INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS WEEKLY SEMINAR

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1. March 6: Hiro Tanaka: Lagrangian cobordisms, the Fukaya category, and dreams about mirror symmetry over ring spectra

So thanks for the invite. I wrote "I" here in case things don't quite finish. Because this is a big picture, it'll be painted with non-fine strokes. If you're an expert, you can raise your hands and complain. The big picture, what is this thing called the Fukaya category anyway? There's a conjecture announced publicly at the ICM by Kontsevich, called the homological mirror symmetry conjecture. Roughly speaking, this conjecture says that given some sort of sypmlectic manifold M, you can find a complex manifold $M^\vee$ which you can think of as an algebraic variety. There's a natural category you can define from this, sheaves, or if you don't like infinite dimensions, you can say coherent sheaves. If you want this to have nice properties, take the bounded derived category $D^bCoh(M^\vee)$. It wasn't clear until a few decades ago that you could get a category. Physicists and Fukaya started studying things and built this so-called Fukaya category and the conjecture is that these are equivalent categories.

What kind of category is this? It's geometric; often this means some sort of analysis. To fully describe it, and this is a reason that grad students had difficulty getting into it, it needs not only analytic language but also A_∞ categorical language.

In case that's the side that's familiar to you, if $M^\vee$ is $Spec\, R$ for some ring R, then D^b is chain complexes of finitely generated R-modules (with a finiteness condition on the homology). From the beginning of mathematics we've been interested in representations of rings, and this is contained somehow in symplectic geometry.

A lot of the motivation in studying symplectic geometry is to try to understand this picture.

Let me give some caveats. One thing that is annoying is that the right hand side still has some technical problems that need to be overcome. When I start describing it, a lot of geometry and analysis go into making this well-defined.

I'd like to list some complaints or questions about this conjecture. The first one is:

(1) Can we get rid of the analytical difficulties of defining the Fukaya category? This involves moduli spaces of solutions to a PDE, and that's annoying as hell.
(2) We can do derived coherent sheaves for any base, not just for $\mathbb{C}$. So can we generalize the Fukaya category of M to accomodate algebraic geometry over the integers or ring spectra?

There's a partial answer over $\mathbb{Z}$, but the second one is less trivial for people to answer. The answer is "who knows?" But there is a conjecture that David Nadler and I proposed a few years ago, which is that something that can do both at once is Lagrangian cobordisms.

I'm not saying there should be an affirmative answer, but maybe the answer could be yes in some cases.

For somebody who has never seen the Fukaya category, let me give some examples.

Let me give a brief introduction to Floer cohomology and the Fukaya category. Here's the setup. In this setup we'll fix a symplectic manifold M with form ω. An example might be $\mathbb{R}^2$ with $dx \wedge dy$. To be symplectic we need $d\omega = 0$ and $\omega^{\wedge top}$ is a top form. I won't just require that it's closed, but also exact. I'll take $\omega = d\theta$ where $\theta = -ydx$.

Now $\mathbb{R}^2$ also has more structure, J, an anti-involution of the tangent bundle. In my general setup I'll require this as well. I'll require that it be compatible with the symplectic form in the sense that $\omega(—, J)$ is a Rimeannian metric.

Now let L_0 and L_1 be half-dimensional submanifolds so that the pullback of θ vanishes. I'll require that $\theta = df_i$ on the Lagrangian submanifold L_i. This is the so-called "exactness" condition.

An example in the case of $\mathbb{R}^2$ is this picture. [Some discussion]

Now the Floer cochain complex $Floer_M(L_0, L_1)$ is a cochain complex generated by $L_0 \cap L_1$ with differential given by counting J-holomorphic strips.

Let me explain this. I promise that this is the most boring part of the talk. What's a J-holomorphic strip? Consider a set of maps $u : [0, 1] \times \mathbb{R} \to M$ such that, well, the infinite strip is some shape in the complex plane so it has a complex structure, and let's say that M is T^*R and I have two Lagrangians. I'll look at maps of the strip so that the top boundary goes to L_1 and the bottom to L_0. We demand that this respects complex structure on both domain and target. Finally, the limit as $t \to \pm\infty$ of u is in the intersection of L_0 and L_1. The constraint is insensitive to 0 on the real axis. I can now act on the strip by translations and still get the same conditions. The $\mathbb{R}$-action is free as long as the strip is non-constant.

What is the differential? The differential is given by, the differential of the point p is the sum over all points in the intersection of L_0 and L_1 of the number of strips from p at ∞ to q at $-\infty$.

In this cochain complex, well, we can consider this in the picture I've given you. For that example, the Floer cochain complex, as a vector space, has two intersection points. You can see that there is one differential from p to q and none from q to p. This is zero. I won't prove it, but it's invariant by Hamiltonian isotopy, that's another way to see it.

Definition 1.1. *The Fukaya category $F(M)$ has objects exact Lagrangians and morphisms the Floer cochain complex from L_0 to L_1.*

Now I need to define composition, which had better be a map from a composition to a multiple. I do this over all points in the intersection with weighted coefficients. We do tis by building a complex triangle

This composition is not associative. I'ts an A_∞ thing. Now we can have some fun.

Let's talk about Lagrangian cobordisms. Fix two Lagrangians.

