INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS WEEKLY SEMINAR

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1. Choi Sung Rak (IBS CGP), Introduction to Birational geometry: 1. the minimal model program 2. Positivity of divisors

In the first part of the talk I will introduce the minimal model program (MMP) and in the second part of the talk I will talk about one application of the minimal model program, to the positivity of divisors.

So in this talk, I will study varieties. In this talk they will always be projective, algebraic, and defined over $\mathbb{C}$. The varieties may not be smooth. They are singular but not too singular.

Definition 1.1. *Say X and X' are varieties. Then we say that X and X' are birational to each other if there exists Zariski open subsets $U \subset X$ and $U' \subset X'$ such that U is isomorphic to U' in the (algebraic) sense.*

This notion defines an equivalence relation $X \dashrightarrow X'$. Roughly speaking, birational geometry is the study of varieties up to birational equivalence. This means we stuy birational invariants.

In this area we can ask the following questions. If we are given a variety X (or the birational equivalence class containing X), can we find a good model birational to X? We use the word model but we mean variety.

[Can you give examples of birational equivalences or the failure?]

For example $\mathbb{P}^2$ and $\mathbb{P}^2$ blown up at one point are birational to one another.

[How about a non-example?]

The easiest one is if the dimensions are different. There is no birational geometry for curves, so if two curves are birational they are actually isomorphic.

So how to describe a good model? Well, you could ask for a smooth model for X. This is well known. The answer is yes (Hironaka 1964). The second question is the origin of the minimal model program. Can we find a simplest (minimal) model? Smooth doesn't mean simple. If you blow up you get something extra over one point. The curve is not related to the birational property of $\mathbb{P}^2$. So X is "minimal" if when $X \to X'$ is a birational morphism, then it is an isomorphism. So there are no interesting birational morphisms out of X. So the algorithm to find a minimal model is the MMP.

As I said, there is no birational geometry for curves. So let's start with dimension 2. The answer was given about a hundred years ago, 1890–1910, by the Italian school. They say that given a surface X we can find a minimal model by contracting over all rational curves C such that $C \cdot K_X = -1$ where K_X is the canonical divisor. These are called -1-curves because $C^2 = -1$.

In dimension two, minimal models are surfaces without negative one curves. An example is a Calabi-Yau surface.

The MMP in dimension two is the following. You are given a surface X. You ask if there is a -1-curve C? If no, then X is minimal. If yes, then contract that curve. Then start over. That is the MMP in dimension 2.

For each round, the Picard rank drops by one. The Picard rank is positive so you can't do this infnitely many times. So in dimension two, it's simple.

Let's go to dimension higher than two. First you need to define minimal models again.

Definition 1.2. *X is minimal if K_X is relatively numerically effective (meaning that $K_X \cdot C \geq 0$ for all curves C)*

[Some discussion about the difference between the motivational definition and the definition, some displeasure with the new definition]

In dimension two, if X has no (-1) curves and the Kodaira dimension is non-negative, then K_X is numerically effective.

In higher dimensions you want something similar to the classical definition for dimension two.

Theorem 1.1. *(Contraction Theorem)*
If there is a rational curve C such that $C \cdot K_X < 0$ then there is a morphism $f : X \to X'$ such that $f(C)$ is a point.

This is true in any dimension. If you contract a curve, not just the curve is contracted. This f may contract a subvariety V containing C.

There are three cases. The codimension of V could be 0, or 1 (these are the good cases) or, well, let me explain these two first. If the codimension is 0 then $V = X$ and f is not birational. In the codimension 1 case, f is called divisorial. The third case is when the codimension of V is larger than one. In this case f is called "small." This does not occur in dimension 2. In dimension 2 we only had the first two cases. In the small case, X' has "bad" singulaity so you don't have intersection numbers for $C \cdot K_{X'}$. So we need to improve the singularity of X'

Definition 1.3. *Let's say we have a small contraction $X \to X'$ where The flip of X is $\phi : X \to X^+$ which is a birational map inducing $X \backslash Z \cong X^+ \backslash Z^+$ for codimension $Z, Z^+ > 1$, with a commuting small contraction $X^+ \to X'$*

$$\begin{tikzcd} X \arrow[rr, dotted] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & X' & \end{tikzcd}$$

and such that $C \cdot K_{X^+} > 0 iff^+(C)$ is a point.

You can define X^+ as

$$proj \bigoplus_{m \geq 0} H^0(X', f_*(mK_X))$$

One feature of the flip is that X^+ has better singularities that X'.

Let's look at the minimal model program in dimension greater than two. Start by asking if X, K_X is numerically effective. Here X is of a special kind, it doesn't have singularities that are too bad. It's up to Q-factorial, klt singularities. Anyway, if so, then X is minimal. If no, then we can find a curve so that $C \cdot K < 0$. We can find a numerically equivalent rational curve such that $C \cdot K < 0$. Ve can use the contraction theorem and now we split into three cases, according to the codimension of V. In the case where the codimension is 0 then f is not birational, you need to stop, and what you get there is called a Mori fibration. If the codimension is 1 then you start again with X'. The problem is what happens when the codimension is greater than 1. So start by flipping f to $f^+ : X^+ \to X'$ and start over with X^+.

