2016 CGP SEMINAR

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1. January 28: Thomas Hudson: Segre classes and Schur polynomials for algebraic cobordism

Thank you for coming. The plan for the talk is to start with a review of how Segre and Chern classes are defined for the Chow ring, please let me know if you cannot read or if you have any questions. Then I want to remind some results about the Grassmannian and for Schubert varieties. Then the idea is that these results can be extended to a more general context where you have oriented cohomology theories. This is the more general thing I'd like to discuss. The idea is about how to extend the ideas about Grassmannians and Schubert varieties to oriented cohomology theories.

I want to start with Fulton's definition of Chern classes. What he does is first defines $c_1(L)$ of a line bundle. Then he considers projective bundles, so $E \to X$, a vector bundle of rank e and associate the projectivization $\mathbb{P}(E)$ and over it π^*E which has a subbundle of hyperplanes H of rank $e-1$ and a cnonical quotient $\mathcal{O}(1)$. We can consider the pushforward onto the base $\pi_* : CH^*(\mathbb{P}E) \to CH^*(X)2$ so $c_1^i(\mathcal{O}(1)) \mapsto s_{i-e+1}(E)$. Then you can put together what you get in a power series and you get $s_t(E) = 1 + s_1(E)t + s_2(E)t^2 + \cdots$

When you compute these classes you get $s_0(E) = 1$, and for dimension reasons $s_{-m}(E) = 0$. Then Fulton can define Chern classes by defining $c_t(E)$ as $\frac{1}{s_{-t}(E)}$. To see this makes sense, you can look at E a line bundle, which is silly because the projecivization is X and the map π is the identity. Then $s_t(L) = 1 + \xi t + \xi^2 t^2 + \cdots$ and moving on to the Chern polynomial you get

$$\frac{1}{1 - \xi t + \xi^2 t^2 + \cdots} = 1 + \xi t$$

So you get $c_1(L) = \xi$.

Now I want to go to Grothiendieck's version of this story. To some extend, Grothiendieck gives a way to define the first Chern class, it doesn't really matter too much what it is, and then he observes that $CH^*(\mathbb{P}(E))$, it's

$$\bigoplus_{i=0}^{e-1} \xi^i CH^*(X)$$

a direct sum of e copies of evaluation on the base, as $CH^*(X)$-modules. Then the next question is, what is ξ^e? This gives the Chern classes. More precisely you have the following relation

$$\xi^e - \xi^{e-1}c_1(E) + \xi^{e-2}c_2(E) + \cdots + (-1)^e c_e(E) = 0$$

Now you can somehow recover the previous relation by pushing forward this equality. Then ξ^e gives $s_1(E)$. The second term gves $s_0 = 1$. So that's $s_1(E) = c_1(E)$. So if we multiply by ξ and push forward, that gives us something about s_2, that

$s_2(E) = s_1(E)c_1(E) + c_2(E) = 0$. If you combine this information together, you get a relationship between the Segre series and the Chern polynomial, which is that $s_t(E)c_{-t}(E) = 1$.

I'm proposing these two because the Grothiendieck is the way you can define these for oriented cohomology theories. On the other hand, the people who developed the oriented cohomology theories didn't think about Segre classes much. The idea is to keep Fulton's definition, and try to see what you can get out of this relationship, that you can still get to define Chern classes. Let me stress another thing. You may know, Chern classes $c_i(E)$ can be understood as an elementary symmetric function, this is the ith, in the Chern [unintelligible]. The equality says that you can see the Segre classes as the complete symmetric functions. Why does this stay true in any context? You don't have the relation between c and s any more.

Now the two concepts are used in Schubert calculus to express the fundamental classes of Schubert varieties. So let k be a field, at some point characteristic zero when we start talking about cobordism but until then it won't matter. We start with a vector space E_n over a point, an affine space, and we can look at the Grassmannian of d-planes inside $\mathbb{A}^n$. Then we have $GR(d,m) \times \mathbb{A}^n_k$, and there we can see the universal bundle of rank d. If we fix a basis of $\mathbb{A}^n$, say $f_1, \ldots, f_n$, we define F_i to be the span of the first i vectors. Sometimes I'll need F^{n-i} to mean the same thing. Once you fix a basis you can define Schubert varieties. If we take a partition $\underline{\lambda} = \lambda_1, \ldots, \lambda_d$, written in weakly decreasing order, then X_λ is all the points p in the Grassmannian (so that the restriction of the universal bundle above it is that point itself), such that the intersection of this fiber with $F^{\lambda_i - i + d}$ is at least i for all i.

This cuts out the Schubert variety.

One question that one can ask is how to express the fundamental class of these varieties in the Chow ring as an element in $CH^*(GR(d,n))$. The answer is by using Schur polynomials. So (where ℓ is the length of the nonzero part of the partition)

$$[X_\lambda]_{CH} = s_\lambda(\underbrace{x_1, \ldots x_d}_{\text{Chern roots}}) = \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1+1} & \cdots & c_{\lambda_1+\ell-1} \\ c_{\lambda_2-1} & c_{\lambda_2} & \cdots & c_{\lambda_2+\ell-2} \\ \vdots & \vdots & \ddots & \\ c_{\lambda_\ell-\ell+1} & \cdots & \cdots & c_{\lambda_\ell} \end{vmatrix}$$

So you can do this for X a smooth scheme, and you get some sort of Schubert variety bundle and this gives you a basis for the Chow ring for the Grassman bundles.

Theorem 1.1. *(Kempf–Laksov) The fundamental class $[X_\lambda]_{CH}$ in $CH^*(\mathcal{GR}_j E_n)$ is*

$$\det[(-1)^{j-1} c_{\lambda_i+j-i}(E_n - U_j - F^{\lambda_i - i + j})].$$

An outline of the proof starts by building a resolution of singularities of X_λ. This is done, recall the setting for a moment, you have E_n and inside of it $F_1 \subset \cdots \subset F_{n-1}$. We pull back this flag over $\mathcal{GR}_d(E)$, and inside we have X_λ and we want to desingularize it. To do this we start with our partition $\underline{\lambda}$ and make some numbers $k_i = n - d - \lambda_i + i$. We consider a projective bundle $\mathbb{P}(F^\vee_{k_1})$, and over this we have E_n with U_d inside of this. Then F_{k_1} is inside of this with a line bundle D_1. We move on and we want to find a bundle of rank 2 in the second element of the flag. We look here and find $(F_{k_2}/D_1)^\vee$, its projectivization, and then inside of this we have

$D_1 \subset D_2$, and one repeats this until you get an element in the partition. At the end you will have $\mathbb{P}((F_{k_\ell})/D_{\ell-1})^\vee)$, and here you have $\tilde{X}_\lambda$, which is your resolution.

You need to compute the fundamental class of this thing in your space of flags. If you are able to do that, since this is a resolution of singularities, and if you can express the pushforward in $\mathcal{GR}_d$, then the pushforward will be the fundamental class of the desired Schubert variety. We need to be able to push forward along projective bundles to do this, which is where Segre classes come in.

This was all so far an old story, now comes something slightly more recent, which is oriented cohomology theories. What we want to some extent is a generalization of the Chow ring. We want pullbacks f^*, pushforwards f_*, and Chern classes.

An oriented cohomology theory is a contravariant functor from smooth schemes to commutative graded rings along with morphisms $f_* : A^*(Y) \to A^*(X)$ of $A(X)$-modules given $Y \xrightarrow{f} X$ whenever f is projective. You want some compatibilities between them. You also want the projective bundle formula, the isomorphism between the value $A^*(\mathbb{P}(E))$ on $\mathbb{P}(E)$ for E a bundle over X of rank e and $\bigoplus_0^{i-1} \xi^i A^*(X)$, where $\xi = c_1(\mathcal{O}(1)) \coloneqq s^* s_*(1_X)$.

Now that you've asked for a projective bundle formula you can use Grothiendieck's formula to define Chern classes.

There is a third property, an extended homotopy property, which says that $A^*(X) \cong A^*(E)$, the pullback under projection is an isomorphism, and really you ask this for E-torsors.

[some discussion of uniqueness]

So I wanted to give some examples of oriented cohomology theories.

(1) Since these are modeled on the Chow ring, your first example is CH^*.
(2) Taking the Grothiendieck ring of vector bundles, you get one which is not graded, so you formally add a grading with a formal parameter β, and modify the formulas for pushforward and pullback to incorporate β, which, if you project $\mathbb{P}^1$ on the point, then the pushforward of the identity is β.
(3) One can consider elliptic cohomology
(4) Algebraic cobordism

This algebraic cobordism is an algebro-geometric analogue of MU. There is a theorem.

Theorem 1.2. *(Levine–Morel) Algebraic cobordism is a universal oriented cohomology theory, meaning that for any A^*, there is a unique morphism $\varphi_A : \Omega^* \to A^*$.*

Suppose you get some formula for cobordism. Then it works everywhere. Then somehow the diagram you can keep in mind is, if you have CH^*, this modified version of K^0, which goes to K^0 by setting $\beta = 1$. Then you can see how high you can push this, can you get to Ω^*?

One way to see how these can differ is to look at the formal group law, to see how A^* differs from CH^*, one looks at $c_1(L \otimes M)$. For the Chow ring, one has that $c_1^{CH}(L \otimes M) = c_1^{CH}(L) + c_1^{CH}(M)$. In general, what is true is that $c_1^A(L \otimes M) = F_A(c_1^A(L), c_1^A(M))$ with coefficients in $A^*(\operatorname{Spec} k)[[u,v]]$, so that the pair $(A^*(\operatorname{Spec} k), F_A)$ is a formal group law. There are requirements, this has to be a commutative formal group law, so for instance $F_A(u,v) = F_A(v,u)$, this is just commutativity, and then you have $F_A(u,0) = u$, and the same for v, thanks to the previous one, and then you have something involving three elements for associativity $F_A(F_A(u,v),w) = F_A(u, F_A(v,w))$. At the level of tensor product this is associativity and commutativity and tensoring by the trivial line bundle gives nothing.

A formal group law gives a formal inverse $\chi_A(u) \in A^*(\operatorname{Spec} k)[[u]]$ which plays the role of expressing $c_1(L^\vee)$ in terms of $c_1(L)$.

[some discussion]

Let me give the formal group laws for the examples. For $F_{K^0}(u,v)$ you get $u+v-\beta uv$, for $F_\Omega(u,v) = u+v+\sum_{(i,j)} A_{i,j}u^iv^j$, for elliptic you get the formal group law of the elliptic curve.

Finally we are back to Segre classes. One first question is how to define Segre classes. As I mentioned before, we can keep the definiton we had before $A^*(\mathbb{P}(E)) \xrightarrow{\pi_*} A^*(X)$ takes $c_1^i(\mathcal{O}(1)) \mapsto s_{i-e+1}(E)$. The second question is how to compute them? Here there is an answer because pushforward along projective bundles was studied by Quillen, we have Quillen's formula, which looks like this:

$$\pi_*(\xi^i) = \sum_{j=1}^{e} \frac{X_j^i}{\prod_{1\le k\le e, k\ne j} F_A(x_j, \chi_A(x_k))}$$

It would be nice to have a formula like before, for $s_t^A(E)$, in terms of Chern classes and maybe other stuff.

There is a key observation, which is that if you write $F_A(z,\chi_A(x))$, this factors, as $(z-x)P(z,x)$, and so in some sense, let me say this in a different way. You have $z -_A x$, and you have $z-x$ the normal one, and the power series turns one into the other. If you use this in the product we have, you can start separating things. It makes sense to look at the products of different P, you can look at $\prod_{i=1}^e P_A(z,x_i)$ and make this a power series, this is

$$\sum_{s=0}^{\infty} w_{-s}(x_1,\dots,x_e)z^s.$$

Then just as

$$c_e^{CH}(E^\vee \otimes \mathcal{O}(1)) = \prod_{i=1}^{e}(z-x_i)$$

we have

$$c_e^{A}(E^\vee \otimes \mathcal{O}(1)) = \prod_{i=1}^{e}(z-_A x_i)$$

If you factor the Chow ring part out, you get the following proposition.

