INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS SEMINAR

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1. January 8: John Terilla: Homotopy probability theory, examples and applications

I don't know how familiar people are with homotopy probability theory. Maybe I'll give a little bit of history. I think it was around November 2011 when Jae-Suk Park gave a lecture at the City University of New York. The way I think of it, in an effort to understand some aspects of quantum field theory, you revisit some basic assumptions and try to understand which things are essential and which things are the accidents of development. In the case of quantum field theory, I think of this as a big lumbering field with many accidents of development. This has analysis, homotopy algebra, and quantum aspects.

In what I thought was a nice bit of housecleaning Jae-Suk was able to decouple the homotopical algebra from the other aspects. These can be separated from the quantum and analytic features and applied to other fields like probability theory. So he envisioned a homotopy theory of probability. Quite independently there have been calls by Terence Tao and Gromov among others to make probability theory more mathematical and categorical and I think homotopy probability theory does it.

In probability theory you have a vector space V of random variables, an expectation map $e : V \to \mathbb{R}$, and then you have a product $\cdot$, and in typical situations V is the measurable functions on a vector space. The expectation map is not a map of algebras. In fact, its failure to be a map of algebras is quite important, for example

$$e(XY) - e(X)e(Y)$$

is important, called the covariance of X and Y. At the center you have an object with structure and a non-structure-preserving map. I think about how probability theory developed with these maps that don't preserve structure, but category theory works with things that preserve structure. So categorical notions never really made an appearance. There's another consequence of having a non-structure-preserving map here. In topology you might have a non-structure preserving map, like integration, and probability theory might be a great source of experience and knowledge for studying a non-structure preserving map. Since probability isn't adapted to homotopy, it can't be a good resource for studying non-structure preserving maps in other settings. Now we have homotopy probability theory that lets you use homotopical algebra to do computations in probability theory using homotopical methods.

I want to talk today about the flip side, where you have a non-structure preserving map and you want to talk about it and by upgrading probability theory you can talk about it in a new way.

Okay, if I think about probability theory on a manifold M, I think of something where I have a measure μ which gives me, I consider the functions on M and I want to integrate the functions over M, $F(M) \xrightarrow{c} R$. Now how does this actually work? You have some sort of a volume form dV on M and a function f maps to $\int_M f dV$. This is like probability theory on a manifold.

Remark 1.1. This volume form is an n-form on the manifold, and I can decompose the functions by first taking the functions (call them the 0-forms) to the n-forms on M, and then map this into the reals by integration.

The map from functions into n-forms, I want to say something under an assumption. If I assume that M has a Riemannian metric g that gives rise to the volume form, so g is an inner product on vector fields which gives an inner product on 1-forms and hence n forms. So this first map is the Hodge star map.

If you know everything of expectation zero, you can take the quotient and you can just know everything about the expectation. If I'm integrating an n-form over the manifold, there's one subset of n-forms, say M is compact, then exact n-forms give zero expectation. So it's natural to look at $\Omega^{n-1}(M)$ and take the differential of that, so how do figure out which functions, when you multiply by the volume form, are exact. In general, $*$ will be a map, an isomorphism $\Omega^j \to Omega^{n-j}$. So then this is a commutative diagram of chain complexes

$$\begin{tikzcd} \Omega^0(M) \arrow[r, "*"] & \Omega^n(M) \\ \Omega^1(M) \arrow[r, "*"] \arrow[u, "*d*"] & \Omega^{n-1}(M) \arrow[u, "d"'] \end{tikzcd}$$

So we have a classical probability space $(\Omega^0(M), \bullet) \xrightarrow{c} \mathbb{R}$ and we extend this to a homotopy probability space $(\Omega^\bullet(M), d^* = *d*, \wedge) \xrightarrow{c} \mathbb{R}$. I should have defined:

Definition 1.1. A homotopy probability space consists of (V, d) a chain complex, a chain map $(V, d) \xrightarrow{c} \mathbb{R}$, and a product $\wedge : V \otimes V \to V$, with no assumption, no relation between $\wedge$ and d or c.

There's one aspect which does respect the algebraic structure. The unit should be sent to 1. So you can make that assumption here too.

In classical probability you have a vector space of random variables and you replace this with a chain complex instead.

Much of the work that has been done has to do with extracting invariant information from this chain complex. I'm not going to do that today.

To check that this is a homotopy probability space you have to check that on the image of $*d*$ it's zero. But that was the calculation I did.

So now I want to do the Gaussian. My manifold M is the reals. My volume form is $dV = e^{-\frac{x^2}{2}\sqrt{2\pi}}dx$ which I'll write as ρdx. The metric $g(dx, dx) = \frac{1}{\rho^2}$. So what is $*$? So $*$ of a function, it's a map $\Omega^0 \to \Omega^1$ and vice versa. So what does $*$ do to a function? It's $f\rho dx$. Then $*f\rho dx = f$ to square to the identity.

What I'd like to do is compute, a typical element $a(x) + b(x)dx \in \Omega(M)$ in my homotopy probability space. Now what is d^* of it? We want to try to understand all of these functions modulo the image of d^*. The expectation map sends $a(x)+b(x)dx$ to $\int_{\mathbb{R}} a(x)\rho dx$. What is d^*? Well, $d^*(a(x) + b(x)dx)$ is, I know what $*$ of a one form which is $f\rho dx$ is, so this is $*d*(a + bdx)$, well, $*bdx = \frac{b}{\rho}$, and I should get

$*(\frac{b}{\rho})'dx = (\frac{b}{\rho})'/\rho$. There are questions of integrability, but if you restrict to an appropriate class this all makes sense. You can see what the outcome is. Let's do a consistency check and integrate d^* of something, this is $\int(\frac{b}{\rho})'/\rho \cdot \rho dx$ and you get the integral of a derivative, you assume this vanishes at infinity and this is zero.

This example is kind of locally famous in a different coordinate system. So $V^1 = \Gamma(TM)$ is isomorphic to $\Omega^1 = \Gamma(T^*M)$. We have a nice vector field on the left which is $\frac{\partial}{\partial x}$. This corresponds under the metric pairing to $\rho^2 dx$. Let me write $\rho^2 dx$ as η.

If I take $d^*(a + b\eta)$, and this is, according to this formula, this is $d^*(a + b\eta \cdot \rho dx)$ and this is $(b\rho)'/\rho$ which is

$$b' + b\frac{\rho'}{\rho} = b' - bx$$

So written another way you get $d^* = \frac{\partial^2}{\partial x \partial \eta} - x\frac{\partial}{\partial \eta}$. This is maybe familiar; whether it is or not, if you work with polynomials then d^* sends polynomial forms to polynomial forms so you can work out polynomial forms modulo polynomial forms and you can compute the class of $[x^n]$ in Ω^0 and you can see that this is like $n!!$ if n is odd and 0 if n is even or something, I guess the other way around so it's $(n-1)!!$ if n is even and 0 if n is odd. If I compute the cohomology of functions with d^*, then this is one dimensional, spanned by the class [1]. So if you want to know the expectation of any polynomial function, you just need to compute its class. You see that polynomials which are odd are cohomologous to zero. You can do the rest of the computation. The point is that you can do the full computation just by doing a little computation in cohomology.

A big part of the story of homotopy probability theory is what you do with correlations here. It's understood what to do there even though I'm not talking about it now.

So I don't have it on the board any more, I have this particular space, so since we're working with vector fields, that's a more convenient way to do it, perhaps I'd like to rewrite this using the isomorphism between vector fields and 1-forms, and you can extend this to an isomorphism between polyvector fields (sections of the exterior power of the tangent bundle) and differential forms. It's common to study differential forms and vector fields but not polyvector fields. I think that's a little odd. In this setting it gives the right, a convenient coordinate, so let me replace this by $V^\bullet$, where V^j is sections of the jth power of the tangent bundel and $V^\bullet$ is the sum of these. I'll rename d^* as ∂_V and I still have $\wedge$. Note that $V^0 = \Omega^0 = \text{Functions}(M)$.

One idea that is in my head but was unspoken, this is an example where you have a classical probability space that maybe you care about sitting inside a homotopy probability space. So you replace your vector space with a chain complex so that the image of d is like the kernel. You take a module with relations that you don't want, so you replace it with something free with the aid of a differential. So you put those relations inside the image of d. The question comes up, you might have many ways of resolving a classical probability space inside a homotopy probability space. You want concepts invariant of the way you resolve things. A physicist uses a gauge symmetry to make a differential, an analyst does things with exact forms, and you want the computations to have meaning in the original probability space.

You might say, what's an isomorphism of homotopy probability spaces? It's not clear to me what an equivalence of probability spaces is. I don't know the

∞-category yet. You don't want your maps to respect the product structure. It's only when you want to compute that the products enter the equation.

Written in terms of vector fields, this ∂_V is familiar. How do I take, d^* maps one forms into functions. So ∂_V maps vector fields into functions. You know a map that sends vector fields into functions, which is the divergence, so Div. This is not a derivation of the wedge product, but its deviation from being a derivation of the wedge product, so $\partial_V(X \wedge Y) - \partial_V(X)Y - X\partial_V(Y) = [X, Y]$. It's convenient to sometimes use in differential forms, if you take the deviation of d^* from being a derivation of the wedge, this defines a bracket on differential forms. This is skew-symmetric and satisfies the Jacobi identity.

There was a discussion during the break, you have $(V, d) \xrightarrow{c} \mathbb{R}$ and you'd like to know how it fails to be an algebra map. You write down $c(XY) - c(X)c(Y)$. What do you write down about $c(XYZ)$, there are multiple ways to write this down, and there are various ways in which you can measure the failure to be an algebra map at the next level. It turns out that this covariance fits into an infinite hierarchy of invariants called cumulants, this is κ_2, so $\kappa_3 = c(XYZ) - C(XY)C(Z) - \cdots + 2c(X)c(Y)c(Z)$.