This flowchart is a hopeful picture. We have some conjectures:

Conjecture 1.1. (1) *Existence of flips. It's unknown in general.*
(2) *Termination. Every round you have codimension $V = 1$, your Picard rank drops.*

So in dimension three, this was done in 80 by Kawamata, Kollár, Mori, Reid, Shokurov, showing that the conjectures hold. So sometimes this is called the Mori program.

In dimension 4, termination was proved in 1987 by Kawamata and existence in 03 by Shokurov. He used the notation perestroika for flip, the economic reform in Russia. Recently Hacon-McKennan in 2005 showed termination in dimension d plus existence in dimension d implies existence in $d+1$.

In 2010, there was a big progress, Birkan-Cascini-Hacon-McKennan showed that for X of general type (Kodaira dimension equal to the dimension of X, then flips exist and minimal models exist.

So they ran an "MMP with scaling" so they specified which curve to contract.

For many applications, this is enough, but there are many problems still. Termination is open for dimensions greater than 4. So of course the MMP for special types where the Kodaira dimension is less than the dimension of X is completely unknown. The last question is the abundance conjecture.

Conjecture 1.2. *Say $X \overset{MMP}{\dashrightarrow} X_{min}$. Suppose you want to contract further to the canonical model X_{can} where instead of $K_{X_{min}} \cdot C \geq 0$, you have $K_{X_{can}} \cdot C > 0$. This will be unique.*

This is again known in the case where the Kodaira dimension coincides with the dimension. In the second part of the talk I will give one application of this result.

So let me talk about positivity of divisors. So we run the MMP on X and get X_{min}, so K_X becomes $K_{X_{min}} C \geq 0$ and then with abundance you get to X_{can} and get $K_{X_{can}} C > 0$.

There is something called the local minimal model program, where you run this on $K_X + B$, a divisor with some mild singularities, Q-factorial and klt singularities. There is also D-MMP, where you run the MMP with an arbitrary divisor.

If you run the D-MMP, then D becomes numerically effective and then in the canonical case, ample. It's too much to expect to run D-MMP with any divisor. In this context, "positivity" of the divisor D means how far D is from being ample.

Let's recall the definition: D is called ample if for some positive integer m the sections of $H^0(, \mathcal{O}(X(mD)))$ define an embedding of X into a projective space $\mathbb{P}^m$.

There are many equivalent characterizations.

Theorem 1.2. *The following are equivalent:*

0 *D is ample.*

1 *The non-ample locus of D is empty. Everything here works for $\mathbb{R}$ divisors, but I'll use $\mathbb{Q}$ divisors for simplicity. So a divisor is $\sum r_i D_i$ where r_i is rational and D_i is prime. The non-ample locus $B_+(D)$ is as follows.*

Choose an ample $\mathbb{Q}$-divisor A. Then take the base locus of $m(D-A)$, $Base|D| = \sqcap(|D| = \{D'|D' \geq 0D' \sim D)$. So we look at $\sqcap\sqcap Base(m|D-A|)$, where the intersections are over all m and all ample $\mathbb{Q}$-divisors A. This $B_+(D)$ is also called the non-Kähler locus). So the empty set, it's the same as saying the dimension of $B_+(D)$ is less than or equal to 0.

2 *(Serre) $H^i(X, \mathcal{O}_X(mD)) = 0$ for all $i > 0$ for all $m \gg 0$.*

3 *(de Fernex-Küronya-Lazarfeld)*

$$\limsup_{m\to\infty} \frac{h^i(X, mD')}{m^{dim\ X}} = 0$$

for all $i > 0$ for D' sufficiently close to D.

Let me generalize ampleness.

Definition 1.4. *So k can be from 0 to one less than the dimension.*

(1) *(Choi) D is numerically k-ample if dim $B_+(D) \leq k$.*
(2) *(Sommese 78) D is cohomologically k-ample if $H^i(X, mD) = 0$ for all $i > k$ and all $m \gg 0$.*
(3) *(Choi) D is asymptotically k-ample if $\hat{h}^i(D') = 0$ for all $i > k$ and all D' sufficiently close to D.*

You might hope these are equivalent, but they are not.

Theorem 1.3. *For big D, $h^0(X, mD) \sim m^{dim\ X}$. Then 1 implies 2 implies 3 for D for all k. That 1 implies 2 is proven by Küronya. It is not true in general that 2 implies 1. There is an example. It is expected that 3 always implies 2.*

Theorem 1.4. *(Choi) If D is numerically effective and big, then all three are equivalent.*

Theorem 1.5. *For $D = K_X$ or K_+B big, the notions 1, 2, and 3 are precisely (L)MMP-invariant, that is, if $X \dashrightarrow Y$ is the map from (L)MMP then K_X is (numerically, cohomologically, or asymptotically) k-ample if K_Y (or $K_Y + B_Y$) is.*

These are too technical, but the next one is easy.

Corollary 1.1. *For $D = K_X$ (or $K_X + B$) big, then 1, 2, and 3 are equivalent for all k from 0 to the dimension of X minus one.*

We only need to show that 3 implies 1. Assume that K_X is asymptotically k-ample. Run the special MMP, and K_Y is still asymptotically k-ample. Since K_Y is numerically effective, the three notions coincide. Then you can go back to X.

