INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS SEMINAR

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1. September 25: Changzheng Li, Primitive forms via polyvector fields

Thank you very much, today I will introduce my work on primitive forms. This work is joint with Si Li and Kyoji Saito.

Let me start with motivation from the mirror symmetry. Why would like to say that something is the same as something, like A model is equivalent to B model. We say the A model should be some symplectic geometry and the B model is about complex geometry. So in the setup, in the story that I will tell today, we can say that the mirror symmetry in this case, on the B side, we consider a Landau-Ginzburg model, which is a pair (Y, W) where Y is a non-compact Kähler manifold and W is a holomorphic function on Y with finite critical points.

On the A side, we consider some theory, either the Gromov-Witten theory or the so-called $FJRW$ theory. We have some formulation of mirror symmetry in different levels. Let me say a conceptual one at the moment.

In level one, on the A side we will have some algebra, the quantum cohomology or the FJRW ring. On the B-side we consider the Jacobi ring of W. The first level statement is about an isomorphism of algebras.

At a different level, let me say level two, for the A side, we have a Frobenius manifold structure which is on the algebra. On this level we may say there is an isomorphism of Frobenius manifolds with the B side, where the structure comes from a primitive form, let me say, a "good" primitive form. There may be more than one, and we make a good form.

There is a higher level, which is, on both sides, we can define the total ancestor potential function. On the A side, let me just say the A side is equal to the B side.

What I want to say is that, the isomorphism of algebras is about the genus zero three point story. The second level is about genus zero and all points. The third level is about all genus and all points. In some situations once we know level two, then level three is deduced automatically because high genus is determined by genus zero. We can see from the motivation of mirror symmetry, that this is the role of primitive forms. In my talk today, I will focus on primitive forms only. My joint work contains two main parts. The first part is the polyvector field approach to this construction, and the essential part is constructing a higher residue pairing via polyvector fields. The second important part is on perturbative formulas. I will explain the first part in detail and show an example by using the perturbative formula.

Before starting the details, let me say slightly more on the A side. Then we will totally skip the A-side parts.

Basically we count genus g curves with some marked points. We have so-called Gromov-Witten invariants. Roughly speaking, it counts for a compact Fano manifold (for my own interest I would like to consider flag manifolds or even more specifically complex Grassmannians,

$$\{\bar{\partial}\phi = 0 | \phi : (\Sigma_g, p_1, \ldots, p_k) \to (X, \gamma_1, \ldots, \gamma_k)\}$$

Where $\phi(p_i) \in \gamma_i \in H_*(X)$.

For $g = 0$, with three points, we obtain the quantum cohomology of X. One ongoing project is to study the complex Grassmanian case. We tried to construct a special, well, for level 1, the Jacobian, there's some construction of W, and an ongoing project is to understand this isomorphism by constructing a Lagrangian fibration on an open Calabi-Yau. I hope to report something on this in the future.

The second approach is FJRW theory. It's like Gromov-Witten. Similarly, instead of counting such maps, it's associated with a pair $(f; G)$ where f is a quasihomogeneous polynomial and G_f is the symmetric group of f. Then roughly speaking, you count the solutions of the equation

$$\bar{\partial}\phi_i + \frac{\overline{\partial f}}{\partial \phi_i} = 0$$

for all i, normally you can get your Gromov-Witten invariants from these.

So for this case, the level one is basically due to [unintelligible]

The second part, there's a central charge invariant. For smaller than one, it's FJR, for equal to one, it was [unintelligible]. As an application for the perturbative formula, we, LG-LG mirror symmetry forces level 3 from level 2, and we can show this for fourteen exceptional cases with charge between one and two.

Let me write down one example before going to the B-side.

First, a *primitive form* (this was introduced around 1979–1983 by Saito) is a family of holomorphic top forms. Let's see the case $Z = \mathbb{C} \times \mathbb{C}^2 \to S = \mathbb{C}^2$. So $F(z, u_0, u_1) = z^3 + u_0 + u_1 z$. The primitive form is unique up to multiplication by a constant. So $\zeta = dz \in \Omega^{top}(Z/S)$.

The starting point is a pair (X, f) where X is a Stein domain in $\mathbb{C}^n$ and $f : X \to \mathbb{C}$. The Stein condition is technical and prevents some kind of critical points of X. We also have f holomorphic with finitely many critical points.

We said that a primitive form is a family of holomorphic top forms. To construct this, there's some ambiguity of coefficients. Either we study all holomorphic top forms and try to specify a particular one or we just choose any one first and then figure out the good coefficient of this top form. Therefore, let us choose any one, and the choice will turn out not to matter. Choose a nowhere vanishing holomorphic top form Ω_X. The choice is not essential.