Definition 1.2. *A Lagrangian cobordism is a Lagrangian inside $M \times \overline{(}T^*(0, 1))$ where the overline should be the reverse of the standard form. You won't die in the* *middle of my talk if you forget about it. If I look at it in small interval near* 0*, it should be* L_0 *times the* 0*-section of my cotangent bundle; on the other hand, near* 1 *it should be* L_1 *times the* 0*-section.*

In my talk I'll assume as well that P is exact. The condition is just a collaring condition. You can go to $T^*(0,1)$. You might notice I write $T^*\mathbb{R}$ instead of $T^*(0,1)$; forgive me. At the beginning and end it should be specified, above the zero section, but in the middle lord knows.

Choose a Lagrangian and cross it with the zero section. That gives "the identity cobordism."

Let's say we have a cobordism in the classical setting, $Y \subset \mathbb{R}^n \times (0,1)$. I can take $T^*_Y(\mathbb{R}^n \times (0,1))$, the conormal, the set of covectors that vanish on Y. This will be a Lagrangian that is collared if Y is.

Let me state a (pretty cool) theorem that shows that Lagrangians have a place in the discussion of Fukaya categories.

Theorem 1.1. *Biran-Cornea have also done amazing things, they came close to proving this theorem. Let* M *be an exact symplectic manifold and suppose we fix two objects* L_0 *and* L_1 *in the Fukaya category, and suppose they admit a compact exact cobordism. Compact means compact on some interval close to* $(0,1)$*. Then* L_0 *and* L_1 *are equivalent in the Fukaya category.*

We know that cobordisms preserved things and what they preserve depends on the structure of the cobordisms. This is like saying "what to Lagrangian cobordisms preserve?" They preserve the Floer theory.

Here's a "proof." Say we have a compact cobordism P, collared by L_0 and L_1. For any object X in the Fukaya category of M, consider the curve γ in T^*R, and consider the Floer cochain complex in $M \times T^*\mathbb{R}$ between $X \times_\gamma, P$.

We see at the cotangent bundle level, they intersect at two points. We have intersections of L_0 with X and L_1 with X. The differential, what is D? Remember, I drew a strip in a similar looking picture. Because we've reversed the orientation, we're counting strips in the opposite direction. You might also see strips that work in the M direction. The upshot of all this is that the differential can be written explicitly.

$$\begin{pmatrix} d_{X,L_1} & \Xi_P \\ 0 & d_{X,L_0} \end{pmatrix}$$

This is the mapping cone of Ξ_P. The $D^2 = 0$ condition implies that Ξ_P gives a map of cochain complexes between $Floer(X, L_0)$ and $Floer(X, L_1)$. Whatever the cohomology of this is, it's zero. The map then is an equivalence. Hamiltonian isotopy of γ implies that the cohomology of the Floer complex is zero, so the linear map is a quasiisomorphism. By the Yoneda lemma these represent the same object in homology.

The issue is that this is just at the level of the objects, this isn't a natural transformation. How can one make a natural transformation of functors.

We can't invoke Yoneda because we don't know that Ξ_P defines a natural transformation.

How can we resolve this issue? I'll do it in a birds eye view way. We're working in the category of functors from the Fukaya category to chains. If we could produce a functor with target this functor category, its image will be a natural transformation that gives me the result I want.

Theorem 1.2. *There exists a functor from $Lag^{compact}M$ (whose objects are exact Lagrangians and whose morphisms are compact exact cobordisms) to the functor category so that any Lagrangian L goes to $Hom(—, L)$ and P goes to Ξ_P.*

There's something dissatisfactory about this theorem that I'll rectify. So first, if I like algebra, I realize the target category has interesting structure. It's a nice algebraic category, but can cobordisms see this structure? Can cobordisms see the triangulated structure of the functor category?

The second disatisfaction is, "are there a lot of compact exact cobordisms?"

What if every one is just the identity cobordisms? Then this theorem doesn't tell us anything.

So the second one is a dissatisfaction, a serious one.

Theorem 1.3. *Let Q be a smooth manifold with dimension at least 5, simply connected. Let L_0 and L_1 be exact compact Lagrangians in T^*Q. Then any compact exact cobordism P from L_0 to L_1 is diffeomorphic to a cylinder. In particular, L_0 and L_1 are diffeomorphic.*

Let me give two comments. For people really into symplectic geometry, this might be reminiscient of the nearby Lagrangian conjecture. Do cobordisms have an application to the nearby Lagrangian conjecture? Maybe. Let me remind you, the cotangent bundle of Q has an exact Lagrangian, the 0 section. You might have other exact Lagrangians, is it isotopic in a symplectic sense to the zero section? This would prove these would be diffeomorphic.

This also suggests that the second concern is serious.

Let's resolve both dissatisfactions at once.

Theorem 1.4. *Fix $\Lambda \subset M$ (I'll explain this part later). Then there exists a category $Lag(M)$ whose objects are not necessarily compact exact Lagrangians and morphisms are not-necessarily compact exact cobordisms both of which have some relation to Λ such that*

- *(Nadler-Tanaka) $Lag_\Lambda(M)$ is stable (Lurie-Toen) or triangulated.*
- *Moreover, there exists a functor from $Lag_\Lambda(M) \to Fun(Fuk(M)^{op}, Ch)$ extending the functor from the previous theorem and respecting the triangulated structure.*

This is a great resolution. Part two said there weren't enough cobordisms, so I threw in more, and part one said there was algebraic structure and throwing in more gives me that algebraic structure.

