy $\bar{\partial}_J u = 0$. You can write this $\frac{\partial u}{\partial s} + J_t\frac{\partial u}{\partial t}$, we want the almost complex structure to vary with t for transversality reasons. We count only rigid J-holomorphic maps. That means we count only those which are finite after modding out by translation.

Let me point out two things.

Remark 3.1. (1) The strip, you can look at u^*TM, and you can trivialize this tangent bundle and then you look at how much the boundary is rotating, which is measured by the Maslov index. What you do, if you have $\mathbb{R}^n$ times some unitary matrix, you look at the determinant square map $U(n)/O(n) \to U(1)$, which measures how much the Lagrangian subspace rotates. The full loop gives Maslov index two. If you have a loop, a loop of Lagrangian subspaces you get a rotation number. You have to connect from one to another when your loop is made of two strips. When we have L_0 and L_1, we get a canonical path by identifying L_0 with $\mathbb{R}^n$ and L_1 with $i\mathbb{R}^n$. There is a symplectomorphism between the two tangent spaces by rotating via $e^{i\pi t}\mathbb{R}^n$. The Maslov–Viterbo index is given by applying the canonical path and then its inverse at the two breaking points. The rotation number is the Maslov–Viterbo index. The expected dimension is the Maslov–Viterbo index. An easy exercise is, well, compute, you can find that this picture has index 1. It's one dimensional, just translation.

This is $\mathbb{Z}_2$-graded. Why is it $\mathbb{Z}_2$-graded? It's kind of important, instead of connecting by the canonical path or anything, when you just have one point, you want a number, which is only in $\mathbb{Z}_2$. You have T_pL_0 and T_pL_1. Choose any path between them preserving orientation. Any two such paths will differ by a number of full rotations, which gives a multiple of 2. I want an integer grading. So then we use the canonical path from L_1 to L_0 (ignoring orientation) and this gives a loop starting and ending at T_pL_0. This is some number which is the winding number, well-defined modulo $\mathbb{Z}_2$. Then if this winding number is η then switching the L_0 and L_1 is $n - \eta$.

Let's give the definition of the complex. Recall that we considered this kind of cotangent bundle, and we looked at the graph of a 1-form. We found that we have some kind of Morse flow downstairs which corresponded, we push it up by some Hamiltonian isotopy s, and this is $u(s,t)$ is a J_t-holomorphic strip. There was some correspondence when f is very small. the boundary is given by summing over the count of points in the moduli space from p to q times the symplectic area of the strip $T^{\omega(u)}\langle q\rangle$ where the difference between p and q is 1 in Maslov index and you're counting J-holomorphic strips from p to q.

So the next question is, is $\partial^2 = 0$? There are roughly three settings you could consider. The first one is the exact case, the second is the monotone case, and the third is the general case, where you eventually need the full machinery of FOOO. In the exact case, an exact symplectic manifold, you have $\omega = d\theta$ for θ a 1-form. This usually has to be non-compact, because the top power of ω has to be a volume form which is nontrivial for compact things. Then you can ask whether $\theta|_L$ is exact, and if this is dK there then we call L an exact Lagrangian. A good thing about exact Lagrangians is that they don't contain holomorphic disks or spheres. If you had a disk with exact Lagrangian boundary, then the L^2-energy $\int ||du||^2 = \int u^*\omega$ which, since the symplectic form is exact by Stokes' theorem is $\int_{\partial D^2} u^*\theta$ which, using Stokes' theorem again, is 0.

In fact, you don't even need to go to the covering space. You can work with $\mathcal{A}(\gamma) = -\int_\gamma \theta$. You can check that gradient flow gives, $\mathcal{A} : \mathcal{P}(L_0, L_1) \to \mathbb{R}$. You don't need the area term. You can define, redefine the generators so that you don't need that term in the differential. It's just an application of Stokes' theorem. [picture]

What is the area of this slice?

$$\int_{\text{strip}} u^*\omega = \int u^* d\theta = \int_{L_1} u^*\theta + \int_{L_0} u^*\theta = K_1(p) - K_1(q) + (K_0(q) - K_0(p))$$

and so you get

$$\langle p\rangle T^{K_0(p)-K_1(p)}$$

, which we rename $\langle\tilde{p}\rangle$ and then the differential goes from $\tilde{p}$ to $\tilde{q}$ without any area terms.