Proposition 1.1. *(Hudson–Matsumura)*

$$s_t(E) = \frac{1}{c_{-t}(E)}\frac{1}{w_{t^{-1}}(E)}\mathcal{P}^A_{t^{-1}}$$

with $\mathbb{P}_t^A = \sum[\mathbb{P}^i]_A t^{-1}$.

So what does this mean? You want to compute the Segre power series. One thing is the Chern classes, exactly as the old version. When you apply the unique morphism to CH^* the other part goes to one. The second one comes from the P_A. The third comes from the projective spaces.

Now if you want to compute, this is at the level of bundles, but one would like to have things for virtual bundles. c_t is not a problem on virtual bundles. You look carefully and w is also not a problem because it's attained as a product. You can get $s_t(E-F)$ by just adding $-F$ in the evaluation of w and c.

Now there is, there is a geometric result that says maybe this definition makes some sense for the virtual one.

Theorem 1.3. *(Hudson–Matsumura) Let E and F be vector bundles of rank e and f over X, and consider $\mathbb{P}(E)$. Look at the top Chern class c_f of $\mathcal{O}(1)\otimes\pi^*F^\vee$ and push this forward along $\mathbb{P}(E)\xrightarrow{\pi_*}X$. That's the Segre class $s^{\Omega}_{f-e+1}(E-F)$.*

So this lets you recover the result for any theeory you like. This suggests the notion has some meaning. Then you can try to express the Kempf–Laksov result in terms of Segre classes of this type.

The application we had in mind was to compute $[\tilde{X}_\lambda]_\Omega \overset{?}{\cong} [X_\lambda]_\Omega$. So you want to do a bunch of pushforwards and need machinery like our theorem. Kazarian showed that everything you want to do this, you just need something slightly more general, the pushforward of $c_1^s(\mathcal{O}(1)c_f^\Omega(\mathcal{O}(1)\otimes\pi^*F^\vee)$, and that's just as easy, you just add s in our index on the right side. Then there's machinery to produce determinants.

Now we have two issues. The first is what do you get for the resolved singularity, and the second is the question of whether it is equal to what you have on the right hand side. The other problem is that this isn't necessarily defined before resolution.

One good property of what comes out of the procedure is that they don't depend on the size of the Grassmannian, they exhibit stability phenomena. There is a similar picture with respect to the flag bundle and if you do it with the flag bundle you get Schubert polynomials instead of Schur. So there you use Bott–Samuelson resolutions, and that's pretty awful.

Let me write a formula, for $\lambda = (\lambda_1,\dots,\lambda_\ell)$

$$[\tilde{X}_\lambda^{KL}] = \phi(t_1^{\lambda_1}\cdots t_\ell^{\lambda_\ell}\prod(1-\frac{t_i}{t_j})\prod_{1\le i<j\le\ell}P(t_j,t_i))$$

where

$$\phi(t_1^{s_1},\dots,t_\ell^{s_\ell}) = \mathcal{S}_{s_1}((F^{\lambda_1-1+d}-E_n/U_d)^\vee)\cdots\mathcal{S}_{s_\ell}((F^{\lambda_\ell-\ell+d}-E_n/U_d)^\vee)$$

and from the point of view of the polynomials involved this should have good properties.

2. February 4: Namhee Kwon :Vertex algebras and their applications to the denominator identity

First of all, I'd like to appreciate Professor Yong-Geun Oh for giving me a chance to stay at this wonderful place. Most of the people in this room work on topology and geometry but most of this will be algebra. I intend to give a more accessible talk so if this is boring, please understand.

Definition 2.1.
(1) A *superspace* is a vector space $V = V_{\bar 0}\oplus V_{\bar 1}$.
(2) A *superalgebra* is a $\mathbb{Z}/2\mathbb{Z}$-graded (not necessarily associative) algebra, meaning that $V_\alpha V_\beta \subset V_{\alpha+\beta}$.

If $a\in V_{\bar 0}$ then we say it has *parity* $p(a)=0$ (and likewise for 1). We say also that $p(a,b) = (-1)^{p(a)p(b)}$.

Definition 2.2. A *Lie superalgebra* $\mathfrak{g}$ is a $\mathbb{Z}/2\mathbb{Z}$ graded vector space with a $\mathbb{Z}/2\mathbb{Z}$-graded superalgebra $[\,,\,]$ such that

(1)
$$[x,y] = P(x,y)[y,x]$$

(2)
$$[x,[y,z]] = [[x,y],z] + P(x,y)[y,[x,z]]$$

If we have no odd part then this is just an ordinary Lie algebra.

Let me give some examples. Let V be a superspace of dimension m for the even part and n for the even part. Then we can think of $\mathrm{End}(V)$ (linear maps), we can give a $\mathbb{Z}/2\mathbb{Z}$ grading to these endomorphisms, $\mathrm{End}(V)_\alpha = \{a \in \mathrm{End}(V) | aV\beta \subset V_{\alpha+\beta}\}$. Then the endomorphisms are a superspace. Then in matrix form we can rewrite this as $\{\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}\} \oplus \{\begin{bmatrix} 0 & * \\ & 0 \end{bmatrix}\}$.

We can define a bracket operation as
$$[A,B] = AB - p(A,B)BA,$$
and then this becomes a Lie superalgebra, which we denote $\mathfrak{gl}(m|n)$.

Definition 2.3. Let V be a superspace. A *field* is a series of the form
$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$$
in $(\mathrm{End}\ V)[[z,z^{-1}]]$ such that for any $v \in V$, we have $a_{(n)}v = 0$ for a high enough.

If $a_{(n)}$ is always even we say $p(a(z)) = 0$.

Definition 2.4. A *vertex algebra* is the following data.

(a) A superspace V
(b) A special vector $|0\rangle$ in $V_{\bar{0}}$ (the *vacuum*)
(c) A field $Y(\quad, z) : V \to (\mathrm{End}\, V)[[z,z^{-1}]]$, given by $a \mapsto Y(a,z) = \sum a_{(n)} z^{-n-1}$

such that

(1) there exists a linear map $T : V \to V$ defined by $T(a) = a_{(-2)}|0\rangle$ satisfying $[T, Y(a,z)] = \partial_z Y(a,z)$,
(2) $Y(|0\rangle, z) = \mathrm{id}$ and $Y(a,z)|0\langle|_{z=0} = a$
(3) $(z-w)^N (Y(a,z)Y(b,w) - p(a,b)Y(b,w)Y(a,z)) = 0$ for sufficiently large N.

Let me give an example. Let $\mathfrak{g}$ be a simple Lie superalgebra and let $\hat{\mathfrak{g}}$ be $\mathbb{C}[t,t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K$ with bracket $[a_m, b_n] = [a,b]_{m+n} + m(a|b)\delta_{m+n,0}K$ where $a_m = a \otimes t^m$ and $(a|b)$ is the non-degenerate Killing form from the simple Lie superalgebra. We set $[K, \hat{\mathfrak{g}}] = 0$.

In vertex algebras usually people write this down in the operator product expansion.

Right now we have a formal series $a(z) \in g[[z,z^{-1}]]$ and $b(w)$ in the same place. We assume that $a(z)$ and $b(w)$ are a local pair, so that $(z-w)^N [a(z), b(w)] = 0$. Then it is known that their product
$$a(z)b(w) = \sum_{j=0}^{N} \iota_{z,w} \frac{1}{(z-w)^{j+1}} a(w)_{(j)} b(w) + :a(z)b(w):$$

So first $\iota_{z,w}$ is, we can expand $f(z,w)$ as a power series, but we take domain $|z| > |w|$ and express it there as a power series.

Then $a(w)_{(n)}b(w)$ for positive n is $\mathrm{Res}_z[a(z), b(w)](z-w)^n$ and $a(w)_{-n-1}b(w) =: \partial^{(n)} a(w) b(w):$ where $\partial^{(n)} a(w) = \frac{1}{n!}\partial^n a(w)$.

We usually write $a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{a(w)_{(j)}b(w)}{z-w}$ and this is the operator product expansion. This is equivalent to the Lie bracket operation $[a_{(m)}, b_{(n)}] = \sum_{j=0}^{N-1} \binom{m}{j} c^j_{(m+n-j)}$ where $c(w) = a(w)_{(j)}b(w)$.

Now by going back, the bracket operation is the same as giving an operator product equation like this.

$$a(z)b(w) \sim \frac{[a,b](w)}{z-w} + \frac{a|b}{k}(z-w)^2.$$

Now let's go to Verma modules. I'll define $\mathbb{C}_k \cong \mathbb{C}$ as a $\mathfrak{g}[t] \oplus \mathbb{C}k$-module as follows.

(1) $\mathbb{G}(t)$ acts trivially on $\mathbb{C}$.
(2) K acts as a scalar k on $\mathbb{C}$

Then $\mathcal{U}(\hat{\mathfrak{g}}) \otimes_{\mathcal{U}(\mathfrak{g}[t]\oplus\mathbb{C}K)} \mathbb{C}_k$, as a vector space this is isomorphic to $S(\hat{\mathfrak{g}}^{<0})$, the symmetric algebra. This is Poincaré–Birkhoff–Witt.

Call this thing F and make it a vertex operator algebra as follows. For any $a \in \mathfrak{g}$, we define $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ which means $a \otimes t^n$, where $a_{(n)}$ is an operator over F.

I'll give a second example, where the bracket will just collapse. Let A be a superspace with a skew supersymmetric bilinear form, this is a bilinear form satisfying $(\varphi|\psi) = -(-1)^{p(\varphi)}$.

Then the *Clifford affinization.* Take $C_A = \mathbb{C}[t, t^{-1}] \otimes A \oplus \mathbb{C}K$. Then (with $\varphi_m = \varphi \otimes t^{m-\frac{1}{2}}$),

$$[\varphi_m, \psi_n] = (\varphi|\psi)\delta_{m+n,0}K.$$

So our bracket collapses and we just get this one term.

This is the same as an operator product expansion $\varphi(z)\psi(w) \sim \frac{(\varphi|\psi)K}{z-w}$.

Now we consider $\mathcal{U}(C_A) \otimes_{\mathcal{U}(C_A^{\geq 0})} \mathbb{C}_k$, where this $\mathbb{C}_k$ still has a module structure as before. Again this is isomorphic to $S^(C_A^{<0})$. Then again the vacuum vector is 1 and the state field correspondence is $Y(\varphi^1_{-n_1+\frac{1}{2}} \cdots \varphi^k_{-n_k+\frac{1}{2}}|0\rangle, z) = \varphi^1_{(-n_1)}(z) \cdots \varphi^k_{-n_k}(z)\mathrm{id}$.

Let's take A as an odd 2-dimensional superspace, spanned by ψ^+ and ψ^- so that $(\psi^+|\psi^-) = 1$. The $(|)$ being supersymmetric is just symmetry.

Then we can think about $F = \mathcal{U}(C_A) \otimes_{\mathcal{U}(C_{A^{\geq 0}})} \mathbb{C}_1$.

Then I'll define a special element

$$\nu = \frac{1}{2}(\psi^+_{(-2)}\psi^-_{(-1)} + \psi^-_{(-2)}\psi^+_{(-1)}|0\rangle$$

and then $Y(\nu, z)$ is the Virasoro field $\sum L_n z^{-n-2}$ where

(1) $L_{-1} = T$
(2) L_0 is diagonalizable over F
(3) $[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}c$

This Virasoro algebra is $\bigoplus \mathbb{C}L_n \oplus \mathbb{C}Z$ and then something that satisfies the above conditions (with c replaced by Z) is the Virasoro algebra.

Consider a new field $\alpha(z) =: \psi^+(z)\psi^-(z) :$ (an even field) and once again we take the normally ordered product $: \alpha(z)\alpha(z) :$ and it turns out that $Y(\nu, z)$ is $\frac{1}{2} : \alpha(z)\alpha(z) :$. It turns out that $[\alpha_m, \alpha_n] = m\delta_{m+n,0}$ and $[\alpha_m, \psi^\pm_{(n)}] = \pm\psi^\pm_{(m+n)}$ and you can figure this all out from the operator product expansion.

This is the definition of the Heisenberg algebra. So $\mathcal{S} = \bigoplus \mathbb{C}\alpha_n$ will act on the vertex algebra F.