Now there's two directions I could go. I could give a full onslaught definition, I could give the functor, or I could give future directions now that this is in hand.

Any question you asked about Fukaya you can now ask about cobordisms. So the first future direction is the analog of Fourier-Mukai transform. Let me remind you what I mean. If I have two varieties X and Y, I can look at coherent sheaves on both of them, and I can also look at coherent sheaves on $X \times Y$. There's a famous push-pull construction I can make. In the case X and Y aren't such nice varieties, I can still do something. This is a common construction. If you believe in mirror symmetry you should be able to do this in the Fukaya category. There's some map, or should be from $Fuk(M \times N)$ to $Fun(Fuk(M), Fuk(N))$. This is a hard thing to do. Let me state a theorem in progress, or I should say work in progress, to construct a functor $Lag_{\Lambda_1 \times \Lambda_2}(M_1 \times M_2) \to Bimod(Lag_{\Lambda_1}(M_1), Lag_{\Lambda_2}(M_2))$.

You have the right to ask me why this is interesting. This connects to two, three, and four manifolds invariants. What does some general philosophy from TQFT tell us? It should be a functor from a cobordism category into a target category $\mathcal{C}$. Evaluating this on a manifold gives an invariant. What people who work in QFT try to do is to construct symplectic manifolds and correspondences. To every 2-manifold Σ you get a symplectic manifold M_Σ. To every three manifold with boundary you get a Lagrangian inside $\overline{M_{\Sigma_1} \times M_{\Sigma_2}}$. What if the category you're mapping into is the category of A_∞ categories? Then to M_Σ I can give the Fukaya category. This should be able to give a functor from one Fukaya category to the next. In other words, a field theory to correspondences should postcompose to give invariants of manifolds. You might think this is a fairy tale. In fact Denis Auroux's ICM talk, this is what they do where the manifold invariants they give are the bordered Heegaard Floer invariants (at least for dimension three). If I can construct this functor of Lagrangian cobordisms gives us a new functor to Cat from $Corr$ so a new family of invariants. These should be able to give new invariants which come with a map to already defined Heegaard Floer invariants.

2. March 13, 2014: Hiro Tanaka, Lagrangian cobordisms, the Fukaya category, and dreams about mirror symmetry over ring spectra II

This is my last talk, I'd like to thank everybody again. Marcia and I brought some cake. Let me review some of what we talked about last time. We're talknig about Lagrangian cobordisms. The theorem I wanted you guys to remember was the following

Theorem 2.1. *Given a symplectic manifold (of a certain type) M and a subset Λ we could define $Lag_\Lambda(M)$. The objects were Lagrangians $Y \subset M$ and morphisms were cobordisms with some conditions. The theorem says there's a functor $Lag_\Lambda(M) \to Fun(Fuk_\Lambda(M)^{op}, Chain)$. This captures geometry but doesn't have nice algebraic properties. There's no sense in which I can take a mapping cone of morphisms in Λ. So we use the Yoneda embedding.*

This isn't just a functor, it repsects the stable (triangulated structures)

One corollary, how does this behave? If I have Y I should get an object in the functor category. It goes to $Floer_M(—, Y)$. As a corollary, we get the following:

Corollary 2.1. *If two compact Lagrangian (branes) admit a compact brane cobordism, they are equivalent in the Fukaya category.*

Let me give you an outline for what I'd want to do today. I'd like to define the category with a little more rigor and consider the role of Λ and the stable structure. So I need to tell you what kernels and cokernels are. I'd also like to give you an idea about the topology on the space of morphisms. In the second part I'd like to describe the functor and the relationship with work of Biran-Cornea, who have a result connecting Lagrangian cobordisms to the K-theory of the Fukaya category. You might expect this at the level of categories. These are both triangulated and you can get the K-theory naturally, recovering the results of Biran-Cornea.

Here's the setup. Fix a symplectic manifold M such that the symplectic form is the derivative of a one-form. So we know our manifold is not compact. We get a vector field X_θ so that $\iota_{X_\theta}\omega = \theta$.

For example, $M = T^*Q$, then $\theta = pdq$ and $\omega = dpdq$ while $X_\theta = p\partial_p$.

If the manifold is the circle, the flow of X_θ is the outward flow.

Let me make a definition.

Definition 2.1. *Let the skeleton Λ^{sk} be the limiting set of the flow $-X_\theta$. Let Λ be a subset containing the skeleton and closed under the flow.*

Now $Lag_\Lambda(M)$ will be defined in a few steps.

Definition 2.2. *$Lag_\Lambda^{\diamond 0}$ has objects Lagrangian branes in M; these are tuples (Y, α, f) where Y is a Lagrangian, α is a grading, and $f : F \to \mathbb{R}$ such that $df = \theta|_Y$.*

The morphisms $hom(Y_0, Y_1)$ are Lagrangian brane cobordisms $Y_{01} \subset M \times \overline{T^*\mathbb{R}}$ so that Y_{01} is collared by Y_0 and Y_1. Toward negative ∞ this looks like Y_0 times the zero section; near ∞ it looks like Y_1 times the zero section.

We also require Y_{01} to avoid Λ near $-\infty dt$.