Corollary 1.2. *If X is Fano (type) including log Calabi-Yau) then 1,2, and 3 are equivalent for any big D.*

Fano type means there exists an effective divisor B such that $-(K_X + B)$ is ample. Equivalently, there is an effective divisor B' so that $K_X + B' \equiv 0$.

This proof is also easy. We need to know that 1, 2, and 3 are numerical properties, so they depend en $[D] \in N^1(X) = Div_{\mathbb{R}}(X)/\equiv$.

If X is Fano type, we can find B' so that $K_X + B' \equiv 0$. Consider $K + B' + D \equiv D$. For this divisor, all of the notions are equivalent sinces we have $K_X + (B + D)$ with $B + D$ big. So if D is asymptotically k-ample then $K_X + B + D$ is, so $K_X + B + D$ is numerically k-ample, so D is.

2. October 31: Dmitry Kaledin, Hochschild-Witt complex

Next week I'll do an introduction to non-commutative geometry, today I'll talk about something completely different, what I've been doing later. If you don't like this, the lectures will be different. This is a work in progress. For now there is no fixed reference. The basic story is, I think, rather simple. So I start by recalling something. We start with X which is a smooth algebraic variety over a field $\mathbf{k}$. We can consider cohomology theories for X. One is, we could take de Rham cohomology, defined as cohomology with coefficients in the de Rham complex Ω^*_X. You could take just holomorphic forms; you have to be a little careful, take a sheaf of these and take its cohomology.

There was a non-trivial theorem of Grothiendieck from 1969 saying that even if you consider just algebraic things, then still the result is the same as the cohomology you're used to. So if $\mathbf{k} = \mathbb{C}$, this coincides with C^∞ de Rham cohomology.

Okay. Now what kind of gives rise to this version of non-commutative geometry that I want to talk about is the following observation. In this setting differential forms are something you just find by experiment and it works, so you go with them. But it turns out that they have a different, homological interpretation, namely, let me stick to affine algebraic varieties. If have some X which is the spectrum of some associative commutative algebra A over $\mathbf{k}$. For any associative algebra A we have an invariant called Hochschild homology. You consider A-bimodules, things which A acts on both on the left and the right so that the action commutes. These are modules over $A \otimes A^{op}$. If you consider A as an A-bimodule, it's fine but it's not flat. You can then get the higher derived "tor" which are the Hochschild homology groups. By choosing an explicit canonical resolution, you can get a presentation which lets you see what this is.

We build these out of the tensor powers of A, so

$$\xrightarrow{b} A^{\otimes 4} \xrightarrow{b} A^{\otimes 3} \xrightarrow{b} A^{\otimes 2} \xrightarrow{b} A$$

I won't tell you about b because I won't need it except that $b(a_1 \otimes a_2) = a_1 a_2 - a_2 a_1$. Let me give you a theorem

Theorem 2.1. *Hochschild Kostant Rosenberg 1962*
If A is commutative, $X = Spec\ A$ smooth, then $HH_i = H(X, \Omega_X)$.

What's interesting is that this invariant is defined for any associative algebra and commutativity is not used. So we can think of Hochschild homology classes can be thought of as some non-commutative generalization of differential forms.

In the eighties people introduced cyclic homology, which includes the canonical functorial ap B on Hochschild homology groups which takes the place of the de Rham differential. There's a point to keep in mind, which is in Hochschild homology you sit in homological degrees while the de Rham sits in cohomological degree. We can recover de Rham cohomology without the grading, so only in the $\mathbb{Z}/2$-graded sense. We also lose the multiplication in de Rham cohomology. You only get that in the commutative setting.

What I want to talk about today is a different generalization. I often hear people talk about this "let's fix a field of characteristic zero" but let's not. Everything up to now works in any characteristic. Let's let $\mathbf{k}$ be a finite field of characteristic p. The only fine point is that Ω^i_X could be either a quotient or a subset of a tensor power, and it characteristic zero it's the same. There's a denominator there and in characteristic p it doesn't work and you really have to take the quotient of the tensor power modulo the relations. You get some kind of cohomology theory. It's not nice for every purpose. It's well-defined but it has coefficients in $\mathbf{k}$, so you want to count something, you only get the answer modulo p.

So the question: how to lift this to a theory in characteristic zero? I want cohomology groups which are modules over the p-adic numbers. One situation is one in which X itself has a lifting. The variety over $\mathbf{k}$, well *Spec* $\mathbf{k}$ can be lifted to *Spec* $W(k)$, the Witt vectors of $\mathbf{k}$, so for example if $\mathbf{k} = \mathbb{F}_p$ then $W(\mathbf{k}) = \mathbb{Z}_p$. So $M_{DR}(\tilde{X})$ is a good invariant, a lifting to the Witt vectors, but unfortunately a lift need not exist and if it does need not be unique.

It turns out it's possible to come up with a theory to deal with this, but non-trivial. The first attempt to refine these into characteristic zero was done by Serre in the 1950s. He tried to consider $M^*(X, W(\mathcal{O}_X))$. Let me for now say that W is a functor from commutative rings to commutative rings. It's a functorial way to lift something from characteristic p to characteristic zero. So $\mathbb{F}_p$ goes to $\mathbb{Z}_p$. This functor is huge. If you start with polynomials in one variable, $\mathbb{F}_p[[t]]$ has as its answer something huge, let me show you.