We want the eigenvalues of α_0, where

$$\underbrace{\psi^-_{(-j_1)}\cdots\psi^-_{(-j_t)}\psi^+_{(-i_1)}\cdots\psi^+_{(-i_s)}}_{\mathbb{V}}|0\rangle = (s-t)\mathbb{V}.$$

Then we call this $(s-t)$ the *charge* and decompose F into $F^{(m)}$ of charge m.Then each of these parts is an irreducible $\mathcal{S}$-module.

Now we're going to think about another eigenvalue, take

$$L_0\psi^-_{(-j_1)}\cdots\psi^-_{(-j_t)}\psi^+_{(-i_1)}\cdots\psi^+_{(-i_s)}|0\rangle = (j_1-\frac{1}{2}+\cdots+(j_t-\frac{1}{2})+(i_1-\frac{1}{2})+\cdots+(i_s-\frac{1}{2}))(\qquad).$$

and now each charge space $F^{(m)}$ decomposes into energy spaces $F^{(m)}_j$ and there is a minimal energy $j\geq\frac{m^2}{2}$. This is the decomposition of the charge space.

Why do we have minimal energy? The smallest energy for $m\geq 0$ will be realized by $\psi^+_{(-m)}\cdots\psi^+_{(-1)}|0\rangle$ and similarly for $m<0$ and what you get is $\frac{m^2}{2}$ in both cases.

For F we consider two bases.

(1)
$$\psi^-_{(-j_1)}\cdots\psi^-_{(-j_t)}\psi^+_{(-i_1)}\cdots\psi^+_{(-i_s)}|0\rangle$$
with $0<i_1<i_2<\cdots$ and $0<j_1<j_2<\cdots$

(2)
$$\alpha_{-j_s}\cdots\alpha_{-j_1}|m\rangle$$
where m ranges over $\mathbb{Z}$ and $0<j_1\leq j_2\leq\cdots$

We need positive subscripts on α so that we don't violate our minimal energy.

Now we'll define a character.

Definition 2.5.
$$\operatorname{ch}F = \operatorname{tr}_F q^{L_0}z^{\alpha_0} = \sum_{j,m}\dim F^{(m)}_j q^j z^m.$$

If you apply this character formula to the first basis, let's thing about what we'll have. So ψ^-_{-j} corresponds to $q^{j-\frac{1}{2}}z^{-1}$. Similarly, we have for ψ^+_{-j}, the monomial $q^{j-\frac{1}{2}}z^1$. The character will be

$$(1+q^{\frac{1}{2}}z)(1+q^{\frac{3}{2}}z)\cdots(1+q^{\frac{1}{2}}z^{-1})(1+q^{\frac{3}{2}}z^{-1}).$$

If you apply to the second case, let's see what you get. The energy, each guy has energy j_s, so q has exponent $j_1+\cdots+j_s+\frac{m^2}{2}$. For each α the charge is 0, since α has both plus and minus. Then we only have charge from $|m\rangle$ so we have z^m. Then this is $q^{\frac{m^2}{2}}z^m q^{j_1+\cdots+j_s}$. In this case our corresponding character is like this.

$$\operatorname{ch}F^{(m)} = q^{\frac{m^2}{2}}z^m\prod_{j=1}^{\infty}\frac{1}{(1-q^j)}$$

because we can have repetition of indices. Then

$$\operatorname{ch}F = \sum_{m\in\mathbb{Z}}q^{\frac{m^2}{2}}z^m/\left(\prod(1-q^j)\right).$$

Therefore we have our identity

$$\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}z)(1+q^{n-\frac{1}{2}}z^{-1}) = \sum_{m\in\mathbb{Z}}q^{\frac{m^2}{2}}z^m/\prod_{j=1}^{\infty}(1-q^j).$$

We can generalize this to A with two even and two odd terms. We have multiple summations and products, but it's like this.

3. February 18: Grigory Mikhalkin: Planar and spatial real algebraic curves: (half-)integer indices

Please interact. I'm planning to make a survey in the first half, what kind of indices, integer and half-integer numbers associated to real algebraic curves and in the second half curves in 3-space and also indices. Let's as an example, since we had yesterday a talk on Legendrian knots. I'll use that as an example. For these knots, the Thurston–Bennequin number is such an index. Algebraically we'll see similar indices. Let's start from the planar case and the nineteenth century, the first part of Hilbert's sixteenth problem.

Just to remind, in its more or less modern form it appeared mostly by Harnack, a student of Klein.

uppose $\mathbb{R}A$ is a curve $f(x,y) = 0$ in $\mathbb{R}^2$, and suppose for simplicity that $\overline{\mathbb{R}A}$ in $\mathbb{RP}^2$ is smooth.

The question is, what is the topology of $(\mathbb{RP}^2, \overline{\mathbb{R}A})$ or $(\mathbb{R}^2, \mathbb{R}A)$.

Hilbert said, if f is a curve of degree 6, what can be $(\mathbb{RP}^2, \mathbb{R}A)$ topologically? We immediately know that all curves are one-manifolds, so they're all just circles, but some circles might sit inside other circles. If you have an oval, the two components are not equal, the interior is a disk, the exterior a Möbius band.

By the time of Hilbert, Harnack had constructed a series of curves with the maximal possible number of components. [Pictures]. The question was what else, in particular, for maximal possible number of components. Here we see the simplest possible version of the index. The simplest possible index is, first and most naively, the number of components ℓ. It's bounded, and as was shown by Harnack, we have $\ell \le g + 1$. Here g is the genus of the complexification. By the adjunction formula this is $\frac{(d-1)(d-2)}{2}$ where d is the degree. So in particular for $d = 6$ we have 11 the maximal number. If ℓ is relatively small, then any configuration of ovals is possible if it doesn't contractict the Bezout theorem. Let's see how the Bezout theorem works in the simplest possible number. Suppose $d = 6$, then we cannot have two disjoint nests. You'd have 8 points of intersection with a line. You *could* have nests of two levels, Harnack studied these in connection with some [unintelligible], these are so-called hyperbolic curves. If you have this, you cannot have any other oval anywhere because you'd have 8 points of intersection with a line.

If it's up to 8 there are no other restrictions, but if ℓ is maximal there are many other restrictions. It would take a long time to survey all possible restrictions. Let me mention one restriction in the case of even degree curves. There was a particular configuration asked by Hilbert, which is just 11 ovals all disjoint from one another. The answer was no, given by Petrovsky in the thirties. The equation, the restriction which he found, a homological restriction, he had a background in differential equations, and what he proved is that $P - N \le \frac{3k(k-1)}{2} + 1$ if $d = 2k$ and P is the number of ovals inside an even number of ovals and N is the number ovals inside an odd number of ovals.

From the modern point of view, we know where this comes from, we take the branched covering, complexify, branch it over the even degree, and we get the $P - N$ is the Euler characteristic. Apparently, not only the 11 ovals were not realizable, they only had two configurations, and after work of Gudkov–Arnold–Rokhlin (70s), only three were possible the two (one nested by Harnack and nine nested by Hilbert) known since the nineteenth century and one with five nested.

Also if $\ell = 10$, there are some restrictions, but now six types, and the smaller, everything starts to be possible. Even though we have, Rokhlin had congruence modulo 8, even up to now a complet classification is only known for degree up to 7 (Vito) in the algebraic case.

There is a symplectic version of the problem, when we ask for real parts of pseudoholomorphic curves. If we do that, then for these curves degree up to 8 is known, for maximal curves, and this is Orevkov.

This is the first and most classical index. There is some other important notion introduced in the nineteenth century, I have the name Klein on the board, he introduced a rather important notion, that of the *type* of the curve. A curve is type I if the complement of the real part in the complexification is disconnected and type II if it's connected. The maximal ones, where ℓ is maximal, always are of type I, the Euler characteristic of the quotient, conjugation reverses the orientation, so it's just a question of orientability, and then maximality [unintelligible]something about a disk with punctures.

So we see that restricting to the maximal possible number of components, things are restricted, and let's see if there are any other numbers we can associate to such a plane. Something that can be thought as a variation of the degree of the Gauss map. Let me give, it will come from the degree of the Gauss map, let me mention one other index which I will, those of you who will be next week in the conference in Seoul, I'll give a more detailed talk about the quantum index of a real curve. Here we will speak about a curve in the torus, so it will be important how it intersects $\mathbb{RP}^2$ (presented as a toric surface) with three lines. Suppose we have inside of this $\mathbb{RP}^2$ our $\mathbb{R}A$.

For simplicity we don't need to consider the most general case. Let me just look at $\mathbb{RP}^2$ (toric picture). We have this curve. Let me assume that $\mathbb{R}A$ is of type I and also that it intersects all coordinate axes in real points. If we have a curve of degree d it should intersect the complexification with d points with multiplicity. They should be on the real line. For me, purely imaginary intersections are also okay, in fact. Okay, and the theorem is the following. Now if I take the logarithmic image $\operatorname{Log}(\mathbb{R}A) \subset \mathbb{R}^2$ where $\operatorname{Log} : (z, w) \mapsto (\log|z|, \log|w|)$, and since I'm talking about type I curves, I can choose one half and then the complex orientation give me an induced orientation on the ovals (switching the choice switches all orientations simultaneously). In particular, the logarithmic image also has some orientation. Any time you have a closed curve on the plane you have a region inside. Then I can take $\operatorname{Area}_{\operatorname{Log}}(\mathbb{R}A)$ which is the signed area inside $\operatorname{Log}(\mathbb{R}A)$. Our curve is not compact but it's easy to see that there is inside and outside and that the inside is finite volume.

Okay.

Theorem 3.1. *This area takes a discrete spectrum of values (if we're in type I and have purely real or purely imaginary intersection with the coordinate axes), in fact, it's in $\frac{1}{2}\mathbb{Z}$, call it $K(\mathbb{R}A)$, the quantum index, the second example of an index whose maximality puts a restriction on the curve.*

Now what can be the maximal index? Let's do examples. If we have $d = 1$, then the area is $\pm\pi^2/2$, and from this Possare determined Euler's formula, if you take one third you get $\pi^2/6$, and then that's $e^x + e^y = 1$, and if you write $y = \log(1 - e^x)$, we'll see this is $\zeta(2)$, and on the other hand you can compare with the complementary map, the argument, you can take the imaginary part, and there this area π^2 is the total area of the argument torus after folding it four times.

The index K is $\pm\frac{1}{2}$. Another examples is that if we don't make those special assumptions, then the area takes continuous values. For instance, if I take $d = 2$, then the conic [picture], the image under the logarithm map will be [picture] and once it starts intersecting the third line it hits the maximal possible value and it will be just, actually, if it is a circle then it's constant. If it's a circle, the definition of a circle in affine geometry. Actually let me make an example that is any conic. It will be a conic not satisfying the coordinate intersection condition. I can take a small conic. If I take a small one it's arbitrarily close to zero, so it's continuous. As the third example, let me take a circle, a circle, an affine circle is equivalent to its projectivization passing through 0, 1, and $\pm i$, the dominant terms are $x^2 + y^2$. So just because it's a circle, the intersection with the infinite axis is purely imaginary, which is okay. If I take the other intersections real it will satisfy the condition and therefore the area will be [unintelligible]. [pictures]. The area is π^2, so the index is ± 1 and as another example, suppose it intersects differently [picture]. Then $k = 0$ and the picture has two parts. If its a circle, the area on the two sides will be the same. With these conditions there is this jump by π^2 in the area.

In this case the index also implies some condition. So it's not hard to see that $-\frac{d^2}{2} \le K \le \frac{d^2}{2}$, and if K is maximal (torically maximal), it implies that the topology of $((\mathbb{R}^\times)^2, \mathbb{R}A)$ or the projectivization is unique. There's only one possible type in the maximal case. Furthermore it is the simple Harnack curve. This was a theorem (M.–Rullgard) of 2000. This is the corollary of maximal possible area.