I didn't use the differential, this is a quantity of interest, their vanishing corresponds to independence. It turns out that you can interpret these, the first point of contact between probability theory and homotopy theory is that there is a version, a way to view these κ maps as an infinite, an L_∞, an infinitely homotopy Lie thing, the κ_n are the data of an L_∞ morphism from an L_∞ algebra defined by (V, d) and the product, and the real numbers with trivial L_∞ structure.

[Missed some]

You have this $(V, d) \xrightarrow{c} \mathbb{R}$ and you can pass from this to $(V, D^\bullet)$ and you can do this on the right and you still get $(\mathbb{R}, 0)$ and you get

$$\begin{tikzcd}
(V, d) \arrow[r, "c"] \arrow[d, "\text{use product}"] & \mathbb{R} \arrow[d, "\text{use product}"] \\
(V, D^\bullet) \arrow[r, "\kappa"] & \mathbb{R}
\end{tikzcd}$$

This is part of the 2011 point of Jae-Suk Park. This is literally true even in ordinary probability theory but the L_∞ picture has a clear generalization beyond ordinary probability theory.

One thing I didn't say because it's a little involved, not every element of the chain complex do you want to take expectations of. Only certain things do you want to call random variables. You can describe those as L_∞ maps from 0-L_∞ structures into V. These are homotopy random variables. It turns out that then you can compose and get numbers out and the numbers, the things are homotopy invariant, so if you take homotopic expectation maps, homotopic random variables, you get the same numbers. What I stated was the sort of onramp idea. Then you have to develop something to make it meaningful.

Let me come back to polyvector fields with divergence and wedge. My descendent L_∞ structure is this differential graded Lie algebra. That's what you get in this example. Now what i'd like to say is, imagine you want to solve the Navier Stokes equation on a three-manifold. This says $\dot{Y} = [X, Y]$, div $X = 0$ (incompressibility), and curl $X = Y$. This is the Euler version of the Navier Stokes equation, I don't have viscosity.

I can write down, the curl of a field in V^1, I can make this a 1-form by g and then apply $*d$ to get a 1-form, and then use g to get back to V^1. This is called the curl. I think, I wanted to think of why it's called the curl. You have the gradient of f, you can take df, that's a 1-form, I can take gdf and that's the gradient. You can see that if I take a gradient, then curl, I get zero. What's the gradient flow on a function? It flows down the function. Curl being zero is basically a version of being the gradient of a function. This is de Rham d, so it depends on the topology of the manifold. Your direction could get you back to where you started even with zero curl.

What does Navier-Stokes have to do with any of what I'm talking about? I think of this lecture in 2011, this was envisioned and went on to have a life of its own, Dennis Sullivan wrote a paper (2014) where he sketched an idea using homotopy probability theory to do a combinatorial version of fluid flow. Gabriel and I are in the middle of a conversation of fleshing out how you would do this. Roughly, the idea is that you take some combinatorial chain complex C, maybe $\mathbb{R}^3$ or a torus, it's not clear you can do this for any manifold, and you take a cell decomposition of it, there might be ways to vary this idea but you use cubical cells for a special reason, and then you have the boundary ∂^C, and this chain complex is quasi-isomorphic to $(V^\cdot, \partial^V)$, you define maps in both directions. On map is ι or B for bump function. If you have a chain, you define a differential form by taking this form and smoothing it out a little bit. You get a map the other way essentially by integration. If you have a differential form, you can differentiate over a chain, you can get from differential forms to cochains and then get from cochains back to chains. If you do ι and then π you get the identity, you have to be careful about your bump functions to make that true, and you also get a homotopy between $\iota\pi$ and the identity.

When you have quasiisomorphic chain complexes and an understood algebraic structure on one, you can use homotopical transfer to get the structure to a homotopy version on the other side. You can promote i and π (and h) to being L_∞ maps (and homotopy). Then I can write down a version of the Navier Stokes equation. Let me use x and y for elements of C, or more generally you want them to live in SC, then you can image that x and y are related to X and Y and you can write down

$$\dot{y} = Ad_x^{L_\infty}(y), \qquad \partial_\infty x = 0$$

and the real question is, what do you do with curl. This is where you have to do something. So first, define a combinatorial star operator on C and once you have that, then you can define a combinatorial curl. I think it's just $\partial_\infty *$. Then step three is to use the "cumulant map" meaning the way you use a product to define the cumulants, this gives an automorphism of SV and you conjugate by it to obtain an operator which will be $(\partial_\infty *)^\varphi$. Then the final equation is $(\partial_\infty *)^\varphi x = y$, where the conjugation uses homotopy probability theory. There's a discussion about measure preserving flows using Eulerian pdes. You're really doing ordinary measure theory, enriching ordinary measure theory to a homotopy probability theory that says something about measures. You're also using the same data to modify the L_∞ structure to get measure-preserving flow on the combinatorial manifestation of the manifold. The conjecture is that the combinatorial solutions converge to a real solution.

What little I know about the Navier-Stokes equation I learned from Scott Wilson. He translates everything to differential forms and uses Whitney forms to get a combinatorial star operator, and shows this converges in an appropriate sense to the Hodge star.

The difference between the choices you have to make is a homotopy that gets small as the mesh gets smaller.

It seems to me that if you have wedge, divergence, and d transported, and curl, and maybe you need the inner product or the volume form or whatever, and you knew your homotopy algebra, then you'd transfer that whole package and you could define the equation there. It's not known what the correct operad for all of that structure is. You have problems with inner products and element in homotopy transfer. There are limitations to it that if you have inner products and units or elements, things like that Hirsh-Mill‘es tell us about some of it. I don't know whether homotopy transfer is the right idea for a pde. For instance, we did a computation yesterday, your manifold is the circle and you choose the bump function, you get quasiisomorphic chain complex, you didn't get better and better approximations, if you take the value at points of a function and then smooth it, then it's not clear that that's a good approximation for the original function.

2. January 29: Dan Berwick-Evans: Field theories and elliptic cohomology

I want to talk to you about two seemingly different objects, one a certain type of two dimensional quantum field theory and the other elliptic cohomology. I'll spend a while talking about each one.

So let me start by talking about a universal elliptic cohomology theory, called TMF. By its construction it is related to number theory and homotopy theory. There's a long story connecting elliptic curves to number theory. There are other things, it might be related to loop group representation theory. Here I mean positive energy representations of the loop group at a fixed level. Another thing these are related to is analysis on loop spaces. You might want to call this the Laplacian on a free loop space. This object talks to this sort of information. There are also higher geometric structures, gerbes, string geometry, and all this stuff. The string geometry should be a generalization of spin geometry.

This thing sits at the heart of a lot of interesting mathematics. Another thing that interacts with all of these things, which are two dimensional $N = 1$ supersymmetricy Euclidean field theory. The relationship of elliptic cohomology to these Euclidean field theories is not clear but the relation of Euclidean field theory to all these other things is clear. So the question is, what if any is the relationship between Euclidean field theories like this and elliptic cohomology?

Let me same some things about elliptic cohomology.

The beginning of the story is about Chern classes of a line bundle. If I have a line bundle L over a space X, then I can build the Chern class $c_1 \in H^2(X, \mathbb{Z})$. If I take the tensor product of two line bundles, then the Chern classes add. Elliptic cohomology does the same type of thing but with a different group and a different tensor product formula.

So given an elliptic curve E you can build a cohomology theory (a functor from spaces to graded Abelian groups with certain properties) Ell with Chern classes so that $c_1^{Ell}(L) \in Ell(X)$. You get an analogous formula for the tensor product formula which is more complicated and encodes the formal group structure on E. Here I mean you're taking a Taylor series at the identity of the group.

So TMF, that is "topological modular forms" is the universal elliptic cohomology theory in the sense that for each elliptic curve there is a map to elliptic cohomology for that cohomology theory, you can evaluate to get a map.

That's hard to wrap your head around, that's too abstract, let me explain why modular forms get mentioned.

2.1. **Modular forms and $TMF \otimes \mathbb{C}$.** This is a standard trick, tensor a complicated ring with the rationals or reals or complex numbers.

Definition 2.1. A (weak) modular form is a holomorphic function f on the upper half-plane such that for all two by two matrices of determinant one, then $f\frac{za+b}{zc+d} = (zc+d)^k f(z)$. This is called weight k.

Let me tell you how to build a cohomology out of these guys. Let me define a graded ring $MF^\bullet$ which is modular forms of weight k in even degrees and zero otherwise. That's a graded commutative ring.

Here's a fact: $TMF \otimes \mathbb{C}$ is $H_{dR}(X, MF^\bullet)$, that is, it's

$$\bigoplus_{i+j=\bullet} H^i_{dR}(X, MF^j)$$

Let me tell you the sense in which this is some kind of universal thing. This requires a transition between points in the upper half-plane and elliptic curves.

Notice that for each point in the upper half-plane you get an elliptic curve over $\mathbb{C}$, specifically the curve which is the plane modulo the lattice generated by 1 and that point τ. Then I can evaluate my modular form at τ. Then evaluation of a modular form at τ gives a map from $TMF(X) \otimes \mathbb{C}$ to my elliptic cohomology theory $Ell(X) \otimes \mathbb{C}$, which is 2-periodic de Rham cohomology.

This tells you the abstractly defined object after you tensor with $\mathbb{C}$. This mixes in this modular form business.