Last time when I tried to give a proof of the corollary, I drew a curve that passed under the cobordism. What does this condition mean? There's an ϵ and a $\tau_0 \in \mathbb{R}^\nu$ so that for all $(y, (t, \tau))$ in $Y_{01} \subset M \times \overline{T^*\mathbb{R}}$ with $\tau \leq \tau_0$, the distance from y to Λ is at least ϵ. So this is how I'm going to use this to make sure I get no intersection.

This category, I've defined a set of morphisms. I want to put some topology on this space. Let me just tell you what a path is.

Definition 2.3. *A homotopy between two cobordisms is a Lagrangian brane W in $M \times \overline{T^*\mathbb{R}_t} \times \overline{T^*\mathbb{R}_s}$.*

I have at Y_{01} a cobordism, and also Y'_{01} and I interpolate between them by W. So W is contained in a box with boundary Y_0, Y_1, Y_{01}a and Y'_{01}.

If I look for instance at $W|_{s\leq 0} = Y_{01} \times [-\infty, 0]$. For $W|s \geq 1$ I want $Y_{01} \times [1, \infty)$. There is also a Λ-avoiding condition.

What's a homotopy of homotopies? It lives in $(T^*\mathbb{R})^3$

Let me give you some examples. Given a Hamiltonian $H : M \times \mathbb{R} \to \mathbb{R}$, let $\phi : M \times \mathbb{R} \to M$ be its flow. For every $Y \subset M$ we can look at $Y \times \mathbb{R} \to M \times \overline{T^*\mathbb{R}}$ by taking (y, t) to $\phi(y, t), t, H(\phi(y, t), t))$. When is this a morphism? It needs to satisfy the collaring conditions. So $H|_{Y\times(-\infty,t_0)} \equiv H|_{Y\times(t_1,\infty)} \equiv 0$. Finally, $H\phi|_{\Lambda\times\mathbb{R}}$ is bounded.

Let me make a digression to discuss grading. If I have M, I have the canonical bundle TM. I can try to put a complex vector bundle on TM. Let's say it's an almost complex structure. I can ask whether c_1 vanishes. Let's assume that it does. What does this do for us? There's a canonical bundle, the bundle of Lagrangian Grassmannians on M. I can look at vector spaces in the tangent space that are Lagrangian. It's a homogeneous space, $U(n)/O(n)$. Of course, there's a natural map, "the determinant squared" to S^1. If $2c_1(TM) = 0$ then you can guarantee that the determinant squared map lands in a trivial circle bundle over M. Fix a trivialization. A Lagrangian gives you a map to this bundle, so you postcompose to give $L \to S^1$.

Definition 2.4. *A grading on L is a lift $\alpha : L \to \mathbb{R}$ lifting the map $L \to S^1$.*

This is the $\mathbb{Z}$ that shows up in shifting the chain complex. This is a constraint on my Lagrangians (and cobordisms). They must admit such a lift.

Now fix a curve c in $T^*\mathbb{R}$ that shoots off to $+\infty$. Then $Y \times c$ is a cobordism from $Y \to \phi$. Usually only a few manifolds are cobordant to the empty manifold. But for us this is good. Every object gives a map from and to the zero object in this way, the empty manifold will be our zero object.

Proposition 2.1. *If Y_{01} is a cobordism such that $Y_{01} \subset M \times \mathbb{R}_t \times [\tau_0, \tau_1]$, then Y_{01} is an equivalence.*

If we prove this, the corollary about compact cobordisms follows. This is an easy proposition to prove. If you're given Y_{01}, let's look at some dumb Lagrangian $M \times \overline{T^*\mathbb{R}_t} \times \overline{T^*\mathbb{R}_s}$. Choose a diffeomorphism from a rectangle of height ϵ to a semi annulus (which sits inside a big rectangle). This induces a map on Lagrangians. What does it look like? It looks like Y_{01} going around to be $\overline{Y_{01}}$. It's switched (t, τ) to $(-t, -\tau)$. Things going to $+\infty$ now go to $-\infty$. In particular it's Λ-avoiding. Because I'm collared on the left by Y_0 and on the right by Y_1, I can cover the rest of my rectangle with Y_0 and Y_1. So this becomes a cobordism between the identity on Y_0 and a composition Y_{01} with $\overline{Y_{01}}$. That's basically the idea of the proof.

This also shows that if your Hamiltonian is bounded away from Λ you get the same sort of thing even if it's not bounded, not compact.

Now I want to talk about kernels. Note that there exists a functor $Lag_{\Lambda}^{\diamond 0}(M) \to Lag_{\Lambda\times\mathbb{R}}^{\diamond 0}(M \times \overline{T^*\mathbb{R}})$. Just cross with a lagrangian in $T^*\mathbb{R}$. I can take $Y \mapsto Y \times R^{\nu}$ or Y_{01} to $Y_{01} \times \mathbb{R}^{\nu}$. It's way easier to study cobordisms in $\mathbb{R}^{\infty}$. In usual cobordism theories, if you take conormals, you get exactly this to go toward $\mathbb{R}^{\infty}$.

It seems natural to pass to this next category. The kernel will exist in the next category. Given a cobordism $Y_{01} \subset M \times \overline{T^*\mathbb{R}_t}$, you can construct another Lagrangian where Y_0 and Y_1 are curved toward $-\infty$. This creates an object in $Lag_{\Lambda\times\mathbb{R}}^{\diamond 0}(M \times T^*\mathbb{R})$.

Let me call this $K(Y_{01})$.