It's always organized in the following way. If I draw [picture], there's a similar story of how it's organized for the odd case. [picture]

In the next hour I'll talk about indices for dimension three. Here let me mention some applications to enumerative geometry, the ideas due to Gottsche. The idea is to replace integer numbers in the classical enumeration porblem with quantum numbers (like how many curves of degree d and enus 0 pass through $3d-1+g$ points). The problem has a well-defined integer answer. The point of view advocated by Goettsche and Kontsevich–Soibelman—[unintelligible]

In this question I asked, I started from the complex case, which is very sturdy, almost always defined, sometimes we're in the superabundant case but in the regular case it's an integer number, invariant, on the choice of constraints. Over real numbers the story is much more delicate. We know it's also an integer number, and it was a breakthrough by Welschinger in 2003 but only for $g = 0$ (even now we don't know how to get rid of this). We take real points passing through a configuration of points, assigning it ± 1 which only depends on the self-intersections. For $d = 3$ and $g = 0$ we get 12 in the complex case and 8 in the tropical case. We get a q-number for all g, if we plug in $q = 1$ we get the projective number and for -1 the real number. This was one of the motivations.

Let me say how the quantum index for real curves gives an approach for real geometry where we don't need any tropical theory. We'll have to modify the problem. So genus equal to zero will still be crucial. Instead of taking $3d - 1$ generic points, we'll take $3d$ non-generic points. Maybe we'll take the points on the real axis. [pictures]

We can note that Menelaus theorem imposes a condition on $3d$ points so that there exists at least one curve of degree d passing through it.

Menelaus' theorem, I have a triangle and I mark points D on AB, E on BC, and F on the continuation of AC. When are D, E, and F colinear? Menelaus says it's true if and only if

$$\frac{|DB|}{|AD|}\frac{|CE|}{|EB|}\frac{-|FC|}{|FA|} = -1.$$

We'll have to take ratios for all of our points, any $3d-1$ points uniquely determine the last one. We'll only consider Menelaus configurations. Then we ask how many curves pass through this configuration. All of the story before makes sense, we just lose a few curves that go to degenerations of this, go to reducible curves consisting of three coordinate axes, three are lost.

Now we can do, set up a real problem, and solve each real cubic which passes through this has a quantum index, we can enumerate all the curves separately. We unfortunately don't get an invariant. There's a $\mathbb{Z}_2$-transformation since $\mathbb{R}$ is not algebraically closed. We can take the square Frobenius map $(x:y:z)\mapsto(x^2:y^2:z^2)$, a generically 4-to one map. Then we ask about curves $\mathbb{R}A$ whose image under the square contains, passes through our Menelaus configuration, which is generic. We ask not about the curve but it's square.

We take the corresponding generating function. Now we define $\mathbb{R}_d$ of the configuration

$$\mathbb{R}_d(C) = \sum_{\mathbb{R}A}\sigma(\mathbb{R}A)q^{K(\mathbb{R}A)}$$

where we take $\mathbb{R}A$ with orientation. This is the corresponding refinement of the real count. It turns out that this agrees with the Block–Goettsche refinement from tropical geometry.

Theorem 3.2. $\frac{\mathbb{R}_d}{(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{3d-2}} = BG.$

If you compute real curves with this index, it's enough to count real curves to compute the complex curves. This means that knowing only real curves, of course heer we go through the doubling, but do it with the refinement by quantum index. So the number of real curves determines the number of complex curves. On the logarithmic scale multiplying disappears.

Let me make a break.

When it's close to maximal there is still uniqueness. There's still, the projective topology is still unique. When we enumerate maximal index curves, the leading power is always one. There's a unique curve of maximal index, this is not quite new, this appeared in 2006, a paper of [unintelligible]and Oukounkov.

We'll also have one with a symmetry and one without in the three dimensional world.

So now we have $\mathbb{R}A\subset\mathbb{RP}^3$ of degree d, and we have two indices, one due to Viro in the 90s and one to Welschinger around 2004. The real enumeration in the plane, each planar curve had a particular sign, plus or minus, and so shortly after his introduction of invariant count of curves in the plane, he generalized it to three-space. Suppose $\mathbb{R}A$ is what he called balanced. That means its normal bundle (this is genus 0), this splits into line bundles, the total degree is $4d-2$, and it means it splits into two line bundles of degree $2d-1$ (example).

All subbundles of this normal bundle $\mathcal{O}(2d-1)\oplus\mathcal{O}(2d-1)$ of degree $2d-1$ form $\mathbb{RP}^1$, the space of all of them, take combinations of one and the other. Geometrically it means, our curve is some knot in projective space, but we have $\mathbb{RP}^3$ as the ambient space instead of S^3, and the subbundle, there are only a circle many of them, the ones of maximal degree, they form a framing, each subbundle gives a framing of the knot. The degree is odd, so the ribbon is the Mobius band. It doesn't matter which one I take. All of them sit in this pencil. This means the knot comes with a framing, and a framing for a knot in a rational homology sphere is a number, half-integer because this is not oriented. This half-integer is half-integer if d is even, then we have an honest circle, and make a Möbius band, this is Welschinger number is zero. We get half-integers or integers depending on whether it's framed orientably if d is odd.

I think it's an interesting index but I cannot say much about this index. The second index is the writhe which we will now consider.

This is originally, Viro called it encomplexified writhe. This doesn't exist for smooth knots. This is a finite type invariant of degree 1; for honest smooth knots these start in degree 2. The Reidemeister I move would kill the writhe. So in degree 1, if we cross, the difference does not depend on the rest of the knots. The second Reidemeister move we should get the same knot, so then it always doesn't change, it's always trivial, there are no interesting Vassiliev invariants of degree 1. Then we can explore and exploit the algebraic version, here we change our projection. This is a spatial real algebraic curve, now we have a planar one. The number of nodes cannot change. So the point cannot disappear. The planar curve will have a point that is not visible or defined but in the algebraic world we have a special point that is still preserved. Viro defined that we have the entire algebraic curve including solitary singualarities $x^2 + y^2 = 0$, and Viro gave these a sign, either +1 or −1, seeing how it separates, and there is a way so that the Reidemeister move preserves the sign. We'll never do this using algebraic deformation, Reidemeister I. So the writhe itself, the encomplexified writhe, is the total sum of all signs of the knot.

So it is an index of a curve, integer valued invariant, and again we can see that V_ρ is between $\frac{(d-1)(d-2)}{2}$ and its negative, again this is a symmetric index by a reflection. So plus and minus behave similarly.

Let me announce a recent theorem, joint work still in progress with Orevkov

Theorem 3.3. *If V_ρ is the maximal (or minimal) number, then $(\mathbb{RP}^3, \mathbb{R}A)$ is unique, and this is a [unintelligible]knot. The smooth isotopy class—$\mathbb{R}A$ can by isotoped (smoothly not algebraically) to a curve on a hyperboloid, we have a generator in one family and $d-1$ in another family, and then smooth all of them consistenly. This is the only knot which is rational. We call these* hyperboloidal *knots.*

If p and q are greater than 1 then different p and q give different knot types. We can make (p, q) hyperboloidal ones, this kind is $(d-1, 1)$, and it's not trivial for us. The universal covering will lift (p, q) to $(p+q, p-q)$. SO we already have several different topological types.

The last remark I'd like to make is that there, for d up to five, this invariant determines the topological type completely.

4. May 12: Yanki Lekili: Koszul duality patterns in Floer theory

Thank you for the invitation, it's my first time in Korea. I'm here for two weeks, so I'll give two talks, interrelated but on different papers, but I'm generally available to talk to. This is joint with Etgu.

This is about Lagrangian Floer theory and a manifestation of Koszul duality there. Let me start by reviewing some classical topology which was an entrance point.

Say X is a smooth manifold. Let me require that it is simply connected, and pick a point p in X, this will be the input, and associated to this I'll consider two augmented dg algebras. The first, which I'll call A, is the cochains on X, and this has a natural augmentation $C^*(X) \to \mathbf{k}$ from the inclusion of the point in X. The second algebra is $B = C_{-*}\Omega_p X \to \mathbf{k}$, which has an augmentation given by contracting everything except the constant loops at p. An augmentation is a dg algebra map to the ground field. Then we consider $\mathrm{Rhom}_A(\mathbf{k},\mathbf{k})$ and get something isomorphic to B, and if we look at $\mathrm{Rhom}_B(\mathbf{k},\mathbf{k})$, we get something isomorphic to A. These are theorems, classical results, I don't know the correct attribution very well, but the first one is maybe the Adams cobar construction and the second one is, I think maybe Milnor–Moore or Eilenberg–Moore or some classical topology. Both are remarkable theorems. For me the first one, I sholud say the second one holds true without simple connectedness. The first one requires simple connectedness. In general, you get some kind of completion of B. One thing I should maybe emphasize is that, I had a finite dimensional compact manifold in mind so A is finite dimensional, and B is infinite dimensional, usually, but this gives us a way to compute it.

Another consequence is that this kind of duality induces isomorphisms at the level of modules of these guys, so that $HH^*(A,A) \cong HH^*(B,B)$. This is a general statement, this, soon I'll change the statement, whenever you have A and B with these two isomorphisms, you have this Hochschild isomorphism. So in this case $HH^*(A,A)$ is the homology of the free loop space $H_*(\mathcal{L}X)$, the free loop space, I'll attribute this to Jones.

Here on $H_*(\mathcal{L}X)$ there is a natural BV structure given by loop rotation, I don't know whether this is the level at which they're isomorphic. At least Gerstenhaber algebras are known.

Now I would like to interpret all of this in the language of symplectic geometry. I'll call these algebras different names, just a reinterpretation.

To study the smooth topology of X, we pass to its cotangent bundle and get a symplectic manifold T^*X, and there's increasing evidence that T^*X captures the smooth topology of X. We have two canonical Lagrangians. The first is the zero section X. Let me remind you that the symplectic form is $dq \wedge dp$. The cotangent fiber T_p^*X at a point in X (this is a different p, our basepoint) is $q = 0$. We have A the Floer complex $CF(X,X)$ and B the wrapped Floer cohomology $CW^*(T_p^*X, T_p^*X)$. The first one here is Fukaya–Oh, and the second is Abouzaid. These are chain level statements.

So now what about the augmentations. Here is our cotangent bundle. We have these cotangent fibers. We have a specific choice (X is the zero section) and geometrically they intersect at one point. Floer cohomology is an intersection theory. You isotope a pair of Lagrangians to intersect at isolated points and these points generate Floer homology. This prescription is true if one is compact, so if

X is compact then $CF(X, T_p^*X) \cong \mathbf{k}_p$. When I have a noncompact Lagrangian, then pushing away from the first copy by a diffeomorphism is ambiguous, I have a noncompact thing. So one does a "wrapped" construction. Let me draw the picture for the cotangent bundle of S^1. [pictures, words]

As I said, if X is compact we don't have to do this, this is only when I do noncompact with noncompact. Now I'd like to emphasize the following picture. When I have Floer cohomology, there is a natural product $CF(X,X) \otimes CF(X, T_p^*X) \to CF(X, T_p^*X)$, since these are hom spaces of our category. In this situation, we have just this $\mathbf{k}$ as our intersection, so we have a natural way of making this one dimensional space a module. We actually have an A_∞ structure, so we have $(CF(X,X))^{\otimes m} \otimes CF(X, T_p^*X) \to CF(X, T_p^*X)$, and this is actually an A_∞ module, and we can do the same in the other case and we get these module structures for B as well, and this gives the A_∞ augmentation. In this way the vector space $\mathbf{k}$ becomes a module for both algebras.

The second isomorphism says the following:

$$\mathrm{Rhom}_{(CW^*(T_p^*X, T_p^*X))}((CW^*(X, T_p^*X)), (CW^*(X, T_p^*X)))$$

this should be $CF^*(X,X)$. This should be obvious from the Yoneda embedding along with the fact that the cotangent fiber generates the wrapped Fukaya category of the cotangent bundle.

I have the wrapped Fukaya category, which has objects given by both compact and noncompact Lagrangians. In particular, the compact part sits inside here fully faithfully. If I look at modules over this, this is modules over the endomorphisms of the wrapped thing. The compact Lagrangian X is an object in the wrapped Fukaya category, and by Yoneda it goes to $CF(X, T_p^*X)$. That is, I could first apply the Yoneda map and then compute its endomorphisms.