Now we'll have a little K-theory interlude. This is about something else, if you zoned out, come back. This is related but has very pleasant descriptions everywhere. So you can look at real vector bundles over X and look at $KO(X)$, the smallest group containing the monoid of vector bundles up to isomorphism and Whitney sum of bundles, this is like $\mathbb{N} \hookrightarrow \mathbb{Z}$. I want to show some torsion that shows up in one of these cohomology theories.

If I look at $KO(S^1)$, I have a lot of vector bundles, there's the trivial ones which give me $\mathbb{Z}$. I also get the mobius bundle and the sum of two Mobius bundles is trivial. This then gives me $\mathbb{Z} \oplus \mathbb{Z}/2$.

That's a fact about KO. I want to give you a little bit more to connect to TMF. The first is, what do you get when you tensor KO with $\mathbb{C}$?

The first fact is, there's a map, called the Chern character, which goes from $KO(X)$ to 4-periodic de Rham cohomology, and this map is a rational isomoprhism. It remembers a lot of stuff but forgets the Mobius bundle.

The second fact which I won't say a lot about, there's an analytical description of KO in terms of Fredholm operators. The homotopy classes of maps from X into Fredholm operators on a Hilbert space is $KO(X)$.

Let me use this to say a little more about the universal elliptic cohomology theory. Let me say a little about the relation to K-theory.

There is a commuting square

$$\begin{tikzcd} TMF(X) \arrow[r] \arrow[d] & KO(X)((q)) \arrow[d, "ch"] \\ TMF(X)\otimes\mathbb{C} \arrow[r, "q-\text{expand}"] & H(X,\mathbb{C})((q)) \end{tikzcd}$$

The top arrow I still haven't told you that, it turns out to be evaluation in a formal neighborhood of the Tate curve (this is due to Haynes Miller).

I get a bunch of vector bundles whose Chern character is modular in the sense that it gives you something in the image of the bottom horizontal map.

This is a close enough approximation for now, it's *like* a sequence of vector bundles with a modularity condition on the Chern character.

I told you two-periodic in one case and four-periodic in the other, and it's a little calculation to see why the modular forms vanish outside multiples of four.

2.2. **Field theories.** Now let me turn to field theories, which have a very different flavor, very physical, and give a connection to this square. The dream is to find a differential geometric or analytic realization or description for cocycles in $TMF(X)$. I didn't say it, but the relationship between Fredholm operators and K-theory give amazing things, like the index theorem. Analysis on loop spaces would have topological control if you could realize this for TMF.

Conjecture 2.1. *(Segal, Stolz-Teichner) There is a category of two dimensional field theories over X, $2|1$ Euclidean field theories, and modulo an equivalence relation you get $TMF(X)$. These are $N = 1$ supersymmetric Euclidean field theories evaluated on X a smooth manifold.*

Why don't I say what I'm going to do next and then we'll take a break. So the next goal is to explain how this square goes where we replace $TMF(X)$ with field theories and see how the arrows are exactly the kind of thing we should expect.

So in the beginning I was talking about TMF, whose existence has been proven, we can't calculate it, we can calculate the modular forms part and the K-theory part.

I want to provide some motivation for why field theory should be related to these things at all. This was before TMF existed, but he was thinking about elliptic cohomology. Let's define a symmetric monoidal category $1 - Bord(X)$, where the objects in this category are finite subsets of X, the morphisms, are compact oriented 1-manifolds with decorations on the boundary and a map to X. They could have boundary and these give source and target. The idea is to look at functors from this category to vector spaces and see what we get. So define 1-dimensional topological field theories over X as symmetric monoidal functors from $1 - Bord(X)$ to vector spaces. Because this is symmetric monoidal, it suffices to specify a vector space for every point in X. So to each $x \in X$ you get V_x, and then on morphisms, I can think of these as paths or paths glued together, so to each path from X to Y we get a linear map $V_x \to V_y$, that sounds like a vector bundle and parallel transport with respect to a connection.

Theorem 2.1 (Folklore, Berwick-Evans–Pavlov)**.** *One dimensional topological field theories in X is the same as vector bundles with connection on X.*

This should be intuitively clear. Because of definitions it's a bit technical, hard for the wrong reasons.

There's this relation between vector bundles and K-theory I mentioned before, you might ask if there's a relationship between field theories and K-theory.

One answer, due to Hahnhold, Stolz, and Teichner, says that there is a classifying space of field theories, $\mid 1|1 - EFT \mid$ homotopy equivalent to Fredholm operators, which implies that maps into this classifying space give K-theory of a manifold.

This is a relatively old result, announced in 2004 or so, this is motivation. Let me say a few more words about these 1|1 Euclidean bordisms. In a little more detail, these are defined by some bordism category. We use a symmetric monoidal category of 1|1 Euclidean bordisms whose objects are points with an odd line bundle on them. The morphisms are compact metrized 1-manifolds with an odd line bundle and source and target data on their boundary. Let me say a little bit about oddness, isometries, we have extra isometries but you might call them supersymmetries.

Then 1|1-Euclidean field theories are functors from this space to vector spaces, as before.

We can throw in maps to X to get 1|1-Euclidean bordisms over X. From that we get the notion of a 1|1-Euclidean theory over a manifold.

Next I want to move this to dimension two and unravel some of the geometry.

Define a symmetric monoidal category $2|1-EBord(X)$ whose objects are metrized circles with an odd line bundle with a map to X. The morphisms are *flat* metrized 2-manifolds with an odd line bundle and map to X. This flatness is a real restriction here because we're in two-manifolds. We want Euclidean.

Here's a problem, compute these field theories, that is, functors from this bordism category over X to vector spaces.

To make progress we can restrict to some subcategories of 2|1-Euclidean bordisms and compute with these. The easiest one is from the empty set to itself via the torus.

So let's talk about subcategories of tori. Let $\mathcal{M}^{2|1}(X)$ be tori with an odd line and a map to X. These are bordisms from the empty set to itself. Our functor should assign $\mathbb{C}$ to the empty set. So this is a map $\mathbb{C} \to \mathbb{C}$, so to each one of these tori a field theory should assign a number. In particular I should get a map $2|1 - EFT(X)$ to functions on $\mathcal{M}^{2|1}(X)$. This is something like the double loop space of X. We'll next restrict to the constant double loops. There is a subspace $M_0^{2|1}(X)$ which consists of constant (energy zero) tori.

Theorem 2.2. *(Berwick-Evans)* $C^\infty(\mathcal{M}_0^{2|1}(X))/\sim \cong TMF(X) \otimes \mathbb{C}$ *(modulo concordance)*

The torus with the map to X being constant must factor through an odd line bundle on the point, $\mathbb{R}^{0|1}$. These give me points in $\mathcal{M}_0^{2|1}(X)$. The torus is irrelevant. The thing that matters is the map from the point. So this is $\mathcal{M}_0^{2|1}(pt) \times Map(\mathbb{R}^{0|1}, X)$. When I take functions this is $C^infty(M^{2|1}) \otimes C^\infty(Map(\mathbb{R}^{0|1}, X)$ which is modular forms tensor differential forms, which is differential forms with values in $MF^\bullet$.

So my left map is restriction to tori. The top map is going to be restriction to annuli.

So next we'll look at maps from a circle to itself. Define a category $2|1-Ann(X)$ with objects a circle with odd line and map to X and morphisms annuli with an odd line with a map to X. Just as before we got an object over a double loop space, here we live over the loop space and restrict to $2|1 - Ann_0(X)$ which is constant loops and thin annuli, where the rank of the map to X is 1.

Theorem 2.3. *(Berwick-Evans) Functors (effective functors, this is an analytic condition) from $2|1$-annuli of this sort to vector spaces up to equivalence is $KO(X)((q))$.*

The idea is that $2|1-Ann_0(X)$ is like paths in X with an internal S^1-symmetry. So for some given circle, the S^1 acts on the vector space, you can decompose as characters and that gives this q-expansion.

I should wrap up. Let me draw one big diagram and then be done. I told you a little bit about $2|1$ Euclidean field theories over a manifold. No matter whether this is the exact definition, you'll always have this subcategory of annuli. Likewise you should always have tori. If my annulus has two circles that meet up, this is like a torus where I noticed where they meet. I'll call this $\widetilde{M}_0^{2|1}$.

$$\begin{tikzcd}
 & 2|1-EFT(X) \arrow[rr, "\text{restrict to annuli}"] \arrow[dl, dotted] \arrow[dd, "\text{restrict to tori}"] & & Fun(2|1-Ann_0(X), Vect) \arrow[dl] \arrow[dd, "\text{trace for annuli that are tori}"] \\
TMF(X) \arrow[rr] \arrow[dd] & & KO(X)((q)) \arrow[dd] & \\
 & C^\infty(\mathcal{M}_0^{2|1}(X)) \arrow[rr, "\text{forget meridian}"] \arrow[dl] & & C^\infty(\widetilde{M}_0^{2|1}(X)) \arrow[dl] \\
TMF(X) \otimes \mathbb{C} \arrow[rr] & & H(X, \mathbb{C})((q)) &
\end{tikzcd}$$

3. February 5, Xun Yu: On smooth isolated curves in general complete intersection Calabi-Yau threefolds

Thanks for the invitation and for attending the seminar. I will call complete intersection Calabi-Yau threefolds "CICY." I work over the complex numbers. What is a Calai-Yau threefold? It is a projective variety Y of dimension three with dualizing sheaf the trivial bundle and $h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0$. In today's talk, singularities will be at most nodal. What do we mean by isolated? Roughly this means they are rigid, they cannot move. The technical definition is that the smooth curve C is isolated in a projective variety Y if $h^0(C, \mathcal{N}_{C/Y} = 0$. By deformation theory, this means that $[C]$, regarded as a point in the Hilbert scheme, is a reduced isolated point in $Hilb\ Y$. For example, a smooth rational curve in a $K3$-surface. Or 2785 lines in general quintic three-fold (meaning it's degree five in $\mathbb{P}^4$). General means that in the moduli space of all quintics, a dense open subset.