Let's do one more step, maybe I'm getting too technical, if you wanted to go to the $\mathbb{Z}$-grading instead of the $\mathbb{Z}_2$ grading, then, I wanted to define exactly, more explicitly, what is this chain complex. You need some data, additional assumptions. This is typically what happens. You have a holomorphic volume form, $dz_1 \wedge \cdots \wedge dz_n$ on $\mathbb{C}^n$, It's not $\mathbb{R}^{2n}$, this additional data. This works like a determinant. Any Lagrangian subspace, any $U(n)/O(n)$, to have a well-defined thing you want to square it, then you get a map to $U(1)$. If you have a Lagrangian, take an orthonormal basis and plug it in and you get a number. That happens, suppose you have a holomorphic volume form, at each point you look at the tangent space and plug in and get some number in $U(1)$ and you can write it as $e^{2\pi i\phi(p)}$. So ϕ, this argument, defines a map $L \to \mathbb{R}/\mathbb{Z}$. Then L is said to be graded if ϕ has a lifting from $L \to \mathbb{R}$. Somehow we already saw, at each point, we have this rotation of which path we choose. That's why, you choose this grading. Morally speaking, what's happening, you look at the point, [picture], this is like a complex plane, but what you are doing, you're going to the universal cover, and your Lagrangian, your tangent space is somewhere here. You can assign the degree of p to be $\tilde{\phi}_0(p) - \tilde{\phi}_1(p) +$ canonical path from $L_0 \to L_1$, which is an integer. Somehow if you know where you are, then you can define a path. Somehow that measures, and to make it into an integer you add the canonical path and then the Maslov–Viterbo, well, the degree of q minus the degree of p is the dimension of the space of u maps.

At each intersection point you have a grading in $\mathbb{Z}$ and the difference in grading accounts for the dimension of the moduli space of holomorphic strips.

If you have a loop in your Lagrangian, when you come back you should be on the same phase. If you come back one step up, you don't get something graded. When you come back the phase changes by 2π.

One more comment is that the exact condition is not homotopical. If you slightly move it, somehow the area, because of the area it's not going to be exact. Exact Lagrangians somehow choose a very particular version. In the exact case

Theorem 3.2. *You have $\partial^2 = 0$*

The proof is that if you have a strip of Maslov–Viterbo index 2, then there is a theorem which says that this can bubble in the compactification, after time translation, this looks like a 1-manifold with boundary, so what are the boundaries, you get a 1-manifold and the possible limits are [pictures]. But because in the exact setting, you don't get any bubbling, so the only possible limits are of broken strips. These now appear, you have a map from the strip to somewhere, and the strip is a noncompact domain. Just imagine what the maps do. Look at the L^2 energy of this map. You have a sequence of maps, and some part of the energy can escape to

∞. You have a sequence of maps where the derivatives move to ∞. If you count with suitable sign, they are supposed to cancel.

Let's go to the monotone case. This is the second easiest case that people discuss. This means the Maslov index μ which is attached to the homotopy class of a disk $\pi_2(M, L) \to \mathbb{Z}$, the rotation number around the boundary, you have the symplectic area, integrating the symplectic form. Monotone means that the Maslov index is a positive λ times the symplectic area. Then μ is positive for J-holomorphic disks. Let's look at the dimension of the moduli space of disks with homotopy class β and one marked point, so this is $u : D^2 \to (M, L)$, and z in the boundary and u is J-holomorphic, $[u] = \beta$, and you divide by the relation that $u : (D^2, \partial D^2, z) \to (M, L, p)$ and if you have another map $(D^2, \partial D^2, z') \to M, L, p)$, and if you have an automorphism, if u and $\tilde{u}$ differ by an automorphism, so you're modding out by an automorphism of the disk. The dimension is supposed to be $n + \mu(\beta) + 1 - 3$, whenever I used to see this dimension estimate I didn't like it very much, it seems hard to understand. To give you a feel, if you have a circle in the complex plane, then the honest disk with one marked point, the space of these is $1+2+1-3 = 1$. Without worrying about marked points and automorphisms, you get $e^{i\theta}\frac{z-\alpha}{1-\bar{\alpha}z}$, three dimensions, and winding twice you get something like $e^{i\theta}\frac{z-\alpha_1}{1-\bar{\alpha}_1 z}\frac{z-\alpha_2}{1-\bar{\alpha}_2 z}$, so you get 5 for Maslov index 4.