Let me define an algebra B in the following way. Take $\mathbb{Q}_p[[t, t^{\frac{1}{p}}, t^{\frac{1}{p^2}}]]$. The relation is that $(t^{\frac{1}{p^n}})^p = t^{\frac{1}{p^{n-1}}}$. So this is not finitely generated.

This is big but not so horrible. adding fractional powers, this is graded by rational numbers of the form n over p^i. So B sits in Ω^1_B. Anyway, $W(\mathbb{F}_p[[t]])$ consists of points in B so that the coefficients of f are in $\mathbb{Z}_p$ as are the coefficients af df.

Informally, I can say this is something like $\mathbb{Z}_p[[t, pt^{\frac{1}{p}}, p^2 t^{\frac{1}{p^2}}, \ldots]]$

Serre's problem was that this didn't deal with differential forms, only with functions. So Grothiendieck brought this back with crystalline cohomology in the late 60s. This is functorial and answers the question in the best possible way. There's a comparison theorem which says that if there is a lifting $\tilde{X}$ then $H_{crys}(X) = H_{DR}(\tilde{X})$.

Unfortunately, the definition is really high tech. Later, in '77 or '78, Illusie, following Deligne, Bloch, et cetera, showed that there is a functorial complex of differential graded algebras for every smooth X over $\mathbf{k}$ which they call the de Rham Witt complex, which comes with a topology, a projectiev system over certain complexes so that on one hand $W\Omega^0_X = W(\mathcal{O}_X)$ but also $H^*(X, W\Omega_X) = H_cris * (X)$.

The proof is ad hoc. There are some miracles. You write some formulas and by a miracle everything works, The complex is rather large but it actually follows.

For the ring $\mathbb{F}_p[[t]]$ the answer is given by the same formula, and it's nontrivial in degrees zero and one. So $W\Omega^i_X \subset \Omega^i_B$ for $i = 0, 1$. Here α and $d\alpha$ have integral coefficients. The definition is different and this is the answer you can compute. But you can see we have lots of cohomology in degree one, so for instance $t^{p-1}dt$ is a non-trivial cohomology class. To kill it off, you get something divisible by p.

My goal for today, I have de Rham cohomology, the de Rham complex and two generalizations of it. One is the Hochschild homology. The other in characteristic p is the de Rham Witt complex. What I want to do is combine them and explain what to do. The construction is easier. There's a comparison theorem that says this will reduce to de Rham Witt in the commutative case, but there are fewer miracles. I'll try to explain some parts of the construction today. I want to start by talking about Witt vectors, have a break, and then talk about how to do the non-commutative version.

[break]

The story about cohomology I told was in historical order. Now I'll say something out of historical order. Witt did this in the thirties but the modern approach is simpler, I think. This is a functor from commutative rings to commutative rings, there are two, both W and $\mathbb{W}$, so-called universal (big) Witt vectors. I'll start with the universal one. As a group it's easy to describe. It's an Abelian group. Take the following. You have something in characteristic p, you want something that is not characteristic p, what do you do? You could try considering invertible elements. Look at the subgroup of $A[[t]]^*$ with leading term 1. There is a map from this to A^* by taking the leading term. This is the group. Now we want to put a product on it. Where do I get a power series with constant term one? One example is a characteristic polynomial. Assume I have some free module $V = A^n$, and something that acts on it, a, a matrix, and then I take the determinant, $det(I - ta)$, almost a characteristic polynomial. You can do lots of stuff with this. Call this determinant $ch(\langle V, a\rangle)$. One observation is that $ch(V_1 \otimes V_2, a_1, a_2\rangle) = ch(\langle V_1, a_1\rangle)ch(\langle V_2, a_2\rangle)$. What else can I do? What about on the tensor product? There exists a unique functorial topological (meaning that it respects the inverse limits of all the truncated versions where you kill t^n) product map $* : \mathbb{W}(A) \otimes \mathbb{W}(A) \to \mathbb{W}(A)$ such that $ch(\langle V_1 \otimes V_2, a_1 \otimes a_2\rangle) = ch(\langle V_1, a_1\rangle) * ch(\langle V_2, a_2\rangle)$.

The matrices you get have characteristic polynomials that are dense in all power series so uniqueness is obvious. But existence you need to prove. This is associative and commutative. This is obvious because tensor product is associative and commutative. So you get this for free.

Let's use this to compute the ring of Witt vectors of $\mathbb{Z}$.

Corollary 2.1.

$$\mathbb{W}(\mathbb{Z}) = \mathbb{Z}\langle \epsilon_1, \epsilon_2, \ldots\rangle$$

with product given by $\epsilon_i * \epsilon_j = \frac{ij}{lcm(i,j)}\epsilon_{lcm(i,j)}$

Every power series with constant term 1 is of the form $\prod(1 - t^n)^{b_n}$. If I let ϵ_n to be the class of $(1 - t^n)$, then the whole group is like this. Now I observe for the product, I just need to do it for generators. Observe that ϵ_n is the characteristic polynomial of σ_n which permutes the basis vectors cyclically. I just need to take the tensor product of these guys. You do the calculation and that's the answer you get. If i and j are coprime, you get $ij\epsilon_{ij}$. But at the other extreme you get $\epsilon_n^2 = n\epsilon_n$ so they're almost idempotent. Assume that A is p-local, so that every n not divisible by p is invertible. Then you can see that $\mathbb{W}(A)$ is also p-local. So then we always have $\mathbb{W}(\mathbb{Z}) \to \mathbb{W}(A)$ by functoriality. So here I have the images of the ϵs. For any n not divisible by p I have a well-defined map which takes ϵ_n to $\frac{1}{n}\epsilon_n$. Then we can define $W(A)$ to be the intersection over all such n of $Ker\frac{1}{n}\epsilon_n$.