Now these lines mean $g = 0$ and $d = 1$.

A natural question is, how about for other pairs g and d? Let $d \geq 1$ and $g \geq 0$ be integers, does a general CICY threefold of a particular type contain a smooth isolated curve of degree d and genus g?

This turns out to be a hard problem in general. You may ask, why do you consider only Calabi-Yau threefolds? Well, Calabi-Yau have, if $C \subset Y$ then $\chi(\mathcal{N}_{C/Y}) = 0$. In other words, by deformation theory, the expected dimension of $Hilb\ Y$ is zero at $[C]$. This Euler characteristic is $h^0 - h^1$ and if this is positive then h^0 is positive and we will never have something isolated. This is why we prefer

Calabi-Yaus. For general Calabi-Yaus, having certain curves is still open, but for CICY we have more control, at least for the moment.

Why general? For example, the Fermat quintic contains continuous families of lines. So we can't expect it for all three-folds and only ask it for general three-folds.

Now some history, okay? The existence results, in the genus 0 (rational curve) case, the first breakthrough was Clemens in '83 for the quintic, for infinitely many degrees. Then Katz showed it in all positive degrees. Then Oguiso in 94 and Ekedahl and Johnsen and Sommervoll in 99 showed some other kinds. We have five types, a quintic (5) in $\mathbb{P}^4$ but also $(2,4)$ (Oguiso) and $(3,3)$ in $\mathbb{P}^5$, for $(2,2,3)$ in $\mathbb{P}^6$, and for $(2,2,2,2) \in \mathbb{P}^7$ it was shown by the other authors.

Knutsen in 2010 showed this for $1 \leq g \leq 22$ for infinitely many (g,d). I showed for finitely many (g,d) between 3 and 29 in 2012.

Let me show you [slides]

The methods to construct examples go back to Clemens' '83 method. He constructed a K3 surface S containing infinitely many rational curves L_n with degrees d_n going to ∞. Second, he constructed a nodal Calabi-Yau containing S which leads to an exact sequence

$$0 \to \mathcal{N}_{L_n/S} \to \mathcal{N}_{S/Y} \to \mathcal{O}_{L_n}$$

and by some argument showed that $h^0(\mathcal{N}_{S/Y})$ is zero. Later researchers did this setup and tried to find a different nodal Calabi-Yau and different K3 containing the same curve. If Y deforms to a smooth one, then the curves L_n deform to isolated curves C_n inside the smoothing of Y.

To make this precise takes a lot of work.

In fact, Kley in 2000 constructed a framework to construct rigid (or isolated) curves. He used Clemens' idea but tried to make a formal framework. But there was a serious gap. I'll mention the gap later. Then Knutsen's method fixes Clemens' gap and also adjusted the framework and he found a method. So [previously written boardwork.]

[Missed some]

Let me roughly sketch the idea of the proof, before that I want to remark that the axiom A5, that $H^0(C,\mathcal{N}_{C/X}) \cong H^0(C,\mathcal{N}_{C/Y})$ for all C, is equivalent to the condition that the set of nodes N imposes independent conditions on $|\mathcal{L}|$, plus a technical condition that is easy to check. What do I mean, independent conditions? I mean for all $S' \subset S$, we have $H^0(X,\mathcal{L}) \to H^0(S',\mathcal{L} \otimes \mathcal{O}_{S'})$ by restriction, this is of maximal rank.

Passing through n nodes means n hyperplanes meet transversally, that's general position somehow.

That's somehow, this is a pretty hard condition. For Knutsen, he found a condition, a numerical one, to check $*$. I adjusted the test for this to go up to genus 29 from genus 23.

Maybe in the next part I want to sketch the proof, not in too much detail, so you can see how this goes and how this method works. Then some conjectures, open problems, and if more time, then some nonexistence results for the quintic.

So let me give a sketch of the proof of Knutsen's method. So we have $|\mathcal{L}| \subset$ *Hilb* $X \subset$ *Hilb* $Y \subset$ *Hilb* P. First of all, the containment is clear. The important part is that there exists an open smooth subscheme $\mathcal{H} \subset$ *Hilb* P so that $\mathcal{H} \cap$ *Hilb* $Y = |\mathcal{L}|$. That's essentially because of axiom A5.

Then you consider the universal family $\mathcal{C} \subset \mathcal{H} \times P$. Two key facts are that

(1) the pushforward of the pullback of $\mathcal{E}$ over P under p_*q^* is a vector bundle over $\mathcal{H}$ with rank equal to the dimension of $\mathcal{H}$. Here p and q are the canonical projections.
(2) The second fact is that $Hilb\ Y_t \cap \mathcal{H} = Z(p_*q^*(s_0 + ts))$ where s is a global section to perturb s_0.

Here $Y_t = Z(s_0 + ts)$. Then because $|\mathcal{L}| = Z(p_*q^*(s_0)) \subset \mathcal{H}$, we have the following so-called normal bundle exact sequence:

$$0 \to \mathcal{N}_{|\mathcal{L}|/\mathcal{H}} \to p_*q^*(\mathcal{E}) \otimes \mathcal{O}_{|\mathcal{L}|} \xrightarrow{\rho} Q \to 0.$$

In this case, $|\mathcal{L}|$ is of bigger dimension. The goal is to show that if you perturb the section, the guy on the left will have an isolated point. This means we want to show that $p_*q^*(s_0 + ts)$ in fact contains isolated points.

So then for general s and t we want $\rho(p_*q^*(S))$ has only isolated reduced zeros, which is implied by our axiom set. That means that $Z(p_*q^*(s_0 + ts))$ contains finitely many isolated reduced points, which in turn implies that $\mathcal{H} \cap Hilb\ Y_t$ (and hence $Hilb\ Y_t$), has isolated reduced points.

Those points correspond to the isolated curves we want.

Next I want to say how to use K3 surfaces to produce examples of isolated curves. Let X be a CI K3 of type $(a_, \ldots, a_{r-3}, a_{r-2}) \subset \mathbb{P}^r$ with $a_{r-3} \geq a_{r-2}$. Suppose wehave that $Pic(X) = 2$, the Picard rank is small, so $Pic(X) = \mathbb{Z}H \oplus \mathbb{Z}C$. Now H is a hyperplane section and C is a smooth curve. Here $\mathcal{L}$ is $\mathcal{O}_X(C)$. In this case, suppose we are in a nice situation. Then due to Knutsen's method, you have this theorem

Theorem 3.1. *A general CICY three-fold contains smooth isolated curves of degree d and genus g if*

(1)
$$d \leq a_1 \ldots, a_{r-3}a_{r-2}^2 \text{ or } da_{r-2} > \frac{a_1 \cdots a_{r-3}a_{r-2}^2}{2} + g.$$
(2)
$$\frac{(2a_{r-3} - a_{r-2})a_1 \ldots a_{r-3}a_{r-2}}{2} \geq \begin{cases} g+2 & a_{r-3} \neq a_{r-2} \\ g+1 & a_{r-3} = a_{r-2} \end{cases}$$

My result is just changing the second condition.

Theorem 3.2. *(Yu, 12) the assumptions and conclusion are the same except the second condition becomes*

$$\frac{a_1 \ldots a_{r-4}a_{r-3}^2 a_{r-2}^2}{11} \geq g + 2$$

and $H^1(X, \mathcal{L}(-a_{r-3} - a_{r-2})) = 0$.

Again, this condition is basically to check A_5.

Let me give an example, just one, plug in numbers. For example, the goal is to show that $(g, d) = (23, 18)$ in the general quintic. To start with we need X our CI K3. So:

Theorem 3.3. *(Mori (84), Knutsen (02)) If $g < \frac{d^2}{4n}$ and $(d, g) \neq (2n+1, n+1)$, then there is a K3 X of degree $2n$ inside $\mathbb{P}^{n+1}$ and $Pic(X) = \mathbb{Z}H \oplus \mathbb{Z}C$ where C is as above.*

This was origiially for quartics alone before Knutsen generalized it.

Now by the above theorem and the existence of curves, $X = (3, 2)$ in $\mathbb{P}^4$ and the degree of X is $6 = 2 \cdot 3$. The Picard rank is as we want. You need to check that $23 < 18^2/4 \cdot 3$. Then r is 4 and $a_{r-3} = 3$ and $a_{r-2} = 2$. You can plug in numbers and see that Knutsen's formula is not satisfied but mine is. So that's how it works.

Let me state a conjecture.

The optimal conjecture is that there is no bound on g. Certainly that needs completely new methods. Another conjecture is, suppose that you have a pair (d, g), if this is okay, then how about $(d + 1, g)$. Knutsen's methods work for this.

4. Theo Johnson-Freyd: Local Poincaré duality and deformation quantization

I want to tell you a story that is also a pleasure to tell hear because I was able to rework the details here in November, a story about local Poincaré duality and deformation quantization. I'll get to deformation quantization by the end of the talk. I want to spend the first part of the talk talking about (a version of) local Poincaré duality.

There's lots of things that Poincaré duality means to different people. You might think that local Poincaré duality is something else.

Whatever Poincaré duality is, it's really big. Something that it includes is the following phenomenon, the simple fact that if M is a compact oriented manifold of dimension d, then, I don't know much about topology, but I know the de Rham cohomology of M. This is a commutative Frobenius algebra. It's not quite commutative, it's graded commutative, but I always mean the graded commutativity. I'll never write that. A Frobenius algebra, there are lots of types, but the most naive notion should have a trace of degree 0. There's a trace out of top forms, let me call this a shifted Frobenius algebra, where the trace map to $\mathbb{R}$ has degree $-d$.