Let's look at the sphere mapping into a Kähler manifold M. Then we have for u^*TM the sheaves $\mathcal{E}$ of holomorphic sections, $\mathcal{A}$ of C^∞ sections, and $\mathcal{A}^{0,1}$ of $(0, 1)$-forms. We have the "fine resolution"

$$0 \to \mathcal{E} \to \mathcal{A} \xrightarrow{\bar{\partial}} \mathcal{A}^{0,1} \to 0$$

and the analytic index of $\bar{\partial}$ which is the kernel minus the cokernel is the global sections, this is H^0 and H^1, the fine resolution says this is the same as $H^0(\mathbb{P}^1, \mathcal{E}) - H^1(\mathbb{P}^1, \mathcal{E})$, and then you apply Riemann–Roch to show that the difference is the degree of the bundle, twice the degree of the first Chern number of $\mathcal{E}$ plus n times $\chi(\mathbb{P}^1)$, which is $2c_1(u^*TM) + 2n$.

I'll just use one minute and stop. In the monotone case what happens is, if you look at the moduli space of disks, if you put one marked point and evaluate the Lagrangian, $n + \mu + 1 - 3$, if this index is bigger than 2, the μ index, then what this means is that this index is at least n. If you have only one marked point, it will cover L, the dimension of the image will be n only when $\mu = 2$. So we need to distinguish $\mu = 2$ and the higher case. Let N be the minimal Maslov index of the monotone Lagrangian L. You get the following theorem:

Theorem 3.3. *If $N > 2$ then $\partial^2 = 0$.*

The proof is that, the boundary has parts with sphere bubbles, but generically they cannot appear because the moduli space of sphere bubbles is N and so the moduli space of strips has dimension $2 - N$.

Next time I'll discuss what happens with $N = 2$ and the toric case.

4. Nov. 10: Changzheng Li: Lagrangian fibrations on $Gr(2, n)$

I'll first talk about Gelfand–Cetlin fibration on $Fl_{n_1,\ldots n_k;n}$. Then I'll talk about a new fibration for $Gr(2, n) = Fl_{2,n}$. This is joint with Kwok-Wai Chan and Naichung Conan Leung, still work in progress.

I wanted to make a review for mirror symmetry for Grassmannians. Today I'll focus just on Lagrangian fibrations.

Let us start with a definition.

Definition 4.1. Let (M, ω) be a symplectic manifold. Then a submanifold L is called Lagrangian if it is half-dimensional and the symplectic form restricted to the submanifold vanishes.

Definition 4.2. Let (M, ω, Ω) be a Calabi–Yau manifold. This could be compact or noncompact. A Lagrangian L is called special if Ω restricts to a constant.

The goal is to construct a special Lagrangian fibration with total space M over some B. Here M will be open and dense in $Gr(2, n)$. The fiber generically should be a special Lagrangian of M.

We have some motivation from mirror symmetry. Why do we consider this problem?

Mirror symmetry says roughly that the A-model and the B-model coincide. Some information on the A-side, the symplectic geometry of the object, will be equivalent to the complex geometry of the mirror object. We'll try to make this more precise in order to say why we consider this.