Proposition 2.1. *$W(A)$ is a subring of $\mathbb{W}(A)$, and in fact $\mathbb{W}(A)$ is a product of rings of the form $W(A)$.*

For example, if you consider $\mathbb{W}(\mathbb{F}_p)$, then the answer will be $\mathbb{Z}_p\langle \epsilon_n | p \nmid n\rangle$, then this will split into copies of $W(\mathbb{F}_p) = \mathbb{Z}_p$. If we put an additional vector the result is huge but that's the story of Witt vectors.

So let's look at the non-commutative story. I don't just want to just get Witt vectors for the ring, but actually the Hochschild homology. I should do them at the same time and I can take coefficients for Hochschild homology of R in an R-bimodule M. Then $HH_0(R, M) = M/[M, R]$ and $HH_i(R, M)$ is the derived functor of this. It's important to include the bimodules. Since I want to take the derived functor it's sufficient to consider projective bimodules. So the setting I want to work in is the following. I have R an associative unital algebra over $\mathbf{k}$ and M a projective bimodule, a projective module over $R^{op} \otimes_{\mathbf{k}} R$. I don't get this when I look at $M = R$ because R is not projective. I want to define some version of Witt vectors for the pair.

Again as a group this is very easy to define. I have a bimodule here. I want to look at the tensor algebra of M/R. I can also take truncations of this guy, which is just the quotient by all things of degree $n + 1$. As a vector space, I have the direct sum

$$\bigoplus M^{\otimes_R n}$$

The invertible elements will be non-Abelian, I could Abelianize but it's slightly better to do a different thing. Let me take $K_1(T^{\leq n}(M/R))$, that is to say, $K_1(A) = lim_\ell GL_\ell(A)_{ab}$. For general definitions it's better to use the limit. This is my replacement for my power series. There's a map to $K_1(R)$ which is split, surjective, and the kernel is $\mathbb{W}_n(M/R)$. The whole thing is the inverse limit of $\mathbb{W}_n(M/R)$.

Lemma 2.1. *For $R = \mathbf{k}$ and M a vector space, then $\mathbb{W}_n(M/\mathbf{k}) \to \mathbb{W}_{n-1}(M/\mathbf{k})$ has kernel equal to [missed]*

This is only for $R = \mathbf{k}$. There's another lemma that explains why that's enough.

Lemma 2.2. *Assume we have two algebras R_1 and R_2, and M_{12} is acted on by $R_1 \otimes R_2^{op}$ and vice versa by M_{21}. Let's say these guys are projective. We can take the tensor product $M_{12} \otimes_{R_2} M_{21}$, and we can take $\mathbb{W}(M_{12} \otimes_{R_2} M_{21}/R_1)$ or we can do the opposite: $\mathbb{W}(M_{21} \otimes_{R_1} M_{12}/R_2)$ and these are canonically the same.*

This uses some argument from K-theory. Now a projective bimodule is a summand of a free bimodule, which is of the form $R \otimes V \otimes R$. If you let R_1 be R and R_2 be $\mathbf{k}$, then $\mathbb{W}(M/R) = \mathbb{W}(R \otimes V/\mathbf{k})$. You can get the corresponding statement for alegbras now. Split this over $\mathbf{k}$.

Now lemma three is more like a proposition, which shows there is also a product that can be characterized in a similar way.

I have some free module $V = R^n$. I also have a bimodule, I don't have a single endomorphism. I have a map from $V \to M \otimes V$. I can also define a characteristic polynomial for that. I can take $P = T^*(M/R) \otimes_R V$. It's a free module generated by V so I can extend a to act by $\tilde{a}$ on P. If I take $Id - \tilde{a}$, I'll write a generalized determinant, but it's a class in K_1. The same is true for truncations so I have a compatible system. This will be $ch(\langle V, a\rangle)$. By general nonsense this is additive and you can also take tensor products of these guys. So let me give you a statement.

Proposition 2.2. *Take R_1, M_1, R_2, M_2. Then there exists a unique functorial topological product of the form $\mathbb{W}(M_1/R_1) \times \mathbb{W}(M_2/R_2) \to \mathbb{W}(M_1 \otimes M_2/R_1 \otimes R_2)$ such that you get the same condition as in the classical Witt vectors. This is a justification for why to consider non-commutative algebras. It's associative and commutative.*

Now, this has a corollary.

Corollary 2.2. *First of all, you need projectivity. If R is just $\mathbf{k}$, then any bimodule is projective, and $\mathbb{W}(k/k)$ is a commutative associative ring. This is the same as the classical ring of Witt vectors of $\mathbf{k}$. For any M and R, $\mathbb{W}(M,/R)$ is a module over $\mathbb{W}(\mathbf{k}/\mathbf{k}) = \mathbb{W}(\mathbf{k})$.*

We know that $\mathbb{W}(\mathbf{k})$ contains lots of those ϵs. That has many canonical idempotents. It's easy to deduce from Lemma 1 that $\mathbb{W}(M/R)$ is p-local when [missed]. You can define $W(M/R)$ exactly as before. This took me half an hour. But there are very few miracles as before.