That's some very small piece of Poincaré duality. For my talk I'll claim that this is what Poincaré duality *is*, because that's what I want to focus on.

You could ask the following motivating question. Does this Frobenius algebra structure come from some sort of local structure, some sort of cochain level Frobenius algebra structure?

As soon as I'm asking this about cochains, I should ask about homotopy algebras, is there a homotopy Frobenius algebra structure on cochains?

This sort of forces me to pose two subquestions.

(1) What are homotopy Frobenius algebras?
(2) Which cochain level operations count as local?

The naive answer is that local things, we're working with cohomology. Cohomology classes are global things. But cochains are local things. Cochains are a local standin for cohomology. This should have something to do with cochains, and what local will mean is that some cochains are spread around the manifold, I'll use de Rham cochains. Using partitions of unity these are sums of locally supported cochains. You could ask that your operations not increase the support too much. I want to control how much the support increases.

Now I want to talk about pure algebra for a moment. What are "homotopy (d-shifted commutative) Frobenius algebras?" So first of all, let's step back and say, what are shifted commutative Frobenius algebras? A commutative Frobenius algebra is a cochain complex (V, ∂_V), equipped with a commutative multiplication

$V \otimes V \to V$ which is commutative and associative. For technical reasons, I'll focus on the possibly non-unital part, the open and co-open part. So I have a multiplication and a comultiplication which is cocommutative and coassociative, well, if you think about it, it has cohomological degree d, so I want it to be a degree zero map $V[d] \to V[d] \otimes V[d]$. That's the map that's commutative and associative. There's a theorem of nature that there are no good sign conventions with degree shifts. If you work really hard you can get a mediocre sign convention and get some signs. When you try to translate commutativity and associativity, if d is odd, you get some signs. Here's a choice. The translation ends up saying that for a choice of sign conventions, the associativity and commutativity is twisted by $(-1)^d$.

There is an even deeper theorem of nature that any sign convention used consistently will be correct and can be translated into any other convention.

Working with the counit the Frobenius relation is easy to write down. Here I should say that multiplication and then comultiplication is the same as comultiplication and then multiplication [picture] and if you use different signs you'll disagree with me about whether there's a sign in this formula.

If M is compact and has Euler characteristic zero, then the cohomology also enjoys the relation that comultiplication followed by multiplication of the outputs is zero.

So the point, unfortunately you can see from the diagrams that I drew, an unfortunate fact, something that happens here is that the notion of Frobenius algebra, and I'll start writing $Frob_d$ for these formulas. I should have said, all of these are cochain maps.

Something to emphasize is that the notion of a $Frob_d$-algebra involves many to many operations. There's a fantastic world of homotopy algebra for algebras with many to one operations, this is called operads. This is outside of operads. Those are things that involve many to one operations. The diagrammatics of operads are the diagrammatics of rooted trees.

Definition 4.1. Whatever it is that operads have to with rooted trees, you could work with directed trees, which are like the ones I've drawn, and you could work with algebras with directed trees and that gives a generalization of operads. Those are called dioperads. This won't do involutivity, so what's a small class of graphs that I can use? I could say connected directed graphs with no directed cycles. These are properads, which is a portmanteau of props, which are not necessarily connected. There are versions allowing directed cycles, but I can't interpret directed cycles in an infinite dimensional vector space. I don't need these for the things I want to do.

operads	rooted trees
dioperads	directed trees
properads	connected directed graphs with no directed cycles
props	directed graphs with no directed cycles
wheeled props	directed graphs
wheeled properads	connected directed graphs

This is a definition, the right hand column defines the left hand column. Here's a theorem of universal algebra. Let me say, I'll never use units. I mean also characteristic zero.

Theorem 4.1. *The category of dg erads, for any notion of erad, has a model category structure, the projective one, such that, I won't tell you everything about it, but*

I'll tell you enough to determine it, the weak equivalences are quasi-isomorphisms (homomorphisms which are the identity on homology) and the fibrations are surjections (in the set theoretic sense).

The cofibrations, I don't have an easy way to describe them. I'll mention one property, which is, like other times that you might be used to, every object is fibrant and the quasi-free ones, with some extra sort of things that you should fill in, are cofibrant. Quasifree means that it would be free forgetting the differential.

Not every cofibrant thing is quasi-free but the quasi-free ones are cofibrant. That defines a model category and then what I'll do is declare that the homotopy theory of "erads" is the one determined the model category of the theorem. There might be other model categories to describe the same homotopy theory.

With that definition, I can say that given some erad P, a homotopy P-algebra is, what's a homotopy algebra? P might be $Frob_d$ or $Inv\ Frob_d$, this is a representation, an algebra is a map $P \to End(V)$, but I don't want it to be a map in the category but in the homotopy category, the $\infty, 1$-category. A map in a model category is a map from a cofibrant replacement of the source to a fibrant replacement of the target. It's an actual map from hP, something I'll call homotopy P, to $End(V)$, where hP is a cofibrant replacement of P.

Now there's a word of warning. We have $Frob_d$, which only used trees, so it's a dioperad. We could, any time you have a presentation in terms of trees, there's a kind of universal enveloping functor from dioperads to properads and get $Frob_d$ as a properad. This functor is not exact. So what that tells you is that cofibrant replacement and then taking an enveloping algebra, that's not a cofibrant replacement, this doesn't preserve homotopy representations. So there's no reason that $h^{di}Frob_d$ and $h^{pr}Frob_d$ are equivalent. That tells you that the notion of homotopy algebra depends on the type of erads you're using.

Now I should tell you what local means. Let me tell you what local means and then the partial answers to the questions that I erased.

Remember, I want to work at the chain level, with operations

$$(\Omega^\bullet_{dR})^{\otimes m} \to (\Omega^\bullet_{dR})^{\otimes n}$$

and for me the tensor product will always be the projective tensor product, so that cochains on M^n is the nth tensor product of the cochains of M. If you wanted you to do this for any other cochain model, I'd pick the version of tensor product that was geometric.

I want the local ones. Which ones are local? The picture that I want to have is something like they don't increase support.

Definition 4.2. A *strictly local* operation is compatible with the sheaf structure, the presheaf structure on the de Rham complex. I mean that if I take $\alpha_1, \ldots, \alpha_n$ in $\Omega^\bullet_{dR}(U)$, for some U on M, and calculate their product, that I end up in $\Omega^\bullet(U)$. I'd also want that if I have disjoint support, the operation gives me zero by linearity.

This isn't good enough. There are not enough of these. They're all differential operators. Strict locality is the same as, I have some map on from forms on M^n to forms on M^m, these are always given by integral kernels, the integral kernel of this map f is a de Rham form on $M^\times n \times M^{\times m}$, possibly distributional, and strict locality is that the support of the integral kernel is inside the diagonal of M.

There's not enough of these. There is not a local lift in any sense of Frobenius algebras to strictly local operations. You can't even do the comultiplication.

I can always lift things with no locality, I can lift in the homotopy category with no problem across a quasi-isomorphism. What happens when you try to lift the cocommutative coalgebra structure on forms? You can lift if you allow the support to spread out a little bit. There's no strictly local homotopy algebra on distributional forms that gets the product right? But you can do it if you're willing to be a little bit less strict.

Definition 4.3. A quasi-local operation is a family depending on a parameter ϵ (a positive real number) such that as $\epsilon \to 0$, the support of the integral kernel gets inside any neighborhood of the diagonal.

What do I mean by an ϵ-family? I could have meant a smooth family of operations, which is a C^∞ map on $\mathbb{R}_{>0}$ valued in maps from $\Omega^{\otimes m} \to \Omega^{\otimes n}$, so I want it to be homotopically constant. So I should ask for a de Rham form. I want to get to the punchline, then a break. A homotopically constant smooth family, what I mean is, I want to tell you the complex of homotopically constant smooth families. That complex is the complex of de Rham forms valued here. I have a part in degree zero, which is a family of smooth things. The part that's degree one is a homotopy between all of them.

This is the space of quasilocal operations. What are the punchlines? Then I'll break.

Theorem 4.2. *Gor any M the space of maps from $h^{di}Frob_d \to QLoc(M)$ inducing the Frobenius algebra structure on cohomology is homotopic to a point. In the sense of homotopy, dioperadic local Poincaré duality is unique.*

I'm erasing my warning, but the more interesting theorem is

Theorem 4.3. *For $M = S^1$, there does not exist a map from the $h^{pr}Frob_1 \to QLoc(S^1)$ inducing the Frobenius algebra structure on $End(H(S^1))$.*

The calculations become too hard for me in dimension two, but in dimension at least two, if there exists a properadic map that is quasilocal inducing the desired structure on cohomology, then the fundamental groups of the space of such maps are nontrivial. They're astronomically big, they're giant solvable groups. I can't build such a map. I need to stop for a break.

So I'd like to make an abrupt change of topic. [missed a little]

Definition 4.4. A pointed *infinitesimal dg manifold* is an $L\langle 1\rangle_\infty$ algebra, a chain complex ("linear cofunctions"), and then look at $Sym(V)$ as a cocommutative coalgebra with a differential which is a coderivation, a dg cocommutative coalgebra. I want to talk about, I could be talking about flat or curved, I'll talk about the curved ones.

It'll be technically convenient for me to say $\partial_V = 1$. This is just a technical convenience. You have a little bit of manifold, I've chosen a linear chart, and then it has some little vector field on it which vanishes at the origin. This is a degree one vector field.

It'll make me too confused if I try to talk about reduced algebras.