Up to maybe 1993, there are some statements and conjectures, but I think the event is about the quintic, about the compact case with $c_1(X) = 0$. In other words this is the compact Calabi–Yau case. After that people wanted to generalize this to the Fano case, $c_1(X) \geq 0$. It was proposed by givental in 1993 and then later by Hari–Vafa. So I just want to say a couple of things about this. For this formulation, Calabi–Yau or Fano, then the mirror object, On the A side is (X, ω) compact and $c_1(X) = 0$ or $c_1(X) > 0$. On the B side the object is a family $\vee X_q, \vee W_q : \vee X_q \to \mathbb{C}, \Omega_q$ parameterized by $q \in H^{1,1}(X)$; $\vee W_j$ is holomorphic functions and Ω_q a holomorphic top form.

In the case of a Calabi–Yau, $\vee X_q$ is compact so $\vee W_q$ is constant. In the noncompact case this is a Landau–Ginzburg model.

In Givental's formalism, the quantum cohomology D-module of X is equivalent to, should be equivalent to the D-module generated by $\int_{\Gamma \subset \vee X_q} e^{\frac{\vee W_q}{\hbar}} \Omega_q$. This is imprecise but I wanted to say something.

For $c_1(X) > 0$, this conjecture holds for toric Fano (Fano just means positive first Chern class) (Givental 1993) or $Fl_n \coloneqq Fl_{1,2,\ldots,n-1,n}$ (Givental 1996).

If the conjecture holds, then the quantum cohomology of X as a ring is isomorphic to $Jac(\vee W_q)$. Then the dimension of the classical cohomology should be equal to the number of critical points of $\hat{W}_q$.

I want to say one more sentence about why we study Lagrangian fibrations. Maybe this is the same question, how can we construct $\vee X$ and $\vee W$? One approach is to construct a (special) Lagrangian fibration on X and then do holomorphic disk counting. It's likely that some wall-crossing phenomonena will be involved. This is the SYZ mirror symmetry approach.

If we want to construct the B-side, then one approach is to construct a special Lagrangian fibration and do disk counting. Later I'll say something about the Grassmannian. But this is our motivation.

We will describe this model for Grassmannians precisely. We'll discuss two fibrations.

Let us start with the Gelfand–Cetlin fibration, which is well-known, on $Fl_{n_1,\ldots,n_k;n}$. We consider the partial flag manifold, and then we have an embedding, well-known, of X in $\mathbb{P}(\wedge^{n_1}\mathbb{C}^n) \times \cdots \times \mathbb{P}(\wedge^{n_k}\mathbb{C}^n)$ so that $(X, \omega_{FS}|_X)$, which makes X a smooth projective manifold.

For example, for $Gr(2,n) = Fl_{2,n}$ this is $X \hookrightarrow \mathbb{P}^(\wedge^2 \mathbb{C}^n)$ where this is defined as $p_{ij}p_{k\ell} - p_{ik}p_{j\ell} + p_{i\ell}p_{jk} = 0$.

We have the natural action of the unitary group on $\mathbb{C}^n$ which induces an action on X so that $X = U(n)/\sim$. In the case $Gr(2,n)$, this is a quotient by $U(2)\times U(n-2)$. We can consider the moment map $X \to u(n)^*$, and X can be identified with the adjoint orbit $\mathcal{O}_\lambda$, where λ is a sequence — for $Gr(2,n)$ it's $(1,1,2,\ldots,2)$, and $\mathcal{O}_\lambda$ is matrices A in $Mat(n,\mathbb{C})$ such that $A^* = A$ and the spectrum of A is λ.

Once we're given a matrix A with fixed eigenvalues, where $\lambda_1 = \lambda_2 < \lambda_3 = \cdots = \lambda_n$, then we can consider square $(n-1)\times(n-1)$ submatrices. We can write down a sequence of square matrices like this; eventually we get to a 1×1 matrix. We keep the inequality; the sequence of eigenvalues is increasing.So we get a map $\mathcal{O}_\lambda = X \to \mathbb{R}^{\frac{(n)(n-1)}{2}}$ Some of the entries are constant, however. If we are considering $Gr(2,4)$, we can choose $1,1,2,2$, and then we get $a = 1$, $c = 2$, so we get a variable b and then d and e and f. So this X maps to a polytope in $\mathbb{R}^m$ which embeds in $\mathbb{R}^{\frac{n(n1)}{2}}$.