Now where do I get my de Rham Witt complex? I want to take the derived functor. This is not additive in M. The usual definition of the derived functor does not work. You can't apply a non-additive functor to a complex. It's actually something that Dold did in 1958 or so. It happened much before [missed]. The basic situation is that you have an additive category. Consider complexes there. You want it to be [missed]-closed. You can consider simplicial objects in your category $\mathcal{E}$. Then you have the Dold-Kan equivalence between simplicial things and complexes in your category. You take the normalized chains and realization D. Here you have a complex so there's a condition that your differential squares to zero. On the other side it's a bunch of bimodules with maps. The conditions are compatibilities. If you have a functor, any functor from $\mathcal{E}$ to Abelian groups or an additive category, it extends to $\Delta\mathcal{E}$ pointwise. So if you want to take a bad functor and extend it to complexes, you can define $F_* = N \circ F^{\Delta} \circ D$ which goes from complexes in $\mathcal{E}$ to complexes in *Ab*. Then F_* sends chain homotopic maps to chain homotopic maps. This is strange because there is a formula which you'd think might be broken down. But it's not. If you have a functor $F : R^{op} \otimes R - mod^{proj} \to Ab$ then $LF(M)$ is the homology of F_* applied to a projective resolution of M. It's the usual derived functor. We know two projective resolutions are chain homotopy equivalent so the Dold theorem implies this is well-defined.

I can now simply define this.

Definition 2.1. *Hochschild-Witt homology is the Dold-derived functor of $W(M/R)$*

The theorem says this gives what I want.

Theorem 2.2. *In the situation of HKR, where A is commutative and finitely generated over $\mathbf{k}$ (and smooth) then there is a canonical identification between $WHH_i(A,A)$ and $H^0(X, W\Omega^i_X)$. You can ask if you can extend the cyclic homology, the de Rham differentials. This comes out, you can extend the simplicial thing to the cyclic thing. You can also do the same thing with the Frobenius [missed]. This is more intelligible, I didn't do any strange calculations, I just did a little bit of standard things, used a little K-theory.*

3. November 7: Jeff Brown, Gromov-Witten invariants of toric fibrations (and beyond)

One thing about the title is what I'm going to talk about is from five years ago. We're working on new things. I think for the first hour I'll basically give a review of some basics of Gromov-Witten theory and probably for those of you who were there for Givental's lectures it's probably redundant, but that was a long time ago. Mid July, so. If you already know it, you don't have to stay for this. In the second hour I'll talk a little about for toric fibrations and manifolds that are enough like toric fibrations. So, uh, I'll start with quantum cohomology of a Kähler or symplectic space X.

So basically, uh, so, we can write the ordinary cohomology product as $a \cdot b = (ab, \phi^\mu)\phi_\mu$ where $\{\phi_\mu\}$ is a basis of $H^*(X)$ with Poincaré dual basis $\{\phi^\mu\}$, so $(\phi_\mu, \phi_\nu) = g_{\mu\nu}$. You can rewrite this as $(ab, \phi^\mu)\phi_\mu = \phi_\mu \int_{X_{0,3,0}} ev^* a \wedge ev_2^* b \wedge ev_3^* \phi^\mu$.

You have maps from a curve with three marked points 0 and 1 and ∞ into X. This is just a point, there are no moduli. There's $PSL_2\mathbb{C}$ automorphisms that fix 0, 1, and ∞. So you map a standard point into X. So $[X_{0,3,0}] = [X]$. This bracket is the virtual fundamental class of the manifold. All the marked points go to the same point in X.

This is really the integral over X of $a \wedge b \wedge \phi^\mu$. We rewrote the cup product in terms of moduli space. So $X_{0,3,0}$ is $M_{0,3} \times X$. Then $M_{0,3}$ is a point when you mod out by automorphisms.

So the idea of quantum cohomology invented by physicists is to deform this formula along all degrees. We worked with constant maps and three marked points, but we could put in more marked points and non-constant maps. In general, for $t \in H^*(X)$, you define $a \cdot_t b$ to be (I'll be more precise about my $H_2(X)$ later)

$$\sum_{d \in H_2(X,\mathbb{Z})} q^d \sum_{n=0}^{\infty} \frac{1}{n!} \phi_\mu \int_{\overline{X_{0,n+3,d}}} ev_1^* a \wedge ev_2^* b \wedge ev_3^* \phi^\mu \bigwedge_4^{n+3} ev_i^* t$$

where f_* of the fundamental class is d.

The Mori cone of X is classes of $H_2(X, \mathbb{Z})$ represented by holomorphic curves in X.

For example, if $\sigma_1, \ldots, \sigma_N$ is a basis of the Mori cone of X (tensored with $\mathbb{Q}$) then if $d = \sum a_i \sigma_i$ then $q^d = q_1^{a_1}, \ldots q_n^{a_n}$. So the monomial in q records the degree.

The formula has potentially infinitely many terms. So it's a series. Just to say it clearly, $X_{0,m,d}$ is the moduli space of stable maps f from a space of genus zero with m marked points whose $f_*([\text{source}]) = d \in MC(X)$. Stable here means there are no infinitesimal automoprphisms of the map. For example, here's something unstable, a map from a twice punctured sphere to the point, you could rotate it.