Definition 4.5. What should a Poisson structure be? It's a bivector field which satisfies some Jacobi relation. Let me remind you, so, I want to talk about Poisson geometry, In homotopy world, the bivector field, well, a *pointed strongly homotopy*

Poisson-infinitesimal dg manifold is an infinitesimal dg manifold $(SV, \partial, \pi, \pi^{(3)}, \ldots)$ where π is a bivector field of degree $1-d$, and $[\pi,\pi] = [\partial, \pi^{(3)}]$, and this together will be an L_∞ coalgebra which annihilates 1.

So infinitesimal geometry, everything in infinitesimal geometry, is given by a Taylor series. The point is that some random polyvector $\pi^{(n)}$, what is it, it ends up in $V^{\otimes n}$, it produces n-many outputs, and it's a linear map $Sym(V) \to V^{\otimes n}$.

Exercise 4.1. A strongly homotopy pointed $Pois_d$ infinitesimal manifold is the same thing as a graded vector space, actually a chain complex, and a system of maps $V^{\otimes n} \to V^{\otimes m}$, one for each m, n where both are strictly positive and not $(1,1)$. If I permute the incoming strands, it's the trivial representation of the incoming strands, and the trivial or the sign on the outputs (depending on d) of degree $1-d(n-1)$, so that the commutator with a big diagram, you get a sum of compositions of degrees with two vertices.

I want this to look familiar. The connection with before, this whole diagrammatics, there is a quasi-free dioperad defined by this, freely generated by these vertices with these combinatorics on the differential. In fact this is the quasi-free dioperad you get the cobar of what you get by reading the sum in the other order, cobar of $(Frob^d)^*$. I'll stop writing cobar of the linear dual, but instead write $\mathbb{D}$, this is dioperadic, this is $\mathbb{D}^{di}(Frob_d)$, and is also $\mathbb{D}^{pr}(invFrob_d)$.

So what's the point? The point is the following. There's a canonical (homotopically) map $\mathbb{D}Frob_0 \to hP \otimes \mathbb{D}(P)$. It's also true if I put not just properads but genus-graded properads. I won't tell you what genus-graded means. What's the point? Remember that we have local Poincaré duality. Look at forms on M. This has a homotopy $Frob_d$ structure. This has a homotopy $Frob_d$ structure in the sense of dioperads. Let's say that you are given a V which is a strongly homotopy $Pois_d$ manifolds. Then V is a $\mathbb{D}^{di}(Frob_d)$-algebra so that the tensor product $\Omega^\bullet_{dR}(M) \otimes V$ is a $h^{di}Frob_d) \otimes \mathbb{D}^{di}(Frob_d)$-alg, and so a $\mathbb{D}Frob_0$ algebra. So this is the derived space of locally constant maps $M \to V$.

Definition 4.6. This is the "Poisson AKSZ construction" If V is strongly homotopy $Pois_d$, then locally constant maps from M to V is strongly homotopy $Pois_0$.

This is the classical Poisson version of an important construction, let's do the quantum version. I have $\mathbb{D}^{pr}Frob_0 \to h^{pr}P \otimes \mathbb{D}P$, this maybe first appeared in Gabriel's paper with Terilla and Tradler

Theorem 4.4. *$\mathbb{D}^{pr}Frob_0$-algebra structures on (W, ∂_W) are the same as homotopy BV structures on $Sym\ W$.*

I want to say that this is some differential operator $\Delta^{(n)}$, where this is an nth order differential operator on the infinitesimal manifold W and the rule is that $\partial_W + \Delta^{(1)} + \hbar\Delta^{(2)} + \cdots$, this whole thing is a differential. This is not a differential operator in the strict sense. I want it to be a differential in the sense that it's degree one and square zero. This is some homotopical version of the BV Laplacian.

In any case, this statement about $\mathbb{D}^{pr}Frob_0$ was true for any genus-graded properad, so if I had a properadic local Poincaré duality, that's a local action $h^{pr}Frob_d$ on forms on M, then for any strongly homotopy $Pois_d$ manifold V, I could take the locally constant maps from M to V, $\Omega(M) \otimes V$, and this would be a quantum homotopy BV space. I'm working over $\mathbb{R}[[\hbar]]$.

So what good is this? One of my punchlines is that it didn't work on the circle. So from now on I'll talk about $d = 1$ for storytelling purposes. Let me be a bit ambiguous about coalgebras versus completed symmetric algebras, and let me pretend that I did have a map $h^{pr}Frob_1 \to QLoc(S^1)$. If I had such a map, then here's the game that I can play. I can think of S^1 as $\mathbb{R} \cup \{\infty\}$ and look at the real numbers, and inside $\Omega_{dR}(S^1)$ are $\Omega_{cpt}(\mathbb{R})$. Let me say this incorrectly, the quasilocal operations on S^1 actually act on the subcomplex of compactly supported forms on $\mathbb{R}$. The reason is that you do some quasilocal operation, for small enough ϵ you stay away from ∞. So now let's pick V to be a strongly homotopy $Pois_1$-space, let's pick, track some degree shifts, let's say V is a strongly homotopy $Pois_1$ manifold, then I can look at $\widehat{Sym}(V[-1])$ is a Poisson algebra. This could be entirely in degree zero. Now I'm doing algebra, completed symmetric algebra, with everything continuous. Any power series Poisson algebra, if you unpack what it means to be a Poisson structure like this, it's the same thing.

So this is a Poisson algebra, let's look at $\Omega_{cpt}\mathbb{R}[1] \otimes V[-1]$, let me call $V[-1]$ by the name W, and this is W, and the point is that $W[1]$ is a homotopy Poisson manifold, $\mathbb{D}^{pr}(invFrob_1)$. We're pretending that $\Omega_{cpt}(\mathbb{R})$ is a $h^{pr}Frob_1$ algebra, and then all of this is $\mathbb{D}^{pr}Frob_0$.

The theorem tells me that then $\widehat{Sym}\Omega_{cpt}[1] \otimes V$ has an interesting differential which looked like the de Rham differential plus extra terms. The point of the theorem, the part I forgot to say, is that the $\Delta^{(1)}$ part vanishes in the power series topology. That was part of Gabriel's theorem.

So what can you do, in characteristic zero, $\widehat{Sym}$ is exact, well, let me say this right, if you look at the real numbers, if you choose a one-formon $\mathbb{R}$ with total integral one, then you can give a homotopy equivalence between $\mathbb{R}$ and compactly supported forms which in one direction is tensoring with the form and in the other direction is integration. From this, including a deformation retraction, you get a deformation retraction $(\widehat{Sym}(\Omega_{cpt} \otimes V)[[\hbar]], \partial_{dR})$ to $\widehat{Sym}(V)[[\hbar]]$, and then you get an explicit formula for the deformation retract between $\widehat{Sym}(\Omega \otimes V)$ with the complicated differential and $\widehat{Sym}(V)[[\hbar]], \delta$, but V is in degree 0 and so this differential is 0. Call the inclusion $\tilde{\imath}_\alpha$ and call the other $\widetilde{\int}$. Here's the game I can play. Now pick α and β, compactly supported 1-forms that are far apart. Then I can take two copies of $\widehat{Sym}(V)$ and map them by $\tilde{\imath}_\alpha$ and $\tilde{\imath}_\beta$ into $\widehat{Sym}(\Omega \otimes V)[[\hbar]]$, and I can multiply these, and do the deformed integration map back. This is a perfectly good chain map except for my product. The point is that my symmetric product is not a chain map but Δ was quasi-local, and so the failure to be a chain map vanishes mod high powers of $\hbar$ for small enough ϵ on forms that are far apart from each other. So there is still a thing to do keeping track mod high powers of $\hbar$. So what happens is that you can look at this $\star$ and it turns out that tracing through, this does not depend on α and β and is associative and deforms the symmetric multiplication in the direction of π.

This is why it's all in yellow, this is a universal deformation quantization, and I never had to take a dual, so it's wheel-free. This is in yellow. These are known not to exist.

You can do this in any dimension as well. Use a torus or something like that.

What can you do? Then I will stop.

So $h^{pr}Frob_1$ does not act quasilocally on $\Omega(S^1)$. You can model $Frob_1$ explicitly, and it has a largest maximal quasifree subproperad that does act, and that's $\mathbb{D}^{pr}(LB_1)$ mod a particular operation of genus two.

So the theorem is that the infinitesimal Poisson manifolds that admit wheel-free universal deformation quantizations are the ones where the Poisson structure $\mathbb{D}^{pr}(Frob_1) = shPois_1$ extends to a homotopy algebra over this quotient.

I'll end with a question for the remaining audience, which is, I have no idea, where else in Lie theory does this operation appear?

5. April 9, Mei-Lin Yau: Lagrangian submanifolds via surgeries

So one of the motivations is to study the isotopy problem of Lagrangian submanifolds. In particular, I want to find, given two Lagrangian submanifolds that are smoothly isotopic. How can you tell if they are still Hamiltonian isotopic. This is like a way to understand the subtle difference between symplectic topology and more general differential topology.

Over the past ten or more years, various Lagrangian submanifolds have been constructed by varying methods and some are smoothly but not Hamiltonian isotopic. Some of these can be understood as simple kinds of surgery.

You can do lots of kinds of surgery, connected sum, things like that. I want to preserve the smooth isotopy type.

The first surgery we would like to consider is called Lagrangian n-disk surgery. Most of the time $n \geq 2$. The case $n = 1$ is not very interesting. Let's start with $L^n \subset (M^{2n}, \omega)$ which is embedded. The surgery, we shall see, can apply to immersed submanifolds as well.