Now we start to say, to define primitive form, we consider the polyvector fields, then the polyvector fields valued in Laurent series $PV((t))$, and then we consider the cohomology, along with the higher residue and the Gauss-Manin connection, $(H^{f,\Omega}, K_\Omega^f(\cdot,\cdot), \nabla^\Omega_{t\partial_t}$.

Once we've done this it's easy enough to define a family version, and then ζ is a section of $(S, H_0^{f,\Gamma})$ satisfying properties. This will give a Frobenius manifold.

For polyvector fields, by this, I mean,

$$PV(X) = \bigoplus PV^{i,j}(X)$$

where

$$PV^{i,j}(X) = \mathcal{A}^{0,j}(X, \wedge^i T^{1,0} X)$$

(here $\mathcal{A}$ is smooth forms).

Let $(z_1, \dots, z_n) \in X$. If we have a sequence $I = [a_1, \dots, a_i]$ then we denote $d\bar{z}_I = d\bar{z}_{a_1} \wedge \cdots \wedge d\bar{z}_{a_i}$ and similarly $\partial_I = \frac{\partial}{\partial z_{a_i}} \wedge \cdots \wedge \frac{\partial}{\partial z_{a_i}}$ we let $\alpha \in PV^{i,j}(X) = \sum \alpha_{I,J} d\bar{z}^J \otimes \partial_I$ Then

$$\alpha \wedge \beta = \sum (-1)^{i\ell} \alpha_{I,J} \beta_{K,L} d\bar{z}^J \wedge d\bar{z}^L \otimes \partial_I \wedge \partial_K.$$

Then we see some operators. Let us assume this, we will use the nowhere vanishing volume form to identify polyvector fields with differential forms. The natural identification identifies α in polyvector fields with $\alpha \vdash \Omega_X$ via Γ_Ω. This identifies $PV^{i,j}(X)$ with $\mathcal{A}^{n-i,j}(X)$. An operator on differential forms induces an operator on $PV(X)$ naturally by $P_\Omega(\alpha) = \Gamma_\Omega^{-1} P(\alpha \vdash \Omega)$.

So we can define $\bar{\partial}_\Omega$ and ∂_Ω on PV.

We know the Leibniz rule for ∂ and we may want this on the other side. We define $\{\alpha, \beta\}$ to be the obstruction to the Leibniz rule for ∂_Ω. Then $\{g, \beta\} \vdash \Omega_X = (\partial g) \wedge \beta \vdash \Omega_X$.

We should write $\{,\}_\Omega$ but it turns out that the bracket doesn't depend on the choices. The same is true for $\bar{\partial}$. The operator ∂_Ω is dependent on Ω. All of these preserve polyvector fields with compact support.

In fact, $(PV(X), \bar{\partial}, \partial_\Omega)$ form a BV algebra.

There are two important observations. Consider $(PV(X), \bar{\partial} + \{f, \cdot\})$. This is equivalent to $(\mathcal{A}(X), \bar{\partial} + df \wedge)$. Secondly, $(PV_c(X), \bar{\partial} + \{f, \cdot\})$ is a subcomplex, and this natural inclusion is a quasiisomorphism. Since this is a quasiisomorphism, the cohomologies are isomorphic to each other. So we'd like a representative that is compactly supported, that is, an inverse.

Okay, and then we have $Tr_\Omega : PV_c \to \mathbb{C}$ which takes α to $\int_X (\alpha \vdash \Omega_X) \wedge \Omega_X$ and we need this compactly supported for this to converge.

The second key observation is that $H^*(PV(X), \bar{\partial} + \{f, \cdot\})$ is isomorphic to the Jacobian ring of f, that is, $\Gamma(X, \mathcal{O}_X)/\partial_{z_1} f, \dots \partial_{z_n} f$. There is also a residue map from $Jac(f) \to \mathbb{C}$ and this commutes up to a constant.

$$\begin{tikzcd}
H^*(PV(X), \bar{\partial} + \{f, \cdot\}) & Jac(f) \arrow[l, "|Crit(g)|<\infty"'] \arrow[d, "Res_\Omega"'] \\
H^*(PV_c(X), \bar{\partial} + \{f, \cdot\}) \arrow[r, "Tr_\Omega"'] & \mathbb{C}
\end{tikzcd}$$

If we know the key observations, the remaining things are much easier. Now we consider the polyvector fields valued in Laurent polynomials, so $PV[[t]][t^{-1}]$. Now we have

$$\begin{tikzcd}
PV(X)((t)) & PV(X) \arrow[l, hook'] \\
PV_c(X)((t)) \arrow[u] & PV_c(X) \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

Now we consider the $Q_f = \bar{\partial} + \{f, \cdot\} + t\partial_\Omega$. This space depends now on Ω. Now the cohomology

$$H^{f,\Omega} := H^*(PV(X)((t)), Q_f)$$

The chain complex is isomorphic to $(\mathcal{A}(X)((t), d + \frac{1}{t}df\wedge)$.