Let me talk briefly about compactification. In $M_{0,4}$, you could fix three points at 0, 1, and ∞. You've got three options as the point degenerates; it could go to any of these. So this is like Deligne-Mumford, but the map itself could also pinch.

Some properties of this cohomology: it's graded homogeneous, keeping the ordinary cohomology grading and, well, I need a degree for q. The degree of q representing σ, well, $c_1(TX) \cap \sigma$, that's the grading.

The quantum product is graded homogeneous, these products depend on d. In order that $a \cdot_t b = a \cdot b$ you work out from the dimension of the space of maps what to do.

So for example, on $\mathbb{CP}^1$, let P be the first Chern class of $\mathcal{O}(1)$ then $P \cdot_t P$ is 0.. The degree of P is one and X is two.

Other properties: it's associative and commutative.

Okay, so in general if you want to compute quantum cohomology, you rally need to do some matrk algebra. This doesn't sound good. Hre's a better plan. You should be able to do this starting from one function. So the plan can be to study [missed]

Let's do the example of $\mathbb{CP}^1$. We have $t = tP$. Write $J(z,t) = e^{(pt_1 + t_0 \cdot 1)/z}$. If you take $(z\partial_t)^2$ of this and we get p^2 times this thing which is zero. Classically you can get this sort of thing but to move on you need a quantum correction. J is

$$e^{(pt_1+t_0\cdot 1)/z}\cdot\sum_{d=0}^{\infty}q^d e^{dt_1}\frac{1}{\prod_{n=1}^{d}(p+nz)^2}$$

You can check what happens when you apply $(z\partial_t)^2$ to this. You get $qe^{t_1}J$. So to relate it all to the property we wanted, but if we take $(z\partial_t)^2-qe^{t_1}$ and apply it to J we get 0.

Let me make one more definition. We can define $z\partial_a J$ to be $S^*(z,t)(a)$. We'll define the value of the operator as the derivative of J.

We're almost to the thing we wanted. We were given the formula, you can check what this double differentiation gives us, we're almost at the characterization. A fact is that $z\partial_a z\partial_b J(z,t)=S^*(z,t)(a\cdot_t b)$. In general it's a fact coming from a more natural definition of J. Substitute the given fact into the above relation.

I differentiate twice so I get $S^*(P\cdot_t P)$. I need one other fact, which is that J is $S_X^*(1_X)$. This is equal to zero. We had an operator that differentiated the formula. Using two properties and the fact that S was invertible, we learn that $P\cdot_t P-qe^t\cdot 1=0$

Let's give a definition for J. An abstract definition

Definition 3.1.

$$J(z,f,q)=\sum_{n=0}^{\infty}\sum_{d\in MC(X)}\frac{1}{n!}q^d\phi^\mu\int\frac{ev_1^*(\phi_\mu)}{z-\psi_1}\bigwedge i=2ev_i^*t$$

Here ϕ_1 is the chern class of the line bundle L_1 whose fiber is this [picture]

[missed some]

Just to clarify, the marked points are distributed over every branch, but the number of marked points on either side can be switched using symmetrization. You sum over the partitions of the integer, not specifically the subsets.

So anyway, we're trying to approach the question, can we always find a nice formula for the J-function of a manifold. If not, can you change the question a little bit and find a nice formula in the setting of Gromov-Witten theory, a question that the formula is the answer to.

Let's write the answer for Fano toric manifolds. First let's say something about toric manifolds. YOu have $\mathbb{C}^N$ with the torus T^N acting, and you embed T^N, with embedding at the level of Lie algebras by, well, given K vectors of length N. Each column, you could say it the other way, K rows of length N. Each row is an element of the Lie algebra, so that gives an embedding of T^K into T^N. The entries should be integral and the matrix should be full rank. Then we have this map $\nu:\mathbb{C}^N\to\mathbb{R}_+^N$ via $(z_1,\dots,z_n)\mapsto(|z_1|^2,\dots,|z_n|^2)$. Then you take, well, I want this to be a column vector. Then I multiply my matrix onto my column vector and get a length K vector. Call the matrix M. Then $M\circ\mu$ is in the Lie algebra of T^K. It's the real Lie algebra because the numbers are real, maybe I should think of it as the dual of the Lie algebra. Then, uh, that's the moment map for the action of T^K on $\mathbb{C}^N$.

I take a regular value ω of $M\circ\mu$. Then I take its inverse image and mod out by T^K and that's my toric variety X_ω. Fixing an element of of T^K gives K conditions and pulling back does another K.

So you could take $\mathbb{P}^{N-1} = (\mathbb{C}^N - 0)/\mathbb{C}^*$. Here you mod out by all the conditions at once. You could think of it as first taking the unit S^{2N-1} and then divide out by the circle bundle, that's another way to get it. This is how we're thinking about it.

So these will be symplectic manifolds. The reason I'm doing this aside, you want to find some cohomology classes to go into the formula for the J-function.