This doesn't use the fact that X is simply connected. The second statement is much more surprising in symplectic topology. It says the following. I have $Fuk(T^*X)$, which sits inside the wrapped Fukaya category which allows noncompact Lagrangians. If I use the Yoneda map, I could compute hom spaces for $Fuk(T^*X)$ which sits inside. If X is simply connected, then the theorem says that $\mod -WFuk(T^*X)$ is fully and faithfully embedded in $\mod -Fuk(T^*X)$. So then L goes to $CF^*(L,X)$, and surprisingly when [unintelligible], this is fully faithful.

So I can compute the maps in the wrapped category using only morphisms between the compact Lagrangians.

Given that we understood this interplay in this situation (and I should remark, this situation also implies that $HH^*(\mathrm{WFuk}(T^*X)) \cong HH^*(\mathrm{Fuk}(HH^*(T^*X)))$ when X is simply connected.)

I'd like to generalize to more symplectic manifolds where there is no classical analogue.

One immediate generalization of this kind of Koszul duality is in working over a semisimple algebra.

Let's work over $k = \bigoplus_i \mathbf{k}e_i$ with $e_i^2 = e_i$ and $e_ie_j = 0$ for $i \neq j$. I learned this from Beilinson–Ginzburg–Soegel. Say that A and B are augmented to something like this, k. Then you can study this situation, do you still have $\mathrm{Rhom}_A(k,k) \cong B$ and $\mathrm{Rhom}_B(k,k) \cong A$? Here A and B will be the Fukaya category of compact Lagrangians and the wrapped Fukaya category.

I have several objects now, instead of one pair of dual objects, I should have several objects dual to each other. Let's say I have a compact (simply connected) Lagrangian, and we have a cotangent fiber dual to it [picture]. I want several objects like this. I'll take several spheres like this, they'll be compact Lagrangians, and then dual Lagrangians. These will be objects in a symplectic manifold. Then $S = \bigoplus S_i$, the compact Lagrangians, and I can construct L the sum of the non-compact Lagrangians. Then A is the endomorphism algebra of S and B the augmentation algebra of L. I can identify the augmentations as maps to $k = CF(S_i, L_i)$. I drew my pictures so that this is one of these idempotent rings.

Now I can still, I have A and B and I have k as a module over A and B, and I want to check if I have $\mathrm{Rhom}_A(k,k) \cong B$ and $\mathrm{Rhom}_B(k,k) \cong A$. I should tell you the geometry behind it, the space and the Lagrangians I mean.

I'll start from a finite tree Γ [pictures] and to each vertex I'll take a copy of T^*S^2. To each edge I'll do a plumbing operation. I'll get a symplectic manifold in which this configuration sits naturally.

So what's plumbing? This is a topological construction that can be done in symplectic topology as well. When I have a cotangent bundle, I locally have p and q coordinates. I pick a point in the base and I concentrate near the point where I have these coordinates. What I'll do is glue this to another cotangent bundle near a point, where the coordinates p and q are interchanged. So $(p,q) \mapsto (-q,p)$. This operation, you do this, you identify these squares by this gluing and then smooth out the corners, interchanging the fiber and base directions. This can be done in a symplectic way. Maybe a helpful example is to imagine the plumbing of T^*S^1 with itself.

When you do the plumbing the zero section survives for each and they intersect transversally. Outside of the plumbing region we see just the cotangent bundle of each piece. This is not the case I'm studying, I'm plumbing T^*S^2 because I want simply connected things. The corresponding picture here is I get configurations of spheres, which intersect in this pattern, according to the tree, and away from the plumbing regions, we see cotangent bundles. So I can take points away from the plumbing regions and get cotangent fibers. These will be my non-compact Lagrangians. This is a symplectic four-manifold with the configuration I like, I built it that way, but for certain trees, this manifold apperas in algebraic geometry as well. For Γ of type A, D, or E, I have these as well, and the symplectic manifold X_Γ that I constructed here can also be seen as the symplectic manifold associated to a smoothing of the corresponding singularity $\{z^{n+1} + xy = 1\}$, this is the A_n singularity. If I smooth this, I get a Milnor fiber, which is precisely this plumbing. These were the ones I was studying originally, and then I came across this Koszul duality story.

We now have these symplectic mainfolds for any tree, and we have $\mathrm{Fuk}(X_\Gamma)$ and the wrapped Fukaya category $\mathrm{WFuk}(X_\Gamma)$. Then we can write $A = \mathrm{Hom}(S,S)$ where S is the sum of the spheres and B as $\mathrm{Hom}(L,L)$, where L is the sum of the cotangent fibers. Then we have $k = \mathrm{Hom}(L,S)$, and this is an A and B-module. Then I'll describe an explicit computation of A and of B and then describe that they are Koszul dual.

So now we'll compute these things and get explicit algebras that come out. One could say that the rest of this talk is almost purely algebraic. So let me describe A and B. The computation of A and B are in some sense very different. We have to introduce some notation. We have A and its cohomology H^*A, and B and H^*B. So let me compute cohomology of A, that's the easiest one to compute. So this is $\bigoplus HF^*(S_v, S_w)$, and by construction these are exact Lagrangians, non bounding disks, in an exact Lagrangian. Then $HF^*(S_v, S_w) = H^*(S^2)$. Furthermore, if I have adjacent ones, I just get a copy of $\mathbf{k}$, and I have to determine the grading, there's an extra structure on the Lagrangians, a grading structure, if I choose one to be in a certain grading, the opposite one will be in 2 minus that grading. I can choose these to be in degree 1. I don't intend to discuss grading or signs in detail. It's just very hard to write or explain on the board. Therefore, additively, I computed this algebra, the product is clear, but what about the product of different objects? I have a Calabi–Yau so I have a Poincaré duality. I should get the degree two generator of H and that determines the algebra structure. I'll let $a_{v,w}$ be, I double the tree and put edges in the opposite directions, and I say $a_{v,w}$ goes from v to w, and the algebra is generated over the semisimple ring by these guys, $a_{v,w}a_{w,u} = 0$ if $v \neq u$ and $a_{w,u}a_{u,w} = a_{w,v}v, w$. This is the algebra. What about the A_∞ structure? Potentially you have an A_∞ structure. I basically see the picture. I don't see any products in the picture, so maybe the higher products you think are zero, you have to perturb and what you can say is the following.

You can't directly compute but in certain situations, if Γ is ADE type, then over a field $\mathbf{k}$ of characteristic zero, $HH^2(H^*A, H^*A)$ is 0. What this means is that because of deformation theory of A_∞ algebras, the higher products on an A_∞ algebra, if the second cohomology is zero, you can always find a gauge transformation to trivialize the higher products. So by deformation theory we conclude that A is quasi-isomorphic to its cohomology, it's formal.

Here's an important thing, I said characteristic zero here. I made a mistake previously. In fact, this is nonformal in other characteristic. There was a correction to the paper because of this. For $\Gamma = A_n$, the characteristic of $\mathbf{k}$ is irrelevant, and $H^*A \cong A$, because the second Hochschild cohomology is zero (Seidel–Thomas). For D_n, the calculation is quite involved, and you compute it and if the characteristic is not 2, then $HH^2_{<0}$ is zero, and so the cohomology of A is quasi-isomorphic to A. In characteristic 2, there is a class in $HH^2_{-3}(H(A), H(A))$, which is just $\mathbb{Z}_2$, just $\mathbf{k}$. There is a two-torsion class and potentially that class could be realized by our structure. We have our geometry, we can check, and see that it does, then over characteristic 2, H^*A is not quasi-isomorphic to A. The smallest example is D_4, as soon as there is a trivalent vertex you get this failure. The Fukaya A_∞ algebra associated to this, is not formal in characteristic 2. If you care about geometry, you can't assume characteristic zero, you'll miss this. You have a holomorphic curve, we can't get rid of it geometrically.

For E_6, E_7, and E_8, in these cases, a similar story, I just have to tell you exceptional characteristics, for E_6 and E_7 you have characteristics 2 and 3, for E_8 you have 5 as well. In these characteristics you have higher products. I can't quite say the precise answer for characteristic 3 and 5 but characteristic 2 is precisely the same.

Away from these problematic characteristics, we have formality by deformation theory and computed A. So let me discuss how to compute B. In the non-ADE cases, the Hochschild homology is huge, I can't decide if A is formal or not.

So B is the wrapped Floer homology, $\bigoplus CW^*(L_v, L_w)$, and there's a result of Bourgeouis–Ekholm–Eliashberg, that this is the same as $LCH(\Lambda_\Gamma)$ for a link Γ, this is a nontrivial result, an anouncement of the result is on the arxiv, 90 pages, I'm just using this, making claims about LCH. So what it says is the following. Any Weinstein manifold is obtained by surgery on a Legendrian submanifold (so a 3-manifold by surgery on a Legendrian link in a connect sum of $S^1 \times S^2$—in our case we will have no one-handles so it will be in S^3). The picture is like this. I have the 4-ball and the boundary S^3 with a standard contact structure, and I hav some Legendrian links. These are embedded 1-manifolds tangent to the contact structure. With such a link, you can attach a handle along that link, a 2-handle. When we attach this handle, what happens? The handle has a core part attached to the knot, and the dual cocore. [pictures]

Therefore the right thing for us is to have links of unknots linked as a Hopf link links, along the tree. So each double point in the Lagrangian projection is a Reeb chord. I get $2e + v$ chords. If I do surgery on these guys I get the manifold I was initially thinking about. This is a surgery description. What Bourgeois–Ekholm–Eliashberg tell me is that now I can just look at LCH on these links.

Now we write down a presentation of our dg algebra that will be our B. So let's just first start from the simple case, and I'll tell you the simple case is too simple. It's generated by double points, so $g_{v,v}$, by $g_{v,w}$, and $g_{w,v}$. For each vertex I get one double point and for each edge I get two. So I get an augmented tensor algebra, $k \oplus V \oplus \oplus V^{\otimes_k 2} \oplus \cdots$ and the differential counts polygons in this picture. There are obvious ones, $dg_{v,v} = g_{v,w}g_{w,v}$, and there is another one, these triangles are the only contribution to the differential. That's how you read this off.

But originally, this presentation depends on the choice of the diagram, you can isotope these guys and get something more complicated. Originally we drew a diagram we thought was clever to not have any higher polygons. As it turns out, this is not true in the D_n case, and this simple situation with triangles, as soon as you have more valent vertices, you have more gons. In the D_n case, here you have D_4, this is the guy responsible for nonformality. [pictures]. You get $dg_2 = g_{23}g_{32} + g_{24}g_{42} + g_{21}g_{12} + g_{23}g_{32}g_{24}g_{42}$. We proved that you can find a quasi-isomorphism from this one to the one without the product in different characteristic. This is the wrapped Floer cohomology of the cocores. The quadratic one contributes to the algebra and this one to the μ_4 in the dual.

So ten more minutes. Maybe I'll say a word about Hochschild cohomology. So A and B are computed this way, this guy has a name, the one without the higher product is the Ginzburg algebra, so this is a deformation of $\mathcal{G}_\Gamma$, considered by Ginzburg, I don't know why, but if you do this computation, we get the Ginzburg dg algebra, and if Γ is ADE and the characteristic is not one of the problematic ones, then $B_\Gamma \cong \mathcal{G}_\Gamma$, by computing Hochschild cohomology of $\mathcal{G}_\Gamma$. So for Γ not ADE, I don't really know, I expect this in characteristic zero, and I can prove this if I complete things, if I take power series. This algebra is just words, but I don't know in general.

Finally, my main motivation was to compute symplectic cohomlogy of ADE plumbing, so $SH^*(X_\Gamma)$ is given by $HH^*(B_\Gamma)$, this is Bourgeois–Ekholm–Eliashberg, but B_Γ is this infinite dimensional Ginzburg algebra, but then we can compute this with the Hochschild cohomology of the A algebra, but by Koszul duality between A_Γ and B_Γ, this is the same as the Hochschild cohomology of A_Γ, but it's a formal A_∞ algebra, a finite dimensional associative algebra. Then you can compute Hochschild homology, there's a periodic resolution. Therefore we obtain the complete computation. You can use different resolutions. So this we computed explicitly for $\Gamma = A_n$ and D_n and the E cases on the way. What's the answer? Can I give it? It turns out that SH^0 is the Jacobian of a function W, which is associated to the corresponding ADE graph. This is, we computed this as an algebra, generators and relations, and we just observe it's this one. If you had a mirror matrix factorization category where the potential had isolated critical points, then [unintelligible]of the matrix factorization category is the Jacobian. The story doesn't end here because the critical points aren't isolated, SH^* is generated by SH^0 and two other variables, in degree 1 and -2, free variables, the odd variable the square is 0 and the even variable. This is the form of the answer. As far as I know this is the first complete computation of symplectic homology, we can say things about the BV algebra structure, away from cotangent bundles, at least to first order.