Let's do a calculation, this is $\Gamma(X, \mathcal{O}_X)((t))/Im(Q_f : \Gamma(X, T_X^{1,0}))((t)) \to \Gamma(X, \mathcal{O}(X))((t))$. We write down everything and you can just see that this is the calculation of the image. We can write down the trace map but let's write down the pairing first; that is $PV_c(X)((t)) \times PV_c(X)((t)) \to \mathcal{C}((t))$. We want to $(\alpha, g_1(t), \alpha_2 g_2(t)) \mapsto g_1(t)g_2(-t)\int(\alpha_1 \wedge \alpha_2) \vdash \Omega_X) \wedge \Omega_X$. This is a pairing from Laurent-valued polyvector fields to $\mathbb{C}(t)$. This pairing descends to the cohomology. Then we claim this is Saito's higher residue pairing. Now this is a non-family version. We can see, once we know the key observation, the pairing. This is the most complicated part.

One more, we define $\nabla_{t\partial t} : H^{f,\Omega} \to H^{f,\Omega}$ where $\nabla_{t\partial t}(t^k\alpha) := (t\partial_t + i - \frac{f}{t})t^k\alpha$ for any α in $PV^{i,j}(X)$. This is well-defined on the cohomology. This sort of thing is kind of standard. It defines an operator from cohomology to cohomology.

One more notation is that $H_0^{f,\Omega}$ is the same except you replace Laurent series with formal power series. We obtain this triple now. We can move to the family version. The whole story is the same, we just need to mention what is a family and then we do all the same thing.

We consider X and a function $X \to \mathbb{C}$ but now we consider a map $Z \to \mathbb{C}$ where Z sits over S and X sits over 0. So now we have and extension of $f : X \to \mathbb{C}$ to $F : Z \to \mathbb{C}$, let me consider that $Z \subset X \times S$, we require $X \times S$ to be Stein. We also want the Kodaira-Spencer map from TS to $\mathcal{O}_Z/(\partial z_1 F, \dots \partial_{z_n} F)$ to be an isomorphism, where V maps to $[\partial_V F]$. The third condition is that the projection to S, restricted to the critical set of F, is proper. This is a technical condition to ensure that the family version is a locally free sheaf. Let us forget this technical part.

Let me write down the relative polyvector fields. That is, what is the meaning of relative? The relative polyvector fields are defined to be, let us say that $S \subset \mathbb{C}^\mu$ and X is dimension n, then $PV(Z/S) = \bigoplus PV^{i,j}(Z/S)$ where $0 \le i \le n$ and $0 \le j \le n+s$. Again, $PV^{i,j}(Z/S) = \mathcal{A}^{0,j}(Z, \wedge^i T_{Z/S})$.

Now our PV_c only needs compact support in the fiber direction. If we do all the same things we get a family version. One remarkable thing, after the integration we move from the smooth to the holomorphic world. Also the Gauss-Manin connection, we had previously defined this along the $t\partial t$ direction. We need to define what happens along the base direction. we define $\nabla_V^\Omega[S] = [\partial_V S + \frac{\partial_V F}{t} S]$.

We've taken more than one hour to come to the definition.

Definition 1.1. *A* primitive form *is a section in* $\Gamma(S, H_0^{F,\Gamma})$ *satisfying the following conditions:*

1. $t\nabla^\Omega\zeta : \mathcal{T}S \to H_0^{F,\Omega}/tH_0^{F,\Omega}$ *is an isomorphism.*
2. *for any* v_1 *and* v_2 *in* $\mathcal{T}S$*, we have* $K_\Omega^F(\nabla_V^\Omega\zeta, \nabla_{Z_2}^\Omega\zeta_2) \in t^{-2}\mathcal{O}_S$.
3. *for any* Z_i *in* $\mathcal{T}S$*, we have* $K_\Omega^F(\nabla_{V_1}^\Omega\nabla_{V_2}^\Omega\zeta, \nabla_{V_3}^\Omega\zeta)$ *and* $K_\Omega^F(\nabla_{t\partial t}^\Omega\nabla_{V_1}^\Omega\zeta, \nabla_{V_2}^\Omega\zeta)$ *are in* $t^{-3}\mathcal{O}_S + t^{-2}\mathcal{O}_S$
4. *There exists a* γ *so that* $(\nabla_{t\partial t}^\Omega + \nabla_E^\Omega)\zeta = \gamma\zeta$

So we learn from the second thing that $g^\zeta(V_1, V_2) = Res_{t=0}(t^{-1}K_\Omega^F(t\nabla_{V_1}^\Omega\zeta, t\nabla_{V_2}^\Omega\zeta)dt)$

From the third we get that $\nabla^\Omega = \nabla^\zeta + t^{-1}A$ so that [something about how this gives the equivalence with the Jocobian ring] $g^\zeta, \circ, \nabla^\zeta, e, E$ is a Frobenius manifold structure on S.