So we have $(m \circ \mu)^{-1}(\omega)$ sitting over X_ω and the fiber of this map is just T^k. The T^k is split into circle factors canonically. You can take the first Chern class of each of these circle factors and get cohomology classes $P_i = c_1(S^1_i)$. These are all Kähler classes as it turns out. And in fact they generate multiplicatively the cohomology of X. So, uh, choose, here's what we can do. Define $P_i(d) = d_i$. Then $q^d = \prod q_i d_i$.

The formula for the J function involves an integral, and we want to write down a formula without an integral in it.

Define the symplectic reduction $\{z_j = 0\}/T^K$, well $\{z_j = 0\} \subset \mathbb{C}^N$, so cap with $(m \circ \mu)^{-1}(\omega)$. Define this to be U_j.

As an example, for $\mathbb{CP}^{N-1}$ you have many toric divisors, but you have only one generator for $H^2(X, \mathbb{Q})$, P, which corresponds to one of these divisors.

If X is Fano, meaning that $c_1(TX) \cap d > 0$ for all d in the Mori cone, if X is Fano, then the formula is that

$$J_X(z,t,q) = e^{(Pt+t_0\cdot 1)/z} \frac{\sum_{d\in MC} q^d e^{dt}}{\prod_j = 1}^{N} \prod_{m=1}^{U_j(d)} (U_j + mz)$$

where

$$t = \sum_{i=1}^{k} t_i P_i + t_0 \cdot 1.$$

Okay, so, the computation was due to Givental. In the non-Fano case the J-function has no nice formula, it's just a mess. Originally it's a question for how much of a future this theory has. It'll be a mess in the general case. This is still the toric case! If you have a nice formula you can find differential operators that annihilate it by inspection. Then you'll have to change the plan.

Why isn't this the J-function for non-Fano things? Well, the first Chern class i-a/vay- $\sum U_j$ for all toric X. So in the non-Fano case, the first Chern class of the tangent bundle capped with some degree d can go negative. So some of these U_j will have negative pairing. Then what does this formula mean? It means, notation, $\prod_{m=1}^n (U_j + mz)$, this is

$$\frac{\prod_{m=-\infty}^n U_j + mz}{\prod_{m=-\infty}^0 U_j + mz}$$

so if n is negative, you get a denominator. Then you get a numerator in our formula, which has a polynomial part.

I wasn't knowledgeable or inquisitive enough at the time to understand this. I wasn't working in this area until 2006. Iritani proved in 2005 that the same formula is a generating series for Gromov-Witten invariants, just not the J-function. So that's great, that shows that you can still, by changing the question you can still hope to find nice formulas. Now the expectation is that for any symplectic manifold you can hope to find nice formulas.

It seems unclear whether the theory can be trivialized, even after a lot of technological progress. I can't say any more without going further. So let me define, which generating series? It seems we got further in the hour talk a week ago, somehow. Anyway. Let's say what the new generating series is. Consider a polynomial part. Polynomial in z with coefficients in $H^*(X)$. At each order in q-variables. Consider a polynomial like that. Plus a z-inverse part.

$$q(z) + \sum_{n=0, d \in MC}^{\infty} \phi_\mu \int_{X_{0,n+1,d}} \frac{ev_1^* \phi_\mu}{z - \psi_1} \bigwedge_{i=2}^{n+1} ev_i^* q_K) \psi_i^K$$

where $q(z) = \sum q_K z^K$. You can kind of think of one side as polynomial and the other as z^{-1}. They're like a q and p. There's one point that's distinguished, the first marked point, which is different, so really this is lying in the cotangent bundle, on the graph of dF^0, the genus zero generating function. The differential accounts for that first marked point.

The same function in the non-Fano case is also on the graph of dF^0, it's just tilted, it's not the J-function any more. It gives a particular q. The z^{-1} is given this way.

This result was suggested or semi-suggested by this being a Lagrangian cone, by being conical, almost flat. This is the framework now for working in mirror theorems, which is what something like this is called. Any set of points on the cone where you get a nice formula. You call it a mirror theorem because the nice family of points has an integral interpretation. You can represent the coefficients of this series by integrals closely related to the Euler gamma function. That J formula, you can write it as ϕ_μ, a Laurent series in t^{-1} times some integrals of μ. That's why it's called a mirror theorem, it's supposed to be a mirror model, B-model, from physics.

$$\Gamma(\lambda + n) = (\lambda + n - 1) \cdots \lambda$$

so you get some properties like that. These are some equivariant cohomology classes. Let's stop here.

If we look at $\mathbb{CP}^1$, it's $\mathbb{C}^2$ mod the diagonal torus, we have the $\mathcal{O}(1)$ bundle over $\mathbb{CP}^1$ with the circle fiber, and you can cross it with some universal classifying space, a total space for a classifying space for the circle. You cross it with a contractible space with the simultaneous circle structure. There are some fixed points in $\mathbb{CP}^1$, you can move all the torus action onto the ET^1 side. I just wanted to give a formula, let me show you that. The integral of P over $\mathbb{CP}^1$ should be 1. The Atiyah Bott formula says you can restrict to the fixed points. You take 0^*P and divide by the weight at zero of the tangent space, w_0, plus the ∞^*P pullback over its weight w_∞, and it just works out combinatorially. You get something like $\frac{\lambda_\infty}{\lambda_\infty - \lambda_0} + \frac{\lambda_0}{\lambda_0 - \lambda_\infty}$. So there's a combinatorial way to do this just looking at fixed points.