5. May 19: Yanki Lekili: Talking about my G-generation

The title that I gave is the name of a song by the Who, but it's not very well-known, so, I'm talking about generating Fukaya categories of Hamiltonian G-manifolds. This is joint with Jonny Evans. Thanks again for the invitation.

Let's set up what I'm talking about. X is a compact symplectic manifold and for technical reasons or convenience I'll assume it's monotone. That will make my Fukaya categories defined over the integers. Monotone means the symplectic form is positively proportional to c_1, it's a useful condition for removing some technical problems. It's crucial in working over the integers or a finite field.

So I have this compact symplectic manifold, the second part of the geometric setup is a Hamiltonian G-action. So G is a compact Lie group and G acts on X in a Hamiltonian fashion, in particular we have a moment map $\mu : X \to \mathfrak{g}^*$. What I will next require is that $\mu^{-1}(0)$, the preimage of zero under the moment map, this will be in general coisotropic, but I'll require it to be Lagrangian, and I'll require that G acts on $\mu^{-1}(0)$ freely and transitively.

In other words, $\mu^{-1}(0)$ will be a Lagrangian submanifold isomorphic to G. This is the geometric setup. The way one can think about it, X is our ambient symplectic manifold, G is a Lagrangian, then T^*G is a neighborhood of this, and the remaining part is a divisor. If G is a torus then this is a toric variety. The preimage of the interior of the toric polytope is the cotangent bundle and the preimage of the boundary is the divisors.

I want to extend this kind of picture to more compact Lie groups because toric varieties have been studied quite extensively. I'll usually think of a simple Lie group like $SU(n)$ but you can also, you can have $SU(n) \times S^1$. For me it will be a torus or a compact Lie group. You can study different ones. I gave lots of examples where G is a torus. Let's look at other examples, when G is not a torus. There is an action of $U(n)$ on the Grassmannian $Gr(n, 2n)$, and so $Gr(n, 2n)$ is of this form. There's an action of $PSU(n)$ on $\mathbb{CP}^{n^2-1}$ and this gives an interesting decomposition. Here's an example, $SU(2)$ acts on the quadric three-fold, and remember quadric n-folds are $T^*S^n \cup Q^{n-1}$, so here that's $T^*SU(2) \cup S^2 \times S^2$. So there are many examples where G acts on the symplectic manifold and you get this kind of definition. The theory is not empty at all.

What we want is to study Floer theory of this particular Lagrangian. Let me try to state the theorem. I have to prepare one more thing before I can state it.

If I have this monotone symplectic manifold X and say I'm working over a field, then look at the quantum cohomology of X, this is a vector space over $\mathbf{k}$ the same size as the regular cohomology but with deformed ring structure, and I can always decompose it into idempotent pieces,

$$QH^*(X) = \bigoplus_\alpha e_\alpha QH^*(X) e_\alpha$$

where e_α are a complete set of primitive idempotents that sum to the identity.

There is a "closed-open map" from $QH^*(X) \to HF(L, L)$, which makes the Floer homology into a module over this ring, and the idempotents decompose this module:

$$HF(L, L) \coloneqq \bigoplus_\alpha HF(L_\alpha, L_\alpha).$$

I can lift this to the chain level up to homotopy so I get a decomposition of the Fukaya category into independent summands

$$CF(L, L) \cong \bigoplus CF(L_\alpha, L_\alpha).$$

The different summands don't talk to each other in the sense that $HF(L_\alpha, L'_{\alpha'}) = 0$. When one studies this, one studies each summand, and now I can state the theorem.

Theorem 5.1. *Let p be a field characteristic such that $H^*(G)$ has no p-torsion (vacuously satisfied if $p = 0$). Then $G = L = \mu^{-1}(0)$ (in the sense of split generation for triangulated categories) the Fukaya category $F(X)_\alpha$ if and only if $HF(L_\alpha, L_\alpha) \neq 0$.*

This last condition means that the summand in the object is nonzero. If it's zero, there's no point of generation. An interesting question is which summands it projects as nonzero. Let's just suppose that there is only one summand. Then this says the object generates the category if and only if the cohomology is nonzero. In particular it applies to toric varieties and then some of this was known.

Lets look at one of the examples I was talking about. $QH^*(\mathbb{CP}^N)$ is $\mathbb{K}[H]/H^{N+1} - 1$. If $\operatorname{char} \mathbf{k} \nmid N + 1$, then this is a semisimple ring, which means it's a direct sum of copies of $\mathbf{k}$ corresponding to roots of unity. If the characteristic of $\mathbf{k}$ divides $N + 1$ then this is far from being semisimple. Then for example, $L = PSU(n)$ in $\mathbb{CP}^{n^2-1}$ generates if $n = p^k$ for some k.

In the toric Fano case, $T^n \subset X$, for any α there is a local system ξ_α so that T^n, ξ_α generates $F(X)_a$.

[another example]

One thing I'd like to discuss is, there is a general generation criterion due to Abouzaid, and a followup by Abouzaid, Fukaya, Oh, Ohta, and Ono, which says, if X is compact, take $HH_*(\langle L \rangle) \to QH^*(X)$ via the open closed map, and if this map, the *open-closed* map. If the open closed map hits 1 then $\langle L \rangle$ generates.

[missed some]

Of course, as I said, geometrically the meaning is about intersection of Lagrangians. But I want to think about representation if I care about mirror symmetry, generating objects on either side could be matched up for this.

Now I want to outline the three steps in the proof.

(1) The first step is to understand the wrapped Fukaya category of T^*G. There will be two important Lagrangians, the zero section and the cotangent fiber at the identity. I want to understand the Floer cohomology of G with itself and the zero section at the cohain level.

(2) I want a correspondence C in $T^*G \times X \times X$. This is easy to write down a formula for, it's

$$(g, \mu(gx), x, gx).$$

This was intoduced by Weinstein, and there is a study of it by [unintelligible]and [unintelligible]but I learned about it from Telleman.

For us, F_C will be $W(T^*G)$ to $\mathrm{Fuk}(X \times X)$. So $F_C(G) = L \times L$ and $F_C(T^*_{\mathrm{id}}G)$ is the diagonal of X in $X \times X$.

(3) Something to do with Koszul duality, it's not easy to explain what it is. What happens, in words, in the domain wrapped category, the cotangent fiber generates the zero section. We push down to the diagonal, so the diagonal generates $L \times L$

Maybe I should in more down to earth way say, this is a consequence of the octahedral axiom that we'll see.

That's the plan, let's begin with 1. This is somehow classical and can somehow be understood in an independent way. We have $CF(G, G) \cong C^*(G)$ and $CW(T^*_{\mathrm{id}}G, T^*_{\mathrm{id}}G) \cong C_{-*}(\Omega_{\mathrm{id}}G)$.

I want to study the right hand sides. When G is T^n, then $C^*(G)$ is $H^*(T^n) \cong \wedge\langle x_1, \dots, x_n\rangle$. On the other hand $C_{-*}(\Omega G)$ is $\mathbf{k}[y_1^{\pm 1}, \dots, y_n^{\pm 1}]$. If G is a non-Abelian simple compact group, then G is rationally equivalent to a product of odd spheres. For example $SU(2) \cong S^3$ and $SU(3) \cong_{\mathbb{Q}} \times S^5$. In fact there's a theorem of Serre that this is true over p for $p \geq \frac{\dim G}{\operatorname{rank} G} - 1$.

Here I'm talking about homotopy groups. I care about the A_∞ structure of the cochains.

Let p be a prime so that H^*G has no p-torsion, then $C^*G \cong_{A_\infty} H^*G = \bigwedge\langle x_{2e_i+1}\rangle$.

This is kind of a strange thing, when the characteristic is zero, there's nothing to do, when the characteristic is large there is nothing to do, so we only had to do small characteristic. There is a paper of Munkolm from which I put together an argument. Similarly, $C_{-*}(\Omega G) \cong \mathbf{k}[y_1, \dots, y_n]$.

Furthermore, there's something called the Koszul resolution, Abouzaid has the result that the cotangent fiber of G generates G. In this case we understand what these are and we can write down what these are. This is the Koszul resolution. Geometrically it looks like a cube. Let me give an example for $G = SU(2)$. This is, $G = \{T^*_{\mathrm{id}}G[2] \to T^*_{\mathrm{id}}G\}$. For $SU(2)$ we'll have a square

$$\begin{tikzcd}
T^*_{\mathrm{id}}G \arrow[r] & T^*_{\mathrm{id}}G \\
T^*_{\mathrm{id}}G \arrow[r] \arrow[u] & T^*_{\mathrm{id}}G \arrow[u]
\end{tikzcd}$$

It's easy to see that G goes to $L \times L$ and T^*_{id} goes to Δ under $F_C : W(T^*G) \to Fuk(X \times X)$.

Let's take $SU(3)$ as before. Then G goes to $L \times L$, this is a triangulated functor. It goes to a square

$$\begin{tikzcd}
\Delta \arrow[r] & \Delta \\
\Delta \arrow[r] \arrow[u] \arrow[ur] & \Delta \arrow[u]
\end{tikzcd}$$

where the maps are $F^1(y_1)$ and $F^1(y_2)$. The diagonal map is $F^2(y_1, y_2) + F^2(y_2, y_1)$.

It's important to see that the directedness is preserved. For our arguments, it suffices to understand what happens on the edge maps.

So we deduce that the diagonal generates $L \times L$. I said I want $L \times L$ to generate Δ. The claim here will be the following. The edge maps are quantum cohomology maps, remember $QH^*(X) \cong HF(\Delta, \Delta)$. This is a commutative algebra on a finite dimensional vector space. Any element x in a summand $QH^*(X)_\alpha$ is either nilpotent or invertible.

Let me restrict to $\Delta \to \Delta$ representing $L \times L$. If the morphism is invertible then this is the zero object. If x is nilpotent then I claim $L \times L$ split generates Δ. If $x = 0$ then $L \times L = \Delta \oplus \Delta$. Then $L \times L$ split generates Δ. Suppose instead $x^2 = 0$. What do we do then? Then we consider the following diagram.

$$\begin{tikzcd} \Delta \arrow[rr] & & \Delta \arrow[dl] \arrow[rr] & & \Delta \arrow[dl] \\ & L \times L \arrow[ul] & & L \times L \arrow[ul] & \end{tikzcd}$$

If I take the cone I get $\Delta \oplus \Delta$. The octahedral axiom says that taking this big cone is the same as taking the cone of the morphism $L \times L \to L \times L$

Now we have something we got in two steps, built out of $L \times L$. If $x^3 = 0$ I'll need to do this three times or whatever.

If any direction is not nilpotent, the complex becomes nilpotent. If all directions are nilpotent you can again play this game.

I'll finish with an application. We have an application about, a non-formality result about the chain complex of quantum cohomology. If X is monotone symplectic, such that $QH^*(X)$ is not semisimple, there are examples of this kind by Ostrover-[unintelligible], there's an example of a Fano 4-fold. Over characteristic zero (for concreteness; it's easy over other characteristics as well), the chain complex qH^* (the chain complex) is not formal. There's an A_∞ structure then, on $HF(\Delta, \Delta)$. This was mentioned in a paper of Ruan and Tian, people early expected that this might be formal for Kähler manifolds, but it's not true. It's not true generically because usually you don't have semisimple quantum cohomology.