This image is a convex polytope and this map is continuous. Its restriction to the preimage of the interior of the polytope is a Lagrangian torus fibration (smooth, even). There is a toric degeneration of the flag manifold to a toric variety so that the generic fiber is the flag manifold and the central fiber is the flag variety. Then we get a fibratino over the same polytope. It's the moment map polytope for [unintelligible].

[picture]

This is like a toric degeneration. But $\varphi^{-1}(\partial\Delta)$ is not complex.

So from this Lagrangian fibration, then we can classify holomorphic disks. This is work by Nishinor–Nohava–Ueda (2009). We reduce this to the central fiber. After classifying this, you can write down the [unintelligible]function. The problem is that the superpotential W obtained in this way is the same as the previous one obtained by Givental for complete flags and also by Eguchi–Hori–Xiang and Kiem, well, by B–C–F–K–S, it's hard to say the names, different methods to get the same superpotential. It's sometimes good. For complete flags, it gives all the information and has enough critical points. But even for $Gr(2,4)$, the function obtained this way doesn't have enough critical points.

On the other hand, we have another superpotential. Rietsch gave $W_{Rie} : \vee X \to \mathbb{C}$ for G/P, conjecturally the correct Landau–Ginzburg model for all Lie types, in a representation theoretic way. Also by Marsh–Rietsch. This is difficult to read; they somehow reformulate it for the Grassmannian case in Plücker coordinates.

This is more readable, and we want to understand this in terms of the XYZ mirror symmetry approach.

We'll need to pick out a divisor. We start from $X = Gr(2,n)$. We know the first Chern class. There are many subvarieties that represent a class, so I should say $c_1(X) = [-K_X]$. For a Grassmannian and also for complete flags, just for these two extremal cases, somehow there is a standard choice, a "canonical" anticanonical divisor, given by $X \cap \{p_{12}p_{23}\cdots p_{n-1p_n}$ inside $\mathbb{P}(\wedge^2\mathbb{C}^n)$.

This divisor is good in several ways. It goes to (X_0, D_{X_0}) under the toric degeneration; that is, the anticanonical divisor degenerates to the anticanonical divisor. This is the first good property. The second good property is that the dual Grassmannian, $\vee Y = Gr(n-2, n)$, it's preserved under this action. Then $\vee W_{Rie} : \vee Y \backslash \vee D$ has a simple pole along the divisor. So $Y = X\backslash D$, there's not a unique choice of volume form. Let me say it in local coordinates, Ω_Y, in coordinates $z_{ij} = \frac{p_{ij}}{p_{12}}$ where $p_{12} \neq 0$. Then

$$\frac{dz_{13}\cdots dz_{1n} dz_{23}\cdots dz_{2n}}{z_2 3}\cdots z_{n-1,n} z_{1n}.$$

The dimension of the torus that is acting here will be much lower than the dimension of the target space. This torus action gives us a moment map. Just one thing, even though we have X in here, the diagonal matrix, the diagonal torus action is trivial. This torus is diagonal matrices in $U(n)$. Then the condition from $X \to \mathbb{R}^n$ is that we have n functions $(\mu_1, \ldots, \mu_n)$, but $\mu_1 + \cdots + \mu_n$ is a constant.

So we have $n-1$ linearly independent functions. We can only find an $n-1$-dimensional torus action. If we want a torus fibration, dimension $n-3$ is missing. We need to look for the remaining $n-3$ fibers.

Our statement is that

Theorem 4.3. *(Chan–Leung–Li)*
If we define $f_i = \frac{p_{1,i+2}p_{2,i+3}}{p_{12}p_{i+2,i+3}}$ *for* $i = 1, \cdots, n-3$, *then*

$$\mu_i = \frac{\sum_{j=1}^n |p_{ij}|^2}{\sum_{1\le j\le k\le n} |p_{jk}|^2}.$$

Then $(\mu_1, \ldots, \mu_{n-1}, |f_1|, \ldots, |f_{n-3}|) : X\backslash D \to \mathbb{R}^{2n-4}$ *is a special Lagrangian fibration on* $(X\backslash D, \Omega_{X\backslash D})$.