Let me introduce the perturbative formula quickly and give one example.

A primitive form is a family and we want to compute this. Say we know the primitive form at a reference point. How do we extend it? Let me make a remark. The construction $H^{F,\Omega}_{(0)}$ depends on Ω but if you use $\zeta\Omega_X$ then it doesn't depend. Another remark, this is a *formal* primitive form. There is nothing about convergence here. Leave convergence as a separate issue. For weighted homogeneous polynomial, the formal and nonformal coincide.

So we want to say something from this initial point. We need one notion. So $\mathcal{L}$ is a subspace of $H^{f,\Omega}$. It's a vector subspace satisfying $H^{f,\Omega}_{(0)} \oplus \mathcal{L}$ and $t^{-1}\mathcal{L} \subset \mathcal{L}$.

We call this an *opposite filtration* if $K^f_\Omega(\mathcal{L},\mathcal{L}) \subset t^{-2}\mathbb{C}[t^{-1}]$. It is further called a *good opposite filtration* if $\nabla^\Omega_{t\partial t}\mathcal{L} \subset \mathcal{L}$.

There should be some property, some primitivity, for $\zeta_0 = \zeta|_{u=0}$. Let us skip this because of the remark that ζ_0 is a constant for weighted homogeneous polynomials. If f is a weighted homogeneous polynomial, [missed].

Then the problem is that, we are given $(\mathcal{L},\zeta_0)$. How do we construct a primitive form $\zeta(\mathcal{L},\zeta_0)$. The data we have is sufficient data to make a bijection. There is a unique primitive form. Therefore we ask how to extend it. The construction of $\mathcal{L}$ is highly nontrivial in general. It's elementary for a weighted homogeneous polynomial. What we need is to find a doable way to obtain the primitive form.

The theorem is that

Theorem 1.1. *(LLS)*
Given $s \in H^{f,\Omega}$, then $e^{\frac{f-F}{t}}s$ makes sense and is flat along the S direction but not in the t direction. So it's a flat extension to $H^{F,\Omega}$.

As a mathematical statement this should be more careful because t goes to ∞ and $-\infty$ but that's technical.

Our theorem says that $e^{\frac{f-F}{t}}\zeta_0$ has a splitting $\zeta_+ \oplus \zeta_-$ and $\zeta_+ \in H^{F,\Omega}_0$ (formal power series) while $\zeta_- \in e^{\frac{f-F}{t}}\mathcal{L}$. Then the statement is that $\zeta_+ = \zeta_+(\mathcal{L},\zeta_0)$.

Let's see one nontrivial example. Let $f(x,y,z) = x^3+y^3+z^3$. This is the so-called simple elliptic singularity. This is a weighted homogeneous polynomial with $q_x = q_y = q_z = \frac{1}{3}$. The central charge $\hat{c}_f$ is $\sum(1-2q_i) = 1$

Therefore let us try to use, well, assume some facts and then obtain the answer. In this case the family, $Jac(f) = \mathbb{C}\{\phi_i | i = 1,\dots 8\}$ where it's spanned by $\{1,x,y,z,xy,yz,xz,xyz\}$. So we have a basis. Then $S = \mathbb{C}^8$. Then $F(x,y,z,\vec{u}) = f(x,y,z) + \sum u_i\phi_i$. This has degree because each basis element has a weight. Each u_i should then have a degree so that the total degree is 1.

So our aim is to find $\zeta = \zeta(\mathcal{L},1)$. One fact is that, for a quasihomogeneous polynomial, the grading of K^f_Ω is $-\hat{c}_f$. This is some fact. From this fact we can say something. First, $\mathcal{L}$ is parameterized by a complex number. Any $\mathcal{L}$ is of the form, let me say $\mathcal{L}$ is a good opposite filtration if and only if $\mathcal{L} = \mathcal{L}_c = t^{-1}B[t^{-1}]$ where B is the span of $\{\phi_1,\dots,\phi_7, xyz+ct\}$. We know what is $\mathcal{L}$, it is parameterized by c. So we have the initial data.

Second, for all weighted homogeneous polynomial, we choose a standard volume form $dxdydz$. Now ζ is homogeneous of degree 0. This is an easy fact. This implies that ζ can only be a function of u_8 because, well, it's a section, it could be a function of the base space and then some power series. We have this special fact that it's degree 0. The only possibility is that it's a function of the parameter u_8.