So how do we prove this? It's an application of our theorem. Say X is one of these guys. If A is a commutative algebra and is not semisimple, then $HH^*(A)$ is infinite dimensional. For semisimple algebras this will be finite dimensional. I assume that quantum cohomology is not zero. If I can show that $QC^*(X)$ is finite dimensional, I'll be done. If they were isomorphic, their Hochschild homology would be isomorphic. So we'd like to show that $HH^*QC^*(X)$ is finite dimensional.

Suppose I arrange things so that L generates. I did this by showing that $\langle L \times L \rangle = \langle \Delta \rangle$. If I look at $\mathrm{Fuk}(X \times X)$ to nonunital endofunctors of $F(X)$. This is full and faithful on product Lagrangians. This is due to Abouzaid and Smith. Then this is fully faithful on Δ, which goes to the identity functor. So $CF^*(\Delta, \Delta)$ goes to $\mathrm{Hom}_{\mathrm{Fun}}(\mathrm{id}, \mathrm{id}) \cong CC^*(X)$. Then at the level of cohomology it's an isomorphism, $HF(\Delta, \Delta) \cong HH^*(X)$. Then $QH^*(X) \cong HH^*(F(X))$. This is because we could generate the diagonal by product Lagranians. We want to show that $HH^*(QC^*(X))$ is finite dimensional. This is the same as $HH^*(CF(\Delta, \Delta))$, this is the same as $HH^*(CF(L \times L, L \times L))$. Now apply our generation criterion to $L \times L$, which is also a monotone toric fiber. Then, up to summands, this is $QH^*(X \times X)$, the result above applied to $X \times X$. Via this we have a computation of this and then we are done because quantum cohomology is finite dimensional.

6. July 14: Craig Westerland: Homology of Hurwitz spaces and the Cohen–Lenstra heuristics for function fields

Thank you for the invitation and thank you for coming to listen. Indeed the title that Gabriel gave you is the long version. The short version is "arithmetic statistics." I hope to talk about Cohen–Lenstra as well as another topic. This is joint with Ellenberg–Venkatesh–Tran. The first is with Ellenberg and Venkatesh and the second with Ellenberg–Tran.

Let me try to tell you the general theme, which is to study arithmetic questions in a probabilistic or distributional sense. I'll give you very concrete examples of what I mean as we go along.

These problems are arithmetic, originally defined over number fields. I have no idea how to address those problems. What I'll be talking about is a reformulation of these problems over function fields and when you do this, one of the things that comes out is a much more geometric picture. The geometric picture allows us to address these problems using algebro-topological tools.

We will sort of solve these problems using algebraic topology. In summary, the problems began in what would be called classical number theory. They get translated into algebraic geometry, but the solution is approached from classical algebro-topological tools.

If I time this correctly, in the first half I'll tell you about the number–theoretic conjectures and translate into algebraic geometry and in the second half talk about using algebraic topology to solve them.

Let me start with the setup, some number theory probably well-known to all of you. I have a number field K over $\mathbb{Q}$ of degree n, a degree n extension of the rationals. I want to write down a few things that are the basics of number theory over this field. I'll let $\mathcal{O}_K$ be the ring of integers, these are the elements $\{x \in K : f(x) = 0\}$ where f is a monic polynomial over $\mathbb{Z}$. Because this is a degree n extension, this is abstractly isomorphic to $\mathbb{Z}^n$, and what I can do is pick a basis, $\mathbb{Z}\{e_1, \dots, e_n\}$, and I know that there are n embeddings $\sigma_i : K \to \mathbb{C}$, and I can put these two things together and let the *discriminant* Δ_K be the square of the determinant of the matrix given by all of the basis elements under all the embeddings

$$\Delta_K = \det(\sigma_i(e_j))^2.$$

The problems we will study is about arithmetic in a distributional sense, and the discriminant is the filter that allows us to do the distribution.

Let me remind you that if $K = \mathbb{Q}(\sqrt{d})$ is degree 2 (for d squarefree) then the discriminant is either d (if d is 1 (mod 4)) or $4d$ (otherwise). The basis is $1, \sqrt{d}$, and the determinant is $2\sqrt{d}$.

I can state a conjecture which I have been told is possibly due to Linnik?

Conjecture 6.1. Let $x \geq 0$. Let $Z_n(0, x)$ be the number of number fields $K/\mathbb{Q}$ of degree n such that $0 \leq \Delta_K \leq x$. This is a finite set. For instance, if n is two, I'm looking at quadratic number fields, then the discriminant is either d or $4d$. So there are at most basically $\frac{x}{4}$ of these. Maybe these should be in $\mathbb{C}$ or something. So the conjecture is about the growth of this as a function of x.

Linnik's conjecture is that $Z_n(0, x)$ grows asymptotically linearly in x.

This is known to be true in several cases. When $n = 2$, you could swap $0, x$ with $-x, 0$, but let's stick for the moment with the positive discriminants, then d should be positive, so real quadratic fields. If $n = 2$ then this is true, then counting $Z_2(x)$, this is approximately the number of squarefree integers less than x or $4x$, and it's fairly easy to show that this is asymptotically $\frac{x}{\zeta(2)}$. This requires that we know how to count squarefree integers.

For $n = 3$ this is a theorem of Davenport–Heilbronn. So for instance $Z_3(0, x) \sim \frac{x}{12\zeta(3)}$ and in the negative direction $Z_3(-x, 0) \sim \frac{x}{4\zeta(3)}$. For $n = 4, 5$, this is work of Bhargava and again you have interesting constants. Bhargava and collaborators improved the Davenport–Heilbronn work to give the second order term with is $x^{\frac{5}{6}}$ times a constant that is zeta-function-y.

Any questions? What I want to do is give two more conjectures and evidence for them and then finally start talking about proving them.

Let $K/\mathbb{Q}$ be degree n. Notice that the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ acts on $\{\sigma_i\}$ transitively. What do we know? This is a transitive subgroup of the symmetric group on n letters. Let's reverse engineer the question. Let $G \le S_n$, and be a transitive subgroup. Then we can ask, let's define $Z_G(x)$ (for simplicity I'll only do positive x) as the number of $K/\mathbb{Q}$ of degree n with $|\Delta_K| \le x$ and with Galois group isomorphic to G. I want to count those.

Conjecture 6.2. (Malle) $Z_G(X) \sim cx^a \log(x)^{b-1}$ where a, b, and c are constants which depend on G. In the paper he tells you what a and b are and asserts the existence of c.

The constant a is defined entirely in terms of the group theory of G and b uses the absolute Galois group of the rationals, its action on G.

I don't know about you, but this looks somewhat innocuous when you first look at it, you say "okay, that's a formula," but that implies a very very strong positive solution to the inverse Galois problem. Pick your favorite group. It's a transitive subgroup of some symmetric group. Not only does it exist as a Galois group, but it does so infinitely many times, in a way growing as the discriminant.

One thing Malle proves in his paper is that if this conjecture is true, then Linnik's conjecture is true (this is not surprising, it comes from adding up Malle's numbers over all transitive subgroups of S_n). This is again stupidly easy if $n = 2$, and more deeply than that, it's a theorem of Wright that this is true if G is Abelian. This is misleading to state this in this way, it was proved before Malle and was a foundation for Malle's conjecture. If you fiddle with Davenport–Heilbronn, then this is true for $G = S_3$ because there aren't many transitive subgroups of S_3.

For a symmetric group a and b are both 1.

One more round of definitions and theorems and we can move on to new things. Let Cl_K be the class grop of K, this is the Picard group of $\mathrm{Spec}\,\mathcal{O}_K$ and more directly it's the fractional ideals of $\mathcal{O}_K$ modulo the principal fractional ideals. It's the Galois group of the maximal Abelian unramified extension of K. This is the version that will be most useful for us in what is forthcoming. What can I say about it? It will always be a finite Abelian group and it is trivial if and only if $\mathcal{O}_K$ is a unique factorization domain. So it knows something about the structure of the ring. In Malle's conjecture, we asked, let's count the number of fields with a specified Galois group. Now you could ask about counting the fields with a specified class group. That's what the Cohen–Lenstra heuristics do, but let me state that in a probabilistic context.

Conjecture 6.3. (Cohen–Lenstra) For $K = \mathbb{Q}(\sqrt{-d})$, the conjecture predicts that if you tell me your favorite finite Abelian ℓ-group (for ℓ some prime), the conjecture tells you the probability that that is the ℓ-part of the class group of K. I want to say that the probability that A is the ℓ-part of the class group of K, I count the number of such K whose class group has ℓ-part isomorphic to A. That's an infinite set. I'll make it finite by asking that $|\Delta_K| \leq x$ as before. Then I'll compare that number to the number of fields K with $|\Delta_K| \leq x$. This is the probability that your field has class group with ℓ-part A, conditional on $|\Delta_K|$ being small. Let $x \to \infty$ and they tell you this is

$$(\prod_{j=1}^{\infty}(1-\ell^{-j}))\frac{1}{\#\operatorname{Aut}(A)}$$

If ℓ is three, this normalizing factor is something between $\frac{2}{3}$ and $\frac{3}{4}$. But what this tells you is that the probability is inverse to the size of its automorphism group. This is the "natural way of counting things in a groupoid." So this has a category–theoretic reason that it's a pleasing answer, sort of an orbifold Euler characteristic answer.

Again, this is not known, but there is lots of evidence, lots of computational evidence. Cohen, who is a computational number theorist, did a lot of these computations, he counted these numbers up to discriminants very large. It started looking like this. He showed this to Lenstra and they started finding lots of reasons this might be true. I went to a conference on these heuristics a couple of years ago. This and related questions were posed. Someone asked such a question in the morning. At lunch, Cohen got out his computer, wrote a program, and computed up to 10^7 and was like "yeah this is probably true." People even hassled him for only going up to 10^7. It blew me away that you could do something like this.

There are variants in other Galois situations. You needn't do imaginary quadratic. You could do real quadratic, you could start doing cubic and quartic fields. Cohen and Lenstra have conjectures but it's clear in their papers that they believe their conjectures less and less as you go up.

I should say to set this up, there is a reformulation, this is equivalent to the statement

$$\lim_{x\to\infty}\frac{\#\{(K,f)\|\Delta_K|<x, f:\mathrm{Cl}_K \twoheadrightarrow X\}/\operatorname{iso}}{\#\{K\|\Delta_K|<x\}}$$

that is, the expected number of surjections from $\mathrm{Cl}_K \twoheadrightarrow A$, and it's equivalent to that being 1.

The previous one states what the probability measure is, and this is testing it on particular values.

I'd like to translate these into algebraic geometry. Linnik's conjecture will be implicit, let me tell you about Malle and Cohen–Lenstra. I want to replace $\mathbb{Q}$ with $\mathbb{F}_q(t)$. So Let $q = p^m$ for p a prime, and I want to look at $K/\mathbb{F}_q(t)$ of degree d. The ring of integers, $\operatorname{Spec}\mathcal{O}_K$ is now the ring of functions on a curve over $\mathbb{F}_q$. call $\Sigma = \operatorname{Spec}(\mathcal{O}_K)$. So $\mathbb{F}_q(t)$ is the ring of functions on the affine line, $\mathbb{A}(\mathbb{F}_q)$, and this becomes a map $\Sigma \xrightarrow{\pi} \mathbb{A}(\mathbb{F}_q)$ which is a ramified cover of degree d.

So how did we begin the talk? We looked at extensions K of $\mathbb{Q}$ and counted things. You can do the same thing, but you can interpret the things we're counting in a geometric setting.

So let's ask the same questions, but what can we do about them? We can translate what the Galois group is, $\operatorname{Gal}(K/\mathbb{F}_Q(t))$,this is the same as $\operatorname{Aut}(\Sigma/\mathbb{A}^1)$, the group of deck transformations. Similarly, the class group $\mathrm{Cl}_K = \mathrm{Gal}(K^{\mathrm{Ab,ur}}/K)$, this is also a group of deck transformations, $\mathrm{Aut}(\Sigma^{\mathrm{Ab,ur}}/\Sigma)$, an *actual* cover, the maximal one so this is Abelian and unramified.

Finally, Δ_k is q^n where there are n branch points of π. Over $\mathbb{Q}$ where does this come from? A prime p divides Δ_k if and only if the ideal (p) ramifies in $\mathcal{O}_K$. In the geometric side, the same thing is true except that ramification corresponds to branching. So we're counting ramifications in the cover.