Let $Lag_{\Lambda}(M)$ be the union of $Lag_{i\geq 0} Lag_{\Lambda\times\mathbb{R}^i}(M \times T^*\mathbb{R}^i)$.

The natural thing to do is take a colimit or homotopy colimit here. In our paper, the model we use is quasicategories. The homotopy colimit is the honest colimit, so it's just union.

Theorem 2.2. *Then $K(Y_{01})$ is the homotopy limit of the diagram $0 \to Y_1 \xleftarrow{Y_{01}} Y_0$. If you don't care about homotopy theory, if you get rid of that word it's the literal kernel.*

The real stable is that this is a stable infinity category. If you go to π_0 then this is triangulated. I'll be showing that the diagram with the kernel goes to a mapping cone in the world of chain complexes.

Now I want to turn to functoriality. I should give a functor $Lag_{\Lambda}(M) \times Fuk_{\Lambda}(M)^{op} \to Chain$ This is like the Hom functor $\mathcal{C} \times \mathcal{C}^{op} \to Set$. This takes (Y, X) to $Hom(Y, X)$. So given $Y_0 \to Y_1$ and $X_0 \to X_1$ we need commutativity of the following:

$$\begin{tikzcd}
Hom(X_1, Y_0) \arrow[r] \arrow[d] & Hom(X_0, Y_0) \arrow[d] \\
Hom(X_1, Y_1) \arrow[r] & Hom(X_0, Y_1)
\end{tikzcd}$$

So commutativity follows from associativity of our category $y \circ (f \circ x) = (y \circ f) \circ x$.

A word of caution. In the A_∞ setting, this associativity only holds up to homotopy. This means that there exists an H such that $dH \pm Hd = m^2(y, m^2(—, x)) \pm m^2(m^2(y, —), x)$

In our setting, given a Lagrangian coboridsm Y_{01} and $x : X_0 \to X_1$ in the Fukaya category, we want a homotopy commutative diagram. Remember the functor gives us $Floer(X_0, Y_0)$. By contravariance we can go to $Floer(X_1, Y_0)$. Last time I defined a $\Xi_{Y_{01}}$ which goes from $Floer(X_1, Y_0)$ to $Floer(X_1, Y_1)$.

Before we go further, the Fukaya category $Fuk_\Lambda M$ is the Fukaya category where every object is contained within an epsilon-neighborhood of Λ. You could fix a uniform ϵ for Lag_Λ if you wanted. If you have a Λ you can define a partially wrapped Fukaya category. If I want a Floer cochain complex, I need a restriction in Lag_Λ on my cobordisms.

Here's a proof sketch that one can construct a functor like this. Consider the following diagram. Given X_0, we crossed it with γ, a manifold in $M \times T^*\mathbb{R}$. We took our cobordism and looked at the Floer cochain complex. There were two Lagrangians. We got maps of Floer complexes from this.

Let's introduce another object and another curve that looks similar, γ_1. Now I'll study the m_2 structure of this equation.

Recall, given objects Z_0, Z_1, and Z_2 in the Fukaya category, there exists $m_2 : hom(Z_0, Z_1) \otimes hom(Z_1 \otimes Z_2) \to hom(Z_0, Z_2)$. This satisfies the Leibniz rule. Disregarding signs, $m^1 m^2(a, b) = m^2(m^1 a, b) \pm m^2(a, m^1 b)$. Here m^1 is the differential in each hom complex. Let's study m^2 inside $M \times \overline{T^*\mathbb{R}}$. It will go

$$Floer(X_1 \times \gamma_1, Y_{01}) \otimes Floer(X_0 \times \gamma_0, X_1 \times \gamma_1) \to Floer(X_0 \times \gamma_0, Y_{01}).$$

Here's what I'm going to do. I want to study the space of holomorphic disks that have boundary x and q where x is a boundary between X_0 and X_1 and q is a boundary between Y_0 and X_1. There are two possible intersections of Y_0 or Y_1 and X_0.

The A_∞ relation tells us that $m^1 m^2(q, x) = m^2(m^1 q, x) + m^2(q, m^1, x)$. There are two terms. The one type of disk happens entirely before the cobordism and that corresponds to $m^1 m^2_M(q, x)$ and the other side is $m^1 \Xi^2_{Y_{01}}(q, x)$.

We can play the same game for the left hand side. There are two kinds of differential of q. One is from M and one is from the interaction with Y_{01}. Likewise for $m^1 x$.

I can split again for m_1 or for m_2. When I take $m^1 \Xi^2_{Y_{01}}$, there's nowhere for it to go except to the degenerate M direction. We can do the same thing on the other side. Finally we get

$$m^1_M m^2_M(q, x) + \Xi^1 m^2_M(q, X) ? m^1_M \Xi^2_{Y_{01}}(q, x) = m^2_M(m^1_M q, x) + \Xi^2(m^1_M q, x) + m^2_M(\Xi^1_{Y_{01}} q, x) + m^2_M(q, m^1_M x) + \Xi^2(q, m$$

You cancel the pure m terms. The terms with m_1 and Ξ_2 are the homotopy terms. The terms with $m^2_M(\Xi_{Y_{01}} q, x)$ and $\Xi^1 m^2_M(q, x)$ are the things you want to compare, and the A_∞structure gives you what you want. The other two terms vanish; one because x should be closed and the other for geometric reasons; a certain space can't be of the appropriate dimension.