Let us use this. We want to find $e^{\frac{f-F}{t}}1 = \zeta_+ \oplus \zeta_-$ associated to $\mathcal{L}c, 1$. We want this splitting. Let us do two things. First, we consider the inverse part, because the splitting is in the family version. But somehow in the calculations we invert more things, so let's reduce to the non-family version. Let's try to write $\zeta_0 = e^{\frac{F-f}{t}}\zeta_+ \oplus \bullet$ where $\bullet$ is in $\mathcal{L}$. So we have

$$e^{\frac{F-f}{t}} - e^{\frac{u_8xyz}{t}} = e^{\frac{u_8xyz}{t}}(e^{\frac{F-f-u_8xyz}{t}} - 1)$$

From 1 to 7 the degrees are less than 1 so the differences are negative. Once we subtract 1, the whole right term is of negative degree. The whole thing on the outside is of negative degree. Then $\mathcal{L}_c$ contains the negative degree, at most degree 0. The highest degree of B is 1 so the highest in $\mathcal{L}_c$ is 0. So $H^{f,\Omega}_{(0)}$ can only have nonnegative parts, so all the negative parts, strictly negative, must belong to $\mathcal{L}$.

On the other side we want to obtain a splitting. We're doing this on a cohomology. Each term is a summation $\sum \frac{1}{k!}(\frac{u_8xyz}{t})^k$. This means that, well, Q_f is exact, is zero in homology. We calculate,

$Q_f(y^2z^3\partial_x)$, this is $(\bar\partial + \{f,\cdot\} + t\partial_\Omega)(y^2z^3\partial_x)$. But $\{f, \sum g_i\partial i\} = \sum \frac{\partial f}{\partial z_i} g_i$ and $\partial_\Omega \sum g_i\partial_i = \sum \frac{\partial g_i}{\partial z_i}$. So we calculate Q_f and get that iti is $x^2y^2z^2$. Then we see that $x^{k-2}y^kz^k\partial_x$, by calculation, Q_f of this is $x^ky^kz^k + t(k-2)x^{k-3}y^kz^k$.

If we do this three times, we get that $x^ky^kz^k = -(k-2)^3t^3x^{k-3}y^{k-3}z^{k-3}$. Therefore, in cohomology, in cohomology the thing we are interested in is of the form

$$e^{\frac{u_8xyz}{t}} = g(u_8) + \frac{xyz}{t}h(\sigma).$$

Then this g and h, they are formal power series, and these are the solutions of the Picard-Fuchs equation $((1+u_8^3)\partial^3_{u_8} + 3u_8^2\partial_{u_8} + u_8)\phi = 0$. In particular it's analytic.

In this way we get an analytic formal power series. The last step is the splitting. The left hand side is equal to $e^{\frac{F-f}{t}} - (g + \frac{xyz}{t}h)$ in cohomology.

Then this is $e^{\frac{F-f}{t}} - (g-ch) - \frac{xyz+ct}{t}h$. We want something in $\mathcal{L}$, which is $t^{-1}B[t^{-1}]$ where X is the span of $\phi_1, \dots, \phi_7$ and $xyz + ct$. So then $e^{\frac{F-f}{t}} - (g-ch)$ is in $\mathcal{L}$ and we are done.

From the theorem $\frac{1}{g-ch}$ is a primitive form. Then $\zeta = \frac{1}{g-ch}$. Now these are both functions of u_8. So that's the primitive form associated to this good opposite filtration associated with $\sigma = u_8$.

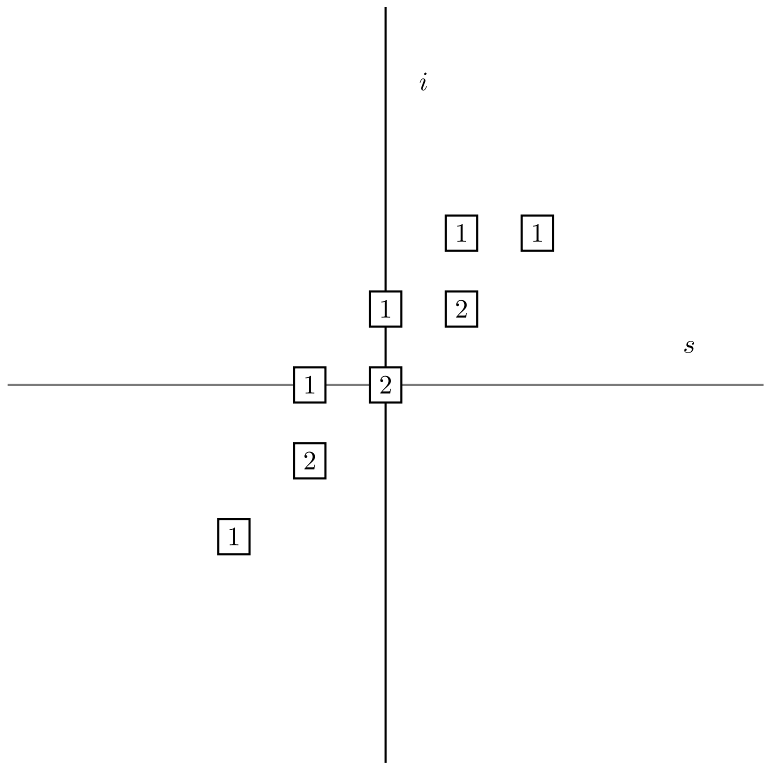
2. Danielle O'Donnol, December 18, Combinatorial Spatial Graph Floer Homology