Now we'll do surgery to this submanifold. I need a second Lagrangian submanifold with boundary attached to this one. We say another Lagrangian submanifold D is an embedded closed Lagrangian n-disk, we say that D is a Lagrangian attaching disk of L if, first of all, $D \cap L = \partial L$, and for each $p \in \partial D$, the tangent space $T_p(\partial D)$, and its symplectic normal $T_p(\partial D)^\omega$ is spanned by the symplectic normals of T_pD and T_pL.

You can find a Darboux chart that contains D and we'll do our surgery there. Let's look at an $n = 1$ example. [pictures].

In a higher dimension you can think of D as the unit ball in $\mathbb{R}^n_\times$ and then this is like the orbit of the interval under $SU(n) \subset GL(2n, \mathbb{R})$. In the real form, elements of G are matrices $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ for $R \in SO(n)$.

In the Darboux chart, the part of L which intersects U is the orbit of γ under the same G.

Now we define surgery along D. We'll make an operation on L along D and call the resulting operation $L' = \eta_D(L)$.

We'll replace γ with γ'. You cut at D and get a manifold which is not unique but depends on the curve you chose. So you remove the part of L inside of U and you glue in the orbit of γ' under the group G.

This is $(L \backslash Orb_G(\gamma)) \cup Orb_G\gamma'$. If you don't touch the origin this will be a Lagrangian.

Proposition 5.1. (1) *L' is Lagrangian, embedded in M*
(2) *L is smoothly isotopic to L.*

(3) *Assume* $c_1(M) = 0$ *and* $\pi_1(M) = 0$. *Then* L *and* L' *have the same minimal Maslov number (I only checked the* $n = 2$ *case but I think it's the same in general).*

(4) *Suppose* $c_1(M) = 0$ *and* $\pi_1(M) = 0$ *(this implies that* ω *is exact). Suppose that* L *is monotone. Then so is* L' *if one of the two conditions are met:*

   (a) *The areas of the two shaded regions in [picture] are the same. This can always be achieved for a judicious choice of* γ'.

   (b) *Suppose* $\sigma \cdot [\partial D] = 0$ *for all* $\sigma \in H_1(L)$ *annihilated by the Liouville class of* L.

I have to explain monotonicity. For each σ you have this Maslov class, $\mu_L \in H^1(L, \mathbb{Z})$, and you also have the Liouville class $\alpha_L \in H^1(L, \mathbb{R})$. We say that L is monotone if these are linearly dependent, if there is $c > 0$ such that $\alpha_L = c\mu_L$.

If you choose this curve wisely you will preserve monotonicity.

I'll give some examples before we move on.

Example 5.1. Say L is the orbit $Orb_G\gamma$ in $\mathbb{C}^n$. If you do the surgery you will get something something like this [pictures]. In particular, when $n = 2$, this is the Chekanov torus and the Clifford torus. I think this Chekanov torus was the first torus that was constructed that is smoothly isotopic but not Hamiltonian isotopic to the Clifford torus.

Example 5.2. If you have many disks you can continue this process. Consider the equation $z_1^2 + \cdots + z_n^2 + z_{n+1}^{m+1} = 1$. This is actually a Stein manifold, a linear plumbing of, let's see, how many spheres? m copies of T_*S^n. [pictures]

So then you can do the surgery on the end of this linear plumbing, and you can do the surgery there to replace the Chekanov torus with the Clifford torus. Then you make another disk. After doing two surgeries, you get a third disk. You can basically keep doing the surgery. In this way you can count, you get at least $m+2$ smoothly isotopic Lagrangian submanifolds. I think they are all monotonic.

So in particular when $n = 2$ you get the torus.

[some discussion of related examples involving a Lefschetz fibration]

Let me mention some further properties. I want to talk about the relation to generalized Dehn twists. Maybe I'll define them later. They are a symplectic diffeomorphism with compact support in the neighborhood of a sphere. A kind of geodesic flow.

Proposition 5.2. *Let* D *be a Lagrangian attaching* n*-disk of* L. *Suppose that* ∂D *inside* L *happens to be the boundary of a disk* Δ*embedded in* L. *Then you can construct an embedded Lagrangian* n*-sphere* S *such that* $[S] = [D \cup \Delta]$, *and* $S \pitchfork L$ *at a point and applying the surgery* $\eta_D(L) \cong \tau_S^{\pm 2}(L)$ *where* $\tau_S \in Symp(M)$ *is the generalized Dehn twist along* S.

The sign in the equation is determined by the symplectic pairing of the outward normals of D *and* Δ *along the boundary. In particular, if* $\omega(\nu_D, \nu_\Delta) > 0$, *then the surgery is the positive square. When it is negative it is the negative square.*

It could be the positive or negative square, because once you get one such disk, you can get the other. Suppose you start with a disk that satisfies the condition for the positive square. Then you can find another disk in L *with the same boundary and you expect that the outward normal pairs with* ν_Δ *as negative, so the surgery along it is the negative square Dehn twist applied to* L.

Maybe I'll leave this picture.

I think, let me draw the picture again [picture]

It turns out that n-disk surgery is related to attaching Lagrangian handles. You can attach such handles but also isotropic handles. We can try to mimic the same type of principle to do other types of surgery related to the handle picture.

At this moment I can think of two kinds of generalizations. Unfortunately, these all look like Dehn twist surgeries, but a parametric version, a smaller dimension of disk multiplied by a handle. One type of thing, everything is inside a Darboux chart. Now I divide into $\mathbb{R}^{2k} \times \mathbb{R}^{2(n-k)}$. I'll pick $D^k \times P$, and P cannot be exact in this kind of picture, but this is still useful. You call the entire product E. Now the boundary of E is $S^{k-1} \times P$. If this is the intersection of $E \cap L$ you can do similar things. You remove again, if you forget P, you have a similar picture to what you saw before, and then you have to modify. Then you glue in something different. This is still Lagrangian and smoothly isotopic.

If you really want this thing to be exact, you can't have this product structure but you can think of something different.

[missed a lot]

6. April 14: Emmanuel Opshtein, Quantitative h-principle and C^0 symplectic geometry, I

[The first half of the talk was a slide talk. I do not take notes on slide talks.]

Now my aim is to give the details of proofs of two statements I made before. The first thing I want to prove is the flexibility theorem.

Theorem 6.1. *(Flexibility) Take D inside $\mathbb{R}^6$. There exists a symplectic homeomorphism φ whchi takes D to $D_{\frac{1}{2}}$.*

Then maybe

Theorem 6.2. *(Rigidity) Lagrangian rigidity*

The main tool is the quantitative h-principle. Let me say very briefly what this is. If you have two symplectic disks in any symplectic manifold (connected) then you can show that, provided they have the same area, there is a Hamiltonian path that takes one disk to the other. This quantitative h-principle says that if you start with disks that are ϵ C^0-close, then you can take the time one diffeomorphism of this Hamiltonian to have C^0 norm less than $c\epsilon$ for some universal constant c.

Before really moving on, let me make two remarks.

Remark 6.1. (1) What do we need for flexibility? We have D and we have $\frac{1}{2}D$ and we want to find a sequence φ_n that will take D to $\frac{1}{2}D$ and converge to a C^0 homeomorphism.

So if you try to send a disk to a point, not to a disk, you can find a symplectic disk of size one in any neighborhood of a point. You can find ψ_i in radius $\frac{1}{2^i}$, and this is because of the standard h-principle. So putting φ_n to be the composition $\psi_n \circ \cdots \circ \psi_1$, and this is C^0-convergent, but not to a homeomorphism.

(2) On the other hand, with the same argument, near the disk of size $\frac{1}{2}$, you can find arbitrarily close disks of size 1. Let $i_k(z) = \frac{1}{2}z, \frac{1}{2}f_\epsilon(z)$, where f_ϵ is a map from D to D_ϵ which preserves area. This is an immersion. You take this to a disk of small radius. Do this for $\epsilon_k = \frac{1}{2^k}$. What you can obviously

do is find a map φ_0 that goes to i_1, and then i_2, and take a φ_1 that goes to i_2 and so on. You get that $\varphi_n \circ i_0 = i_m$, which converges to $i' = (\frac{1}{2}z, 0)$. What is the problem with this? There is no obvious reason that the sequence C^0 converges. The h-principle will help with C_0-convergence.

Let's go to the proof. The first attempt, we have these shrinking $i_1, i_2, \ldots$ going to i' which is one half the length of the final disk. I can find ψ_1, and because of the quantitative h-principle, well, I can build all the φ_n as above. Then the C^0 norm of ψ_n can be made less than $\frac{1}{2^n}$. Therefore φ_n converges in C^0 to something.

If you argue like this, it's very hard to see that this is a homeomorphism. Now we need to make it a homeomorphism.

The claim is the following. The problem is injectivity. On the source, let me introduce some $\frac{1}{2^k}$ neighborhoods of the disk D in $\mathbb{R}^6$. Then let me take i' and define W_ϵ to be the 2ϵ-neighborhood of i'. So I defined this so that W_k contains $i_k(D)$.

The claim is the following. If you can find inductively ψ_k such that

(1) $\psi_n \circ i_{k_n} = i_{k_{n+1}}$,
(2) The support of ψ_n lies inside $\varphi_n(U_n)$,
(3) The image $\varphi_n(U_n)$ contains W_n, and
(4) $\sum ||\psi_n||_{C^0}$ converges.

By the first and last item, we get convergence to φ and the problem is injectivity. Take $x \neq y$. Then x and y may both be in D; then $\varphi(x) = i'(x)$ and this is not the same as $\varphi(y) = i'(y)$.