So all of these conjectures, Malle and Cohen–Lenstra, what are they? They are asymptotic enumeration problems. In the number field setting, they are enumerations of number fields, which are not points in something. But this geometric thing, we can ask the same exact question, but these are now points in a scheme. These are asymptotic enumeration problems in a moduli scheme, and I may as well say, in a Hurwitz moduli scheme. We'll come up with a scheme whose points are in bijection with this data and count those points. The number of those points is what the Malle and Cohen–Lenstra conjectures will be asking about.

So let's just define the scheme.

Definition 6.1. Let G be a finite group and $c \subset G$ be a union of conjugacy classes, and n an integer greater than or equal to 0. Then the *Hurwitz moduli scheme* (I should probably say moduli stack) has L-points $\mathrm{Hur}^c_{G,n}(L)$ the set of branched Galois G-covers $\Sigma \to \mathbb{A}^1(L)$ with n branch points and monodromy around the branch points in c.

This is a scheme whose points are exactly these ramified covers. When I go around the branch points I shouldn't use an arbitrary element of G but this restricted subset c. Let me define, also, $C\,\mathrm{Hur}^c_{G,n}(L)$ to be the same data but with Σ connected. In principle the first thing I defined could be a stupid cover which breaks up.

Let me tell you how to rewrite Malle and Cohen–Lenstra in this form, then take a break and switch to algebraic topology.

So Malle is easy to write down in this setup. What were we doing? We were counting, I claim that $C\,\mathrm{Hur}^c_{G,n}$ is just what Z_G was counting.

Malle's conjecture over $\mathbb{F}_q(t)$ is

$$\#\Big(\bigsqcup_{m=0}^{n} C\,\mathrm{Hur}^G_{G,m}(\mathbb{F}_q)\Big) \sim c q^{an} n^{b-1}.$$

Let's take a moment to unpack this. Without dijoint union, this is connected branched covers with m branch points. That's part of the Malle thing, because the discriminant is $q^{\#\text{branch points}}$. So if $x = q^n$ I'll count up to the number of branch points. That's the left hand side, the $Z_G(x)$. I've got an exponential in n times a polynomial in n, but the log happens because x is an exponential in n.

Maybe I've inflicted enough of this on you, let's take a break and then I'll tell you about Cohen–Lenstra and how do prove some of these conjectures.

Okay so we've rewritten Malle's conjecture about asymptotic counting in these function spaces. Remember the reformulation of Cohen–Lenstra was that the expected number of surjections was 1. Waht happens if I have a surjection from

$\mathrm{Cl}_K \twoheadrightarrow A$, let's draw the diagram of covering spaces.

$$\begin{tikzcd}
& \Sigma^{\mathrm{Ab,ur}} \arrow[ddl, "\mathrm{Cl}_K"] \arrow[dr] & \\
& & S \arrow[dll, "A"] \arrow[ddl, "\mathbb{Z}/2 \ltimes A"] \\
\Sigma = \mathrm{Spec}(\mathcal{O}_K) \arrow[dr, "\mathbb{Z}/2"] & & \\
& \mathbb{A}^1 &
\end{tikzcd}$$

and imaginary here means ramified at infinity. The semidirect product, $\mathbb{Z}/2$ acts on A by inversion. This is a highly ramified thing, in fact ramified at n points. The two covers over Σ are unramified. So I can reformulate this about the cover $S \to \mathbb{A}^1$ as things that are ramified in the $\mathbb{Z}/2$ direction and unramified in the A direction. So the Cohen–Lenstra heuristic, I want to write down that expectation value, well, let me state the theorem.

Theorem 6.1. *(Ellenberg–Vankatesh–W.) If $\ell > 2$ and q and $q-1$ are coprime to ℓ, there exists a $Q(A)$ such that, if q is odd and greater than Q, there exists $B(A)$ such that the limit*

$$\lim_{n\to\infty} \left| \frac{\#C\,\mathrm{Hur}^c_{G,n}(\mathbb{F}_q)}{\#C\,\mathrm{Hur}^c_{\mathbb{Z}/2,n}(\mathbb{F}_q)} - 1 \right| < \frac{B(A)}{\sqrt{q}}.$$

The denominator counts [unintelligible]. The numerator counts these extensions ramified in the $\mathbb{Z}/2$ direction $S \to \mathbb{A}^1$. If we were better algebraic topologists, the left hand side would be zero, because we'd be saying that the expected number is 1, but we're not good enough, this only works for a constant which gets smaller as q grows, and this messes up for some q.

These conjectures, in the end, boil down to enumeration of the points of these schemes. As I imagine you may know better than me, enumerating points on schemes is hard. Maybe I'll say our version of Malle as well.

Theorem 6.2. *(Elleberg–Tran–W.) There exists Q such that*

$$\lim_{n\to\infty} \frac{\#C\,\mathrm{Hur}^c_{S_r,n}(\mathbb{F}_q)}{q^n n^s} = 0$$

where c is the conjugacy class of transpositions in S_r and s is some constant, probably bigger than $b-1$ for Malle's conjecture. We're working on getting it down.

Malle says the number of these things grows, for S_r, all possible monodromy, and we're doing this to include the restricted monodromy, and it predicts the number of points grows asymptotically exponential in n times polynomial in n, and if I take a polynomial of a degree bigger than what Malle conjectures, this goes to 0. So we have an upper bound that's a little weaker than Malle's conjecture. There's the extension to larger classes of conjugacy classes and larger groups, but this is a start.

All right. So we need to find a way of enumerating $\mathbb{F}_q$ points on these schemes. The standard trick as we learned from Grothendieck is the Grothendieck–Lefschetz fixed point theorem. I won't write it down in full generality, but morally I need to count the number of points on $X(\mathbb{F}_q)$, I should compute the étale cohomology of X and the action of Frobenius on the étale cohomology. I will do nothing so advanced. I'll compute the singular cohomology of the complex points, and I'll do it with rational coefficients. Under assumptions on X this has the same rank as étale cohomology. This tells me nothing about Frobenius. But Deligne gives me bounds of the eigenvalues. This gives me bounds on the trace of Frobenius. This is a weak bound for what's going on but it suffices for these approximations.

I want to do this now, I've seen my collaborator give a talk about this, he presented a lot of the number theory, and the way he did what I'm about to do, he said, "we're going to transition to topology, and this is going to look like a David Lynch film. All the actors will stay the same, but the characters they play will change and everyone will continue as if nothing happened." So let's study this Hurwitz space $\mathrm{Hur}^c_{G,n}(\mathbb{C})$. What I need to do to tell you about this, I need to tell you about the configuration space of points in $\mathbb{C}$. Let $P\operatorname{Conf}_n(\mathbb{C})$ be the n-tuples of distinct complex numbers. This has an action of the symmetric group on the indices, let $\operatorname{Conf}_n(\mathbb{C}) = P\operatorname{Conf}_n(\mathbb{C})/S_n$. It's a very well-known gadget. We know that its fundamental group is the braid group B_n, the nth Artin braid group because, if I have a configuration of n points and stretch my path through time it draws a braid in three-space. This has a presentation

$$\langle \sigma_1,\ldots,\sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i, |i-j|\neq 1; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle$$

There is a covering map

$$\mathrm{Hur}^c_{G,n} \to \operatorname{Conf}_n \mathbb{C}$$

which takes $\pi : \Sigma \to \mathcal{C}$ to its branch locus, which is a set $\underline{z}$ in $\mathbb{C}$, a set of cardinality n. This is a forgetful map, it forgets the data of the cover and only remembers where the cover is branched. You can reconstruct the Hurwitz space from the branch locus if you know how the sheets come together around the branch locus. Why is that? We notice that $\pi : \Sigma \to \mathbb{C}$ is equivalent to $\pi' : \Sigma' \to \mathbb{C} - \underline{z}$, where $\Sigma' = \Sigma \backslash \pi^{-1}(\underline{z})$, that is you can fill this in uniquely knowing it's a Galois cover. This is an actual cover, $f_{\pi'} : \pi_1(\mathcal{C}\backslash\underline{z}) \to G$, but this fundamental group is a free group on n letters. So this is in $G^{\times n}$. But since the monodromy is in c, this is really in $c^{\times n}$. So this is the fiber of the covering space. How does the braid group act on $c^{\times n}$? So $\sigma_i(g_i,\ldots,g_n)$ is $g_1,\ldots,g_{i-1},g_{i+1},g_i^{g_{i+1}},g_{i+2},\ldots,g_n$.

So what does that tell me? If I want to compute the homology $H_*(\mathrm{Hur}^c_{G,n},\mathbb{Q})$, that's the same as $H_*(B_n,\mathbb{Q}c^{\times n})$, with this action of the braid group. I'm implicitly using the fact that this thing is $K(\pi,1)$ for the braid group. In the paper we used very classical techniques, the arc complex, that you use to study the homology of braid groups. One thing we've learned since then, there's a surprising connection between, there's a *new* approach to compute this in terms of quantum groups. I don't begin to, can't begin to claim that I'm an expert in these things, but I'd like to give you a hint of where this comes from.

The first thing to notice is that this is a special case of something called a braided vector space. If $\mathbf{k}$ is a field (really $\mathbb{Q}$), then a braided vector space over $\mathbf{k}$ is a finite dimensional vector space with a map $\sigma : V\otimes V \to V\otimes V$ which is an isomorphism and satisfies the braid equation on $V^{\otimes 3}$:

$$(\sigma\otimes 1)(1\otimes\sigma)(\sigma\otimes 1) = (1\otimes\sigma)(\sigma\otimes 1)(1\otimes\sigma).$$

This first data gives me an action of the integers, the second braid group. This relation then means I have an action of the third braid group. In general, you're going to get an action of B_n on $V^{\otimes n}$. This is an example, let $V = \mathbb{Q}c$, and I'll braid by $\sigma(g \otimes h) = h \otimes g^h$. This satisfies the braid equation and $V^{\otimes n}$ with this braid action is the Hurwitz action.

Okay, I've made the problem harder, not easier, but when you have this braided vector space, you can produce something called the quantum shuffle algebra.

Definition 6.2. The *quantum shuffle algebra* $A(V)$ is a braided Hopf algebra (unfortunately not a Hopf algebra), which I can tell you in a concrete sense what it is, it's the tensor coalgebra T^cV with the deconcatenation coproduct and the multiplication is given by

$$[v_1|\cdots|v_n]Ш[w_1|\cdots|w_m] = \sum_{\tau\in(n,m)-\text{shuffles}} \tilde{\tau}[v_1|\cdots|v_n|w_1|\cdots|w_m]$$

where $\tilde{\tau}$ is a particular lift to the $(n+m)$th braid group.

Let me give an example, a shuffle is a permutation that keeps the first n and last m in fixed order but intermingles the two. If I look at a $(2,3)$-shuffle, an example is like [picture]. If I have such a τ, how do I make a braid? I make a choice once for all time [picture], and that's a braid. So that's the multiplication.

So when you look at the formula, there's no reason to believe this is associative, but it is. The diagonal is not an algebra map in the usual sense, but it is if you incorporate the braiding into the target.

Maybe what I'll finish with, and this is a misleading place to finish with, it's what justifies the calculations.

Let $V_\epsilon = V$ with $\sigma_\epsilon = -\sigma$.

Theorem 6.3. *(Ellenberg–Tran–W.) There is an isomorphism* $H_*(B_n, V^{\otimes n}) \cong \operatorname{Ext}^{n-j,n}_{A(V_\epsilon)}(\mathbf{k},\mathbf{k})$ *where the first grading is the homological degree and the second the internal degree.*

That's misleading as a place to stop because I've given you a terrifying algebra, and said "compute its cohomology to get the answer you want." The quantum shuffle algebra is not extremely well-known, but it has the quantum symmetric algebra sitting inside of it, which is a very well-known gadget. This is also called the Nichols algebra. So you can leverage what you know about that. This is a central object in the Hopf algebra community for the classification of Hopf algebras. For transpositions in S_n, this is isomorphic to the [unintelligible]–Kirillov algebra, a non-commutative form of [unintelligible]for the flag variety. You can push what is known just far enough to get what you need. Thank you very much.