What we'll look at today is combinatorial spatial graph Floer homology. We're looking at knot Floer homology. That's an invariant for an oriented knot in S^3. We want to extend this to spatial graphs, where we have oriented edges.

To be able to define it we need a special class of graphs, I was planning on getting to that later. I'll start with knot Floer homology, why it's successful and strong, and then talk about the setting of various different Floer homologies and how our results fit in to that, then the specifics of our version.

For knot Floer homology, which was defned by Ozsváth and Szabó and independently by Rasmussen, to K you associate $\widehat{HFK}_i(K,s)$ a bigraded vector space where i and s are $\mathbb{Z}$-gradings and $\mathbb{F}$ is $\mathbb{Z}/2$. You might look at this as some ordered pairs (s,i), and then you have some ranks. s is the Alexander grading and i the Maslov grading.

This is one way you can represent what is going on, and we'll come back to this looking at all of our nice properties.



(1) The graded Euler characteristic gives the Alexander polynomial. This is one of these things where in most of these homology theories you get some polynomial invariant as your Euler characteristic, you have $\sum(-1)^i rk\widehat{HFK}_i(K,s)t^s$, this is the Alexander polynomial $\Delta_K(t)$. In our example we can just read this off, in this example $\Delta_K(t) = t^-2 - t^-1 + t^0 - t + t^2$.

(2) Another thing that Floer homology can tell us is the genus of a knot. A Seifert surface of a knot is an orientable surface with boundary the knot. The easy example is [trefoil example]. The genus of a knot is the minimal genus of any Seifert surface for the knot.

A theorem of Ozsvath and Szabo from 2001 says that $g(K) \max\{s|\widehat{HFK}_i(K,s) \neq 0\}$. In our example, $g(K) = 2$.

Corollary 2.1. *K is the unknot if and only if our Floer homology just has one group $\widehat{HFK}_0(K,0)$*

(3) We can detect if a knot is fibered. A knot is *fibered* if there is a fibration $f : S^3\backslash K \to S^1$ which is well-behaved near K.

Theorem 2.1. *(Ghiggini, Ni(2006)) K is fibered if and only if $\widehat{HFK}_*(K, g(K)) \cong \mathbb{F}$.*

That's our situation in the example which implies that K is fibered.

Now I'll talk about more the overall picture of Floer homology. First of all, Ozsvath and Szabo defined Floer homology for Y a closed oriented 3-manifold, defined $HF^o(Y)$ which are graded groups.

Then Ozsváth-Szabó and Rasmussen independently defined knot Floer homology. We just talked about the hat version $\widehat{HFK}_i(K,s)$. This a bigraded $\mathbb{F}$-vector space. This also has different versions, and the one that's interesting to us is $HFK^-(K)$ which is a bigraded $\mathbb{F}[u]$-module. This has some more information in it and actually what happens is, to go from $HFK^-(K)$ to $\widehat{HFK}_i(K,s)$ you set $u=0$. So Jukász defined sutured Floer homology. You have (M,γ), where M is a sutured manifold, and this in particular extends to a manifold with boundary. This is usually denoted $SFH(M,\gamma)$. Now sutured Floer homology is an extension of $\widehat{HF}(Y)$ to manifolds with boundary. This gives us our homology but in the simplest version.

I'll talk now about what a sutured manifold is and how this is not just a generalization of the original Floer homology but also how you can get the knot Floer homology.

Definition 2.1. *A* (balanced) sutured manifold *is a pair (M,γ) where M is a compact oriented three-manifold with boundary and γ is a subset of the boundary, a set of disjoint finite oriented simple closed curves that split our boundary into two pieces $R^+(\gamma)$ and $R^-(\gamma)$ such that $R(\gamma)=\partial M\backslash\gamma$ and γ are consistently oriented. This is the definition for a sutured manifold. To be balanced you need the additional condition which is that there are no closed components and the Euler characteristic of R_- is the same as the Euler characteristic of R_+.*

There's a nice example. Take a knot K in a three-manifold Y. Then let $Y(K)$ be Y minus a neighborhood of K and then we'll add some sutures in an appropriate way. The sutures are two oppositely oriented meridians. We look at these being our sutures and so then we have R_+ and R_- and in this case, the sutured Floer homology $SFH(Y(K))\cong\widehat{HFK}(Y,K)$. So sutured Floer homology is really nice and not only generalizes what's happening in Floer homology for a general three-manifold, but also the hat version of knot Floer homology.