I've constructed by this sketch a functor $Lag^{\diamond 0}_\Lambda \to Fun(Fuk^{op}_\Lambda, Chain)$. The question is then whether I can lift to a functor from $Lag^{\diamond 1}$. To make the extension, givesn an object $X \in Fuk_\Lambda(M)$, we can map it to $X \times \mathbb{R}$, we can do Floer theory. What does this mean? I just need to show that this diagram commutes. If I start in $Lag^{\diamond 0}$ I go to $Y \times \mathbb{R}^\nu$. If I have X and multiply by the zero section, then I go to $Floer(X \times \mathbb{R}, Y \times \mathbb{R}^\nu)$. Then the Floer complex will be precisely $Floer(X, Y)$. So the functor is compatible with the procedure for stabilization. Now I need to show that it respects the triangulated structure.

Does this preserve kernels? Say we have a map $Y_0 \to Y_1$. I go to a natural transformation $Floer(—, Y_0) \to Floer(—, Y_1)$. Take now $K(Y_{01})$. What does it go to? It goes to $Floer(\text{—} \times \mathbb{R}, K_{01})$. I should show that this is the kernel of the Floer map. This is actually the same computation we did last time.

Recall that the kernel (homotopy kernel) of a map between chain complexes $A_0 \to A_1$ is given by the chain complex $A_0 \oplus A_1[-1]$ with differential $d_{A_0} + d_{A_1} + f$, maybe with a sign, the shifted mapping cone. I just need to show that these two chain complexes are equivalent, $Floer(\text{—} \times \mathbb{R}, K(Y_{01}))$ and $Floer(X, Y_0) \oplus Floer(X, Y_1)[-1]$.

So draw the same picture and compute the hom complex. By assumption, X is within ϵ of Λ. I can move by Hamiltonian isotopy to a part where Y_{01} must be ϵ away from Λ. So we get $Floer(X, Y_0)$ and $Floer(X, Y_1)$ and if you look at slopes you see the degree shift. I get the regular Floer disks in each fiber and then f is exactly what you get going to the other kind of holomorphic disk.

This completes the proof. It's a cool result showing that the geometry of cobordisms captures a lot of the algebra of the Fukaya category.

Now let me just relate this to results of Biran and Cornea. They are more general and more restrictive. They don't just assume exact symplectic manifolds. They also consider something more constricted where everything in sight is compact. By using non-compact things I can prove theorems. In the exact setting where everything is compact our results are compatible.

Their theorem is:

Theorem 2.3. *(paraphrased) Given the Fukaya category of the symplectic manifold, you can look at $T^sFuk(M)$, which captures exact triangles. The objects are ordered sequences in the Fukaya category. If I fix one such object, what's a morphism? It's a triangular decomposition. Look at a sequence of exact triangles—*

[too fast]

They construct a functor from cobordisms to $T^sFuk(M)$.

If they have cobordisms with multiple ends, it captures the data of a resolution of x by y_i. How do you capture exact triangles? Look at the kernel.

How are these compatible? Let's do it for two objects. If you're given a cobordism, geometrically it's the same as the cobordism except bending some curves up to $+\infty$. If you only have two outgoing things on the left, I get something in Lag with one end escaping to $+\infty$.

Let's take the kernel. That should give an exact triangle. It gives me a picture like this. I claim that this kernel recovers exactly the exact triangle decomposition. How is that so? If I take a test object $z \in Fuk_\Lambda(M)$, look at the intersection of $z \times \gamma$ with my kernel. We have $Floer(z \times \gamma, p) \cong ker(Floer(z, y_1) \to Floer(z, x))$. by a Hamiltonian isotopy, this is equivalent to $Floer(z, y_2)$. I've messed up the indexing somehow. This was supposed to show that by turning their legs toward positive infinity, you get exactly their result.

Whatever is happening in the universe, we agree on what things should encode, so maybe we're doing the right things. I'll stop my talk there.

3. March 20: Joan Millès: Complex structures as homotopy algebras

It's a pleasure to be here and visit Korea. I'll speak about complex structure as homotopy algebra. First of all I'll ask a general question, which is the following. Consider a geometric structure. By geometric structure I mean something like a Poisson manifold, complex manifold, others, maybe symplectic manifold, I'll make this precise a little bit more. We'd like to know if there's a way to present or understand this structure in terms of algebra.

I'll explain later what I mean by this. But first there's is something related to this, a method used by Kontsevich, who likes to think of L_∞ algebra in geometric terms. This is Lie algebra up to homotopy. It's a kind of algebra, well, he wants to consider them as pointed formal manifolds, endowed with a formal vector field. This means there might be a link between the geometric and algebraic notion. Following and reversing this idea, Merkulov explained in some sense or some case how to associate to a geometric structure (in some case) an algebraic data. The examples are the following:

Geometric structure	Algebraic data
Poisson manifold	Lie_1 bialgebra
Nijenhuis manifold	"Nij" algebra
F-manifold	Gerstenhaber algebra
Complex structure	?

In this talk I'll try to explain what to put here and how to find what to put there in general. Then we will show that it really is a good notion.

[Jae-Suk: Witten gave this idea before Kontsevich was born mathematically. This idea should be attributed to Witten. All of his work has been variation of this idea. The association of algebra to geometry is unfair to give to Merkulov, others including me did this much earlier. Also, I think it should be BV and not Gerstenhaber for the F-manifold case.]