We're considering in our case $1 < 2 < i+2 < i+3$. We have $p_{12}p_{i+2,i+3} - p_{1,i+2}p_{2,i+3} + p_{1,i+3}p_{2,i+2} = 0$. We choose the first two terms to span. We have a torus T^{n-1} "in" T^n which acts on $Gr(2,n)$. We can consider the holomorphic vector fields of this torus action.

A key calculation is that if we contract the given volume form by the holomorphic vector field of the [unintelligible], it turns out to be the exact volume form of [unintelligible]. As a consequence, assuming this is Lagrangian, it's special. We follow Tyurin's idea [unintelligible]pseudotoric structures. The setup is also very similar. The point is to prove. You can somehow, f_i looks like $X//T$. Something looks like here. [too fast]

5. November 24: Cheol-Hyun Cho: Introduction to Fukaya category and mirror symmetry III

I'll change the subject a little bit. In the beginning I was planning two lecture series, two different topics. It seems like I'll only give one, so I combined the two and I'll go back and forth. Today I'm going to speak about quadratic differentials. Recently this has played an important role in studying the "stability conditions" for categories appearing in mirror symmetry. Today's lecture is a gentle introduction to this.

So let's start with the canonical bundle. S will be a Riemann surface and K_S is the canonical bundle of S. The fiber at p is the set of complex linear maps

$T_pS \to \mathbb{C}$. The cotangent bundle T^*S has fiber $\mathbb{R}$-linear maps $T_pS \to \mathbb{R}$. So these two are different. In local coordinates, what we're interested in, if we have K_S, then a section s locally looks like $f(z)dz$.

An Abelian differential or holomorphic differential is a holomorphic section of $\pi : K_S \to S$. We could also have a meromorphic differential, which is a meromorphic section. Before we proceed, let me tell you the Euler characteristics of these bundles.

The cotangent bundle, if you think of T^*S, the Euler number is, you think of a generic section and count the number of zeros with multiplicity. In this case, what you can do is choose a Morse function on S. Then the intersections happen at critical points. Then df at critical points will look like $-x_1dx_1 - x_2dx_2 - \cdots - x_\mu dx_\mu + \cdots + x_ndx_n$, and so I'll get $\sum(-1)^\mu$ which is $\chi(S)$.

Now K_S is a complex line bundle over S. I choose a meromorphic section ω over S. This will have zeros and poles. Then the Euler number is the sum of the index over all critical points. It will look like $z^n dz$ for some $n \in \mathbb{Z}$, and we try to write this as complex numbers, if $z = re^{i\theta}$ then this is $r^n(\cos n\theta + i \sin n\theta)(dx + idy)$.

We'll take the real part of ω, which will be a section of the contangent bundle (with poles) $r^n \cos n\theta dx - r^n \sin n\theta dy$, this is our section, and you can look at the intersection, when $r = 0$ this also intersects the zero section of the cotangent bundle. The intersection index, the multiplicity is $-n$.

[A lot of discussion]

A quadratic differential is a section of the tensor $K_S \otimes K_S$ which we just write as K_S^2, if it's meromorphic or holomorphic it is called a meromorphic or holomorphic quadratic differential on S.

Of course, you can define equivalence of quadratic differentials, if you have two surfaces S_1 and S_2 and a biholomorphic map h such that $h^*\phi' = \phi$ then it is equivalent.

In local coordinates it looks like $f(z)dz \otimes dz$ where f is either meromorphic or holomorphic.

Then the critical points of ϕ are the zeros and poles of ϕ. There is a natural coordinate, I think this is rather important. We're given the expression $f(z)dz \otimes dz$ away from critical points. We want to write this as $dw \otimes dw$ in some other coordinates. If $w = g(z)$ then $dw = g'(z)dz$, so $dw \otimes dw$ will look like $(g'(z))^2 dz \otimes dz$. So what should g' be? It should be $\sqrt{f(z)}$, and then we need to integrate, $g(z) = \int \sqrt{f(z)}dz$. The natural coordinate is well-defined up to sign, if $dw \otimes dw = du \otimes du$, then $w = \pm u$ plus a constant.