You get a nice interpretation of the graded Euler characteristic, due to Friedl, Juhász, and Rasmussen, and they define the torsion invariant $T_{(M,\gamma)}\in\mathbb{Z}[H_1(M)]$ which is the maximal Abelian torsion for the pair (M,R_-). This is equivalent to the Alexander polynomial we were talking about before in this setting.

We can do something similar with graphs. There's a natural way to associate sutured manifold to a oriented spatial graph. So what's going to happen is we have a vertex in an oriented graph, we'll take the exterior, $E(f)=S^3\backslash\nu(f(G))$. Then there's a particular way you put sutures on this. Separating incoming and outgoing edges you put a suture, and then on each edge you put a suture. The sutured Floer homology for this sutured manifold is equivalent to the hat version of our theory.

So what happens, for our thing, we have a relationship between our thing and their Alexander polynomial. This defines an Alexander polynomial $\Delta_f := \bar{T}_{(E(f),\gamma(f))}$. There have been Alexander polynomials defined for graphs. In 1958 there was a definition that was just the Alexander polynomial of the exterior, this also uses half of the boundary and retains more of the information.

Putting this in context, we define a minus version of HFG using grid diagrams. We extend this thing in the context of sutured Floer homology to a more general theory using a combinatorial approach. This is a good place to take a break.

All right. So originally the definition for was using holomorphic disks. This was complicated and kind of hard. So Manolescu-Ozsváth-Sarkar introduced a combinatorial description of HFK with grid diagrams. Now the major advantage to this is that in this setting it's very easy to tell when you have a holomorphic disk that you should be counting. In the other setting it's much harder to tell. As a result you don't have to worry about the complicated counting part.

Then Manolescu-Ozsváth-Szábo-Thurston gave a self-contained combinatarial proof of invariance of the grid diagram setup.

The general plan is that we have some sort of objects we're interested in and we encode these in our grid diagrams. Then there's a standard way to turn a grid diagram into a chain complex. This then gives us our homology invariant.

The passage from a grid diagram to a chain complex is standard. The tricky thing is to pass from objects to grid diagrams in ways that preserve the eventual homology.

The natural choice for what to start with is what we call transverse spatial graphs, which I'll define in just one minute. Then we get our grid diagrams that we associate with that, and for each spatial graph there are a lot of grid diagrams which are related by grid moves. Then we get our chain complex and from that we obtain our homology invariant.

So the steps that are involved are showing that our transverse graphs and our grid diagrams are well-defined and that we can pass among different ones via grid moves. Then the other important thing to show is that grid moves correspond to quasi-isomorphisms.

The objects we're working with are transverse spatial graphs.

Definition 2.2. *A* transverse spatial graph *is an oriented spatial graph where each vertex locally looks like, the incoming and outgoing edges are separated. There is a disk that separates our incoming and outgoing edges. This is halfway rigid. You can't have interactions among incoming and outgoing edges.*

In a diagram, we can always move things so that this disk is perpendicular to our plane of projection, so that when we have a diagram, our incoming edges should be separated by a vertical line in the diagram. We don't write the disk because it's implied. In the original example of the trefoil with an extra edge, it looks like this.

There are standard Reidemeister moves for pliable graphs with one restriction based on the vertex restriction.

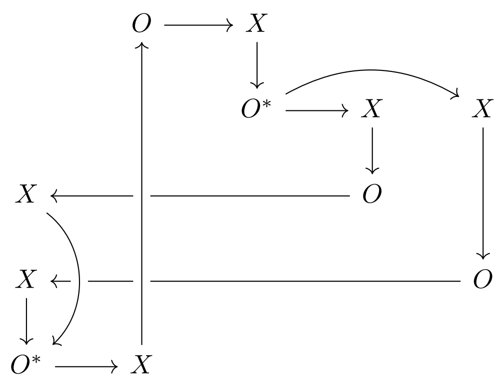
[pictures]

Let's talk about grid diagrams. For a graph, a grid diagram is an $n \times n$ grid where the squares contain either X, O, or nothing along with a subset of the Os which are marked with $*$. For knots you have no $*$. There are certain conditions on what can happen. We require

- There is exactly one O in each row and column
- There is at least one X in each row and column (if I want a homology theory)
- If an O does not have exactly one X in each row and column it is marked with a $*$
- $O*$s are called vertex Os.
- When there are multiple Xs in a row or column, it must look like $O*$ with a line of X above and to the right of the $O*$.