So let's formalize things.

3.1. **Formal complex manifolds.** We'll eventually make this intelligible in terms of algebras over an operad. That's later. A *formal* manifold I don't want to give the full definition, it's the same as in algebraic geometry. I'll recall what I need. The word "formal" refers to the fact that we will deal with Taylor series of functions around a point. This means, take a smooth function, look around the point, you can take the information of the differentials at the point. If the geometric notion is defined only from the differentials, you can difine everything in these terms.

Thanks to this, we don't have to deal with convergence problems. We'll use this to make something not algebraic into something algebraic.

I should first explain a complex structure, at least in the smooth case.

A complex manifold; there are two ways (at least) to describe one. Either you can take copies of $\mathbb{C}^n$ and glue with holomorphic transition functions. Or you can start with a smooth manifold M and add what is needed to make it complex, which is an endomorphism $J_x : TM \to TM$ so that $J_x^2 = -Id_{T_xM}$. This has the interpretation that J_x should be thought of as multiplication by i. So locally you can provide a complex coordinate. To have a complex coordinate locally is not enough; we'd like to glue these locally to get global complex coordinates. The gluing is not algebraic but there is a theorem due to Newlander–Ninenberg, which expresses the gluing condition in a way that is only local.

Theorem 3.1. *J is a complex structure if and oly if*

(1) $J^2 = -\mathrm{id}$ *and*
(2) $N_J = [J, J]_{F\text{–}N} = 0$.

I'll be a little more precise about what it means.

Remark 3.1. *The Frolïcher-Niejenhuis bracket is a generalization of the classical bracket on vector fields to vector forms. On X and Y vector fields, $N_J(X,Y) = J^2[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY]$.*

In the formal case, let V be a finite dimensional vector space. We fix a basis $\{e_\alpha\}$. A coordinate system on V is given be $x^\alpha = (e_\alpha)^*$. Now I can define a formal (pointed) manifold as $\mathcal{V} = (pt = 0, \mathcal{O}_\mathcal{V} = \widehat{SV^*} = \mathbb{R}[[x^\alpha]])$.

So you can think of this manifold as a point, but in a formal neighborhood. We want local information around this point. It's bigger than a point but smaller than a neighborhood.

Everything for the first two sections will work in characteristic zero.

To speak about J we need a notion of the shifted tangent sheaf. I'll denote it $T\mathcal{V}[1]$. It's the $\mathcal{O}_\mathcal{V}$ module freely generated by $\{\frac{\partial}{\partial x^\alpha}$ to $T_\mathcal{V}[1] \cong \emptyset_0 \otimes V[1]$.

The differential forms are built by means of the dual basis $\{sdx^2 = (s^{-1}\frac{\partial}{\partial a^\alpha}^*)\}$. We have $\Omega_\mathcal{V} = \wedge_{\mathcal{O}_\mathcal{V}}(\mathcal{O}_\mathcal{V} \otimes V[1]^*) = \mathbb{R}[[x^\alpha, dx^\beta]]$. An endomorphism $\tilde{J} : T_\mathcal{V}[1] \to T_\mathcal{V}[1]$ is the same thing a an element J in $\Omega^1_\mathcal{V} \otimes_{\mathcal{O}_\mathcal{V}} T_\mathcal{V}[1]$. That is, $J = J^b_a(x) dx^a \otimes \frac{\partial}{\partial x^b}$. Here $J^b_a(x) = \sum_k j^b_{\alpha_1,\cdots,\alpha_k;a} x^{\alpha_1} \cdots x^{\alpha_k}$.

The Taylor series at a point is given by these coefficients j. We're just taking the Taylor series of my endomorphism J.

Now I will describe the Frölicher-Nijenhuis bracket just for this $J = J^b_a(x) dx^a \otimes \frac{\partial}{\partial x^b}$.

$$[J,J]_{\text{F–N}} = 2\left(J^b_{a_1}(x)\frac{\partial}{\partial x^b} J^d_{a_2}(x) s J^d_b(x) \frac{\partial}{\partial x^{a_1}} J^b_{a_2}(x)\right) dx^{\alpha_1} dx^{\alpha_2} \otimes \frac{\partial}{\partial x^d}$$

We should check others but I'll just check this one here.

3.2. **Operadic interpretation.** It's hard to study this formula because there are too many coefficients. We'll assume that J is $J(0) = J^b_a(0) dx^a \otimes \frac{\partial}{\partial x^b}$, a constant. So I just have something in $\mathbb{R}$ which is finally given by a matrix. To see what it means for this particular example to be a complex structure.

This means that $J_0 : V[1] \to V[1]$ is characterized by $V[1] \to V[1]$. Remember that J was a morphism of $O_\mathcal{V}$ modules $\mathcal{O}_\mathcal{V} \otimes V[1] \to \mathcal{O}_\mathcal{V} \otimes V[1]$. We have J_0 which can also be interpreted as a map from $V \to V$. Let me use the picture $+^V_V$ to mean J and the picture $|^V_V$ to mean the identity. Then to be an almost complex structure means that the composition of $+^V_V$ with itself is minus the identity.

If I do some representation theory, I have $T(\mathbb{R}+)/(\mathbb{R}(\ddagger + |)) = \bigoplus(\mathbb{R}+)^{\otimes n}/\mathbb{R}(+\otimes + + 1) \to Hom(V,V)$, a morphism of associative alegbras, is the same as having a J_0 such that $J_0^2 = -Id$. The integrability condition is obvious because we're doing a partial derivative of a constant.