There's an associated metric $|f(z)|(dx^2 + dy^2)$. Then in the natural coordinates this is the regular flat metric $dx^2 + dy^2$. It's known that curvature doesn't change under holmorphic changes of coordinates. So it's this process, if you want to define a surface using $A4$ paper, you can make a cylinder. If you have a quadratic differential, then locally you see how to make it. Then the question is how you glue, and this is a more difficult part, this happens near zeros and poles.

To investigate the singular loci, what's important is the horizontal foliation. In natural coordinates, then it's really the horizontal foliation. When you transfer it to z coordinates, it will look more interesting, if $f(p)(dz(v))^2 > 0$

Let me show you an example. To draw them we need to identify these coordinates, locally suppose $\phi(z) = \lambda z^n dz \otimes dz$, with $n \in \mathbb{Z}$. Then the natural coordinate is given by taking square roots. Then $w = \int \sqrt{\lambda} z^{\frac{n}{2}} dz$ which is $\sqrt{\lambda} z^{n+2} 2$ when

$n \neq -2$. We'll discuss $n = -2$ separately. When $n = 0$, this we already know, then $w = \sqrt{\lambda} z$. [pictures, other examples]

Lemma 5.1. *If ϕ is a meromorphic quadratic differential, then the number of zeros of ϕ minus the number of poles of ϕ with multiplicity is $-2\chi(S)$ which is $4g-4$.*

This formula you can prove using Poincaré Hopf line fields instead of vector fields.

Let me mention, this is a standard result by Strebel. Assume that $g \geq 0$, $n \geq 1$, and $n \geq 2-2g$, and S a smooth compact Riemann surface and $p_1, \ldots p_n$ marked points on S. Choose an n-tuple of positive numbers. Then there exists a unique quadratic meromorphic differential ϕ such that

(1) ϕ is holomorphic on $S \backslash \{p_1, \ldots p_n\}$ and has a double pole at p_i (with imaginary $\sqrt{\lambda}$)
(2) The union of noncompact horizontal leaves form a closed subset of measure zero.
(3) All compact horizontal leaves are closed trajectories near marked points with $\int_{\alpha_j} \sqrt{\phi} = a_j$, the prescribed real number, where the branch is chosen so that the integral is positive.

[picture]

The horizontal foliation between zeros gives you a ribbon graph.

Let me tell you about the GMN differential. Here all zeros are simple and there is at least one pole and at least one zero or simple pole. This is Goresky–[unintelligible]–[unintelligible]. What's good about them is, a saddle trajectory is a horizontal leaf of finite length. Sometimes people use the word "saddle connection" but that's different, that's a phase θ leaf of finite length. Given this foliation, you can measure the angle of something that crosses it, so consant phase is constant angle.

Lemma 5.2. *A saddle free (no saddle trajectory) GMN differential which has a pole with an order greater than 2, then there is no closed trajectory and no recurrent trajectory.*

[pictures]

So you have a space of quadratic differentials which has a natural cell structure. The cells are when the polygonal decomposition does not change. You integrate something and get some coordinate. If you move to another cell the decomposition changes and you get a very strange coordinate change, and the whole together it may define [unintelligible]varieties. People want to understand, you can imagine, this is a Riemann surface. You're studying some algebraic or geometric category parameterized by this Riemann surface. People want to understand what happens as you move around this space. You have stability structure for the category and when you move to the next cell, the stability condition changes, but it doesn't change wildly, but there are wall-crossing phenomena.

Last week I had this slide in the last part of my talk, I had a Calabi–Yau threefold that was a conic fibration over this surface. At each zero you have a thing like this, and if you connect these two things, you get this vanishing sphere. These connections, you have a constant angle trajectory which corresponds to a Lagrangian three-sphere, and you can consider a quiver whose vertices are midpoints of this triangulation. [pictures]