We'll focus on these. For example, [picture]

	O	X		
		$O*$	X	X
X			O	O
X				
$O*$	X			



We write in edges horizontally from O to X and vertically from X to O, letting vertical going over horizontal and choosing a convention for $O*$ vertices.

It's possible to put any spatial graph in a form like this.

You've made a lot of choices and that's where grid moves come in.

Theorem 2.2. *(Harvey, O'Donnol) If g and g' are grid diagrams representing $f(G)$ then they are related by a sequence of graph grid moves.*

These are, first of all, cyclic permutation, where we move a row from the left to the right or the top to the bottom.

Second of all, they are commutation, you can sometimes commute adjacent rows generally if the O and X in a column are separated or contained but not if they are interspersed.

$$\begin{array}{cccc} \cdots & X & & \cdots \\ \cdots & O & & \cdots \\ \cdots & & O & \cdots \\ \cdots & & X & \cdots \end{array} \rightarrow \begin{array}{cccc} \cdots & & O & \cdots \\ \cdots & & X & \cdots \\ \cdots & X & & \cdots \\ \cdots & O & & \cdots \end{array}$$

We also allow you to interchange where the two X overlap or there are multiple X in a row or column.

The other standard grid move is stabilization, where we have an X and O and add a new row and column. We need to know what to do if there are multiple Xs in a row or column and what will happen when we split that column up.

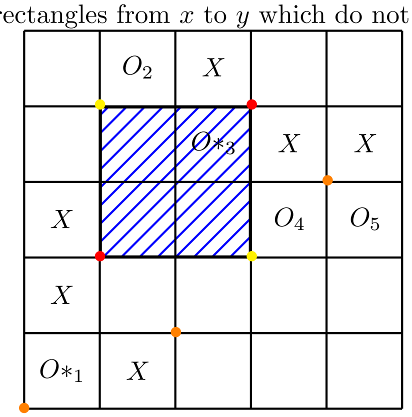
[pictures]

Let g be the grid diagram for $f : G \to S^3$. Then $g \rightsquigarrow HFG_i^-(g, s)$ is a bigraded $\mathbb{F}[U_1, \ldots, U_v]$-module where v is the number of vertices in G.

So i is our Maslov index which is in $\mathbb{Z}$, which is an absolute grading. The $s \in H_1(S^3 \backslash f(G))$ is our Alexander grading which is relative.

What will happen is that $HFG_*^-(g, s)$ is the homology of the chain complex we define. Then ∂ will count certain rectangles in the toroidal grid diagram, and $\widehat{HFG}_*(g, s)$, the sutured Floer homology, is the homology of $(\widehat{CFG}(g, s), \partial)$ setting $U_i = 0$.

Define (CFG^-, ∂) as follows. Take your toroidal grid diagram. Then the generators are n-tuples of inetresections of the horizontal and vertical grid circles. Then to define our boundary map, we need to know what a rectangle is. These are rectangles from x to y which do not contain x_i or y_i



So the boundary of such a tuple $\underline{x}$ is sum over all $\underline{y}$ of

$$\sum_{r \in Rect(x,y), r \cap \mathbb{X} = 0} U_1^{O_1 r} \cdots U_n^{O_n r} y$$

Then label each O with $\{1, \ldots, n\}$ then $O_i(r)$ is the number of O_i in r. Some of these are equivalent and this will eventually be a module over the right thing.

So in our example, $\partial \underline{x} = U_3 y + \cdots$

The Maslov grading is $M(x) - M(y) = 1 - 2n_O(r)$ where $n_O(r)$ is the number of Os in the rectangle. In our example, if we look at $M(x) - M(y) = -1$.

Then the Alexander grading, giving a weight to each O and X it is given by its class in $H_1(S^3 \backslash f(G))$. So $A(x) - A(y)$ is

$$\sum_{X \in r} w(X) - \sum_{O \in r} w(O).$$

This is the only thing that is slightly different in this setup but it is really following the same ideas of what happens in the other theories.

This is the complete definition of how we get our chain complex and like I said before the homology is taking the homology of the chain complex. The final theorem is that

Theorem 2.3. *(Harvey-O.) $HFG^-(g, s)$ as a bigraded $\mathbb{F}[u_1, \ldots, u_v]$ module, is independent of the choice of grid g for our $f(G)$.*

For knots moving along an edge you see that the generator u_i is equivalent to u_j if these O are along the same edge. When you hit a vertex you act as zero and so only the ones for vertices act. It's just not even. You have a nice one-to-one correspondence for knots.