If I do a change of basis I should still get a complex structure. Since the notion of a complex manifold is geometric, when we change the base $\{e_\alpha\}$ by an element of GL_V we should again get a complex structure.

We want this to be true for more general diffeomorphisms from V to V. What is a diffeomorphism? A diffeomorphism φ is a map from $(0, \widehat{S(V^*)})$ to $(0, \widehat{S(W^*)})$, a map sending 0 to 0 and an algebra map $\widehat{S(W^*)} \to \widehat{S(V^*)}$. It is characterized by $W^* \to \widehat{SV^*}$. I can rewrite this as a collection of maps $V^{\odot n} \to W$.

We want to characterize the complex structure in terms of algebra of type P. The transfer property means that the morphism of algebras of type P should contain some such diffeomorphisms.

So now to operads. This is a collection of vector spaces $P(n)$ with an action of S_n, the symmetric group which is a “monoid” meaning that there is a unit $\mathbb{R} \to P(1)$ and a composition rule which tells you how to compose elements. If we represent elements in $P(n)$ by trees with n entries. I can give some examples. Let me put $\tilde{C}$, which is $(\mathbb{C}, 0, \dots,)$. In more generality an algebra is an operad. Another example is endomorphisms of V. This is given by morphisms from $V^{\otimes} n$ to V. Composition is the composition of functions.

I'll give you a new one, which is Lie, and the operad is given by one generator, binary trees in the single generator, with one relation, the Jacobi relation.

One can define

Definition 3.1. *A P-algebra is a morphism of operads $P \to End(V)$.*

This means in particular it's a collection of maps $P(n) \to End_V(n)$.

A $\tilde{\mathbb{C}}$-algebra is a constant complex structure on V. A $Lie - alg$ is a Lie algebra.

Let me go back to the notion of diffeomorphism. In this setting it is well-understood and such a collection of maps gives exactly an L_∞ morphism. The diffeomorphisms are exactly the $Lie_{(1)}$-algebra maps up to homotopy. Therefore to have a geometric notion we need to add some new algebraic element, which will be the Lie algebra structure.

Why should we do this? When we consider a new algebraic notion, it will contain some of the data from this.

So what will we get? We get the following algebraic data. First we get the generators $+$ and the bracket Y along with the relations between these. We want $+$ to be an almost complex structure so $\ddagger + 1 = 0$. Secondly, Y satisfies Jacobi. You don't want all the diffeomorphisms, only the holomorphic ones. For this you have the compatibility relation, that Y commutes with J. A priori, we need the integrability condition. We have no extra information from that at this point.

So we get an operad $\mathcal{C}_X$ with these generators and relations. We want to study this operad. A $\mathcal{C}_X$-algebra structure on $\mathbb{R}^n$ is a constant complex structure on $\mathbb{R}^n$. The bracket will be zero automatically. This notion is too easy. Now we want all complex structures. For those who know L_∞ or A_∞ algebra that's what we'll do now.

3.3. **Homotopy algebra.** I will need curved Koszul duality, which is a previous work with Joey Hirsh. What it does, if you have an operad with a presentation $T(E)/R$, you want to resolve this to find a new one $\tilde{P}$ which has good properties. A way to do that is to use Koszul duality, and works when R is quadratic, in $T^{(2)}(E)$. This is the classical picture. The $J^2 = -1$ relation is not quadratic. You should treat the constant and linear terms as well. So for example, if you want to treat $\mathbb{C}$ or $\mathcal{C}_X$.

I won't say a lot of things about this, but we can construct in a general way some element $P^{¡}$¡ which is a “curved” cooperad such that if P is Koszul, we get a resolution $\Omega P^{¡} \xrightarrow{\sim} P$ via the cobar construction.

I won't be explicit, let me just cite this theorem.

Theorem 3.2. *$\mathcal{C}_{X\infty} = \Omega \mathcal{C}_X^{i} \xrightarrow{\sim} \mathcal{C}_X$; that is, $\mathcal{C}_X$ is Koszul.*

Remark 3.2. *$\mathcal{C}_X^{i}$ are trees with stacks of dots on the leaves. The degree of such a thing is one less than the number of dots.*

Proposition 3.1. *A $\mathcal{C}_{X\infty}$-algebra structure on V is given by a collection of linear maps $j_{n+1} : V^{\odot n} \to V$. These should satisfy*

(1) $\ddagger + 1 = 0$
(2) *some sum with signs of compositions of these is zero.*
(3) *some other sum with signs of compositions is zero.*

The first two are $J^2 = 0$ and the last is $[J, J] = 0$.

Theorem 3.3. *There is an equivalence of categories from $\mathcal{C}_{X\infty}$-algebras on $Vect$ with $\infty - (\mathcal{C}_{X\infty})$-morphisms to complex structures on $\mathcal{V}$ with holomorphic maps.*

To explain a bit how this is given, let me make the remark, the map $j_{m+1} : V^{\odot}m \to V$ is characterized by its values on a basis $j_{n+1}(e_{\alpha_1} \cdots e_{\alpha_k} \otimes e_a) = j^b_{\alpha_1,\ldots,\alpha_k,a} e_b$. So in the same way you can give a formula for morphisms.