What if they both don't belong to D? Then they don't belong to some U_m. Therefore, by the second point, $\phi(x) = \phi_m(x)$ and likewise for y. But ψ_m is a diffeomorphism. Finally, if x is in D but y is not, then we use the third point, and $\varphi(y) = \varphi_m(y)$ which is not in $\varphi_n(U_n)$, so is not in W_{k_m} so is not in D'.

Let me now explain how to get these ψ_n. This is by induction. This is something to take i_{k_n} which I'll call i_k to $i_{k_{n+1}}$. I have i' and I'll introduce another symplectic disk along $i'_{k,\ell}$, which is [missed]. [picture].

This i_k is the image of φ_k, and therefore you can look at U_k and push it forward. For large enough ℓ, the $i'_{k,\ell}$ (it's the same one that I told you how to get), it's D is contained in $\varphi_k(U_k)$. I'll look first at ψ'_k which takes $i'_{k,\ell}$ to i', and has C^0 norm less than $\frac{1}{2k}$.

The effect is what? After ψ'_k, you have the image of $\psi'_k\varphi_k(U_k)$. What I know is that I stay in that neighborhood. Now I'm almost done. Notice that inside this neighborhood I can find some i_{k+p}, and therefore you can apply your quantitative h-principle and get a ψ''_k which has support inside $\psi'_k(\varphi_k(U_k))$ and which takes $\psi''_k \circ \psi'_k \circ i_k = i_{k+p}$, and with small norm.

Now you're done because if you put $\psi_k = \psi''_k \circ \psi'_k$, then the support lies in W_k, the sum of the norms converges, and you chose this so that W_{k+p} lies inside $\psi'_k(\varphi_k(U_k))$, then you get the last necessary containment of W_{k+p} in some $\varphi_k(U_{k-p'})$. Now you are ready to continue your induction. You have the first item, you have the second item up to shift of indices, and the third one again up to a shift of indices. Finally the sum converges.

You need to do pingpong like this, to see that you get the convergence. That's all for flexibility. I could explain one statement of rigidity but maybe you already know a lot. I can show for instance one of the results in five minutes to give the idea.

This is Laudenbach-Sikorav. If you take a closed Lagrangian L and f a symplectic homeomorphism, then $f(L) = L'$ is Lagrangian. Assume that N is middle dimensional and *not* Lagrangian, then there is a Hamiltonian on $M \times T^*S^1$ such that $\phi^t_{X_H}(N \times S^1) \cap N \times S^1 = 0$ for arbitrarily small t.

If f takes L to L', then you could cross with S^1. In M you have the Weinstein neighborhood $W(L)$. You have f_n which is a symplectic diffeomorphism that takes L to L'_n. Then $f_n(W(L))$ is a Weinstein neighborhood for L'_n. This doesn't depond on n, you can let n get large. Assume L' is not Lagrangian. Then disjoint it from itself, at some point it will be disjoint but still inside that neighborhood. Then what happnes is you have a Weinstein neighborhood and you found a φ inside this that disjoints these two from each other. But then you have a theorem that says the zero section in T^*L cannot be displaced. So this is all.

Rigidity results, this is the moral of the story, this rigidity comes from deep results in symplectic topology. You use this theorem that you have, you cannot disjoint things. What is central for instance in invariance of the area is that you cannot disjoint an exact Lagrangian from itself. On the other hand, what you see with the flexibility is that you need, well, what uses a deep result, for Lagrangians, coisotropics, you get ridigity. The C^0 geometry will not see the rest. I don't know deep statements about codimension two submanifolds? Does that mean that we forgot some of them? In some sense, the C^0 symplectic geometry points us not only toward what symplectic structures are really but also a guid toward what objects might be subject to rigidity statements or not.

7. April 23: Vijay Ravikumar: The cotangent bundle of a Grassmannian

So as Changzheng said, I'll talk about the cotangent bundle of the classical type A complex Grassmanian, at least for the first part of the talk.

What I'll talk about first is a recent result that the cotangent bundle has a nice compactification which is a Schubert variety which [missed]. I'll write down this work (which I'm not involved with) and spend 45 minutes talking about it.

Let $G = SL(n, \mathbb{C})$, let B be the upper triangular matrices (the Borel subgroup of G). I can draw a Dynkin diagram of type A:

$$\bullet^{s_1}{}_{\curlywedge}[r] \quad \longleftarrow \bullet^{s_2}{}_{\curlywedge}[r] \longrightarrow \bullet^{s_d}{}_{\curlywedge}[r] \longrightarrow \quad \bullet^{s_{n-1}}[r]$$

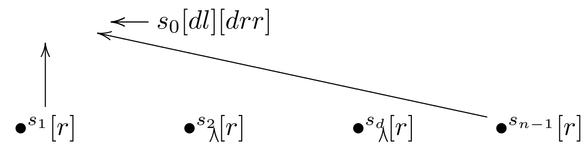
Let W_0 be the Weyl group generated by S_0 these reflections. Let $J = S_0 \backslash \{s_d\}$ and $(W_0)_J$ the subgroup generated by J, with P_J the parabolic subgroup with Weyl group $(W_0)_J$, that is, $B(W_0)_J B$.

If we're just looking at $SL(4, \mathbb{C})$, if we let $J = \{s_1, s_3\} \subset S_0 = \{s_1, s_2, s_3\}$, then the parabolic subgroup P_J will look like matrices with 0 as a lower left 2×2 block. So $(W_0)_J$ can be thought of in terms of matrices, well, the Weyl group itself is permutation matrices.

Let $X_J :- G/P_J$ and since G is S_0 minus a single reflection, this will be $Gr(d, n)$. This is all in the finite dimensional case.

Now I want to use similar notation for the affine case. I'll use calligraphic font for infinite dimensional groups.

Let $\mathcal{G} :- SL(n, F)$ where F is the field of Laurent series $\mathbb{C}((t))$. This is a Kac–Moody group, an affine Kac–Moody group of type $\tilde{A}_{n-1}$. Although it's an infinite group and could be intimidating at first, a lot of the same formalism works. The Dynkin diagram is



I'll denote W as the Weyl group of $\mathcal{G}$. I'll denote the set of reflections $\{s_0, \ldots, s_{n-1}\}$ by S. This generates an infinite group which models the loop group of $SL(n, \mathbb{C})$. I'll keep J the same as before so it doesn't contain s_0 or s_d.

Let me identify $W^J \subset W$ which is the set of minimal length coset representatives of W/W_J

We can define the parabolic subgroup $\mathcal{P}_J \subset \mathcal{G}$ to be equal to the parabolic subgroup with Weyl group W_J. I'll give a more precise definition in a few minutes. People call this the *parahoric* subgroup sometimes.

Now I'm almost ready to state my main theorem. Let me define $\mathcal{X}_J :- \mathcal{G}/\mathcal{P}_J$, a two-step affine partial flag variety. The elements of W^J index Schubert varieties in this.

Let me write it down, W^J is in one to one correspondence with Schubert varieties $\mathcal{X}_J(W)$.

Theorem 7.1. *(Lakshmibai 2015) There exists an element $y \in W^J$ such that $T^*X_J \hookrightarrow \mathcal{X}_J(y)$ as a dense open subset. Furthermore these are both fiber bundles over X_J. The fibers all embed as dense open subsets of the fibers, which are the Grassmannian itself.*

A consequence is that this $\mathcal{X}_J(y)$ is smooth.

The map is a G-homogeneous fiber bundle map, where G is $SL(n, \mathbb{C})$.

Although there are infinitely many elements of W^J, they're all finite dimensional, which makes this infinite dimensional manifold less intimidating.

Let me give a few definitions.

Definition 7.1. In $SL(n, \mathbb{C})$, let's fix a maximal torus of diagonal matrices inside B, and let N be the normalizer (unimodular matrices) of the torus. F as before is the field of Laurent series. Let A be the ring of formal power series, and define a map $\pi : G(A) \to G$ which sends t to 0.

Now define $\mathcal{B}$ to be $\pi^{-1}B$, this is the Borel subgroup of $\mathcal{G}$. At the diagonal and above you have arbitrary power series. Below the diagonal you have no constant term.

Also I'll just note here that the affine Weyl group can be thought of as $N(C[t, t^{-1})/T$, basically permutation matrices with Laurent polynomial entries and determinant one, so that imposes a lot more conditions.

Then $\mathcal{P}_J = \mathcal{B}W_J\mathcal{B}$.

I still have a Bruhat decomposition, $\mathcal{G} = \bigsqcup_W \mathcal{B}w\mathcal{B}$, which descends to a decomposition of the Grassmanians, the affine flag varieties. For any K a set of reflections in S, then $\mathcal{X}_K = \mathcal{G}/\mathcal{P}_K$ is $\bigsqcup_{W^K} \mathcal{B}w\mathcal{P}_K \pmod{\mathcal{P}_K}$. Furthermore, for $w \in W^K$, then $\mathcal{X}_K(w) :- \bigsqcup_{v \leq w} \mathcal{B}v\mathcal{P}_K \pmod{\mathcal{P}_K}$, under the Bruhat order.

Imagine that w doesn't contain s_0, then it looks like an element of the finite Weyl group. In that case, this is equal to the finite Schubert variety $\bigsqcup BvP_{K\cap S_0}$ mod $P_{K\cap S_0}$. Then we think $w \in W^{K\cap S_0}$, so it indexes a finite Schubert variety.

[some discussion]

Now let's talk about a specific concrete example. In particular, well, all Schubert varieties of the finite Grassmanian can be interpreted in the affine Grassmanian. Let w_0 be the longest word in $W_{S_0}^{J\cap S_0}$, which is $W^{J\cap S_0} \cap W_{S_0}$. Similarly, let w_d be the longest word in $W_{S_d}^J$, where $S_d = S - \{s_d\}$. Then we have the following lemma:

Lemma 7.1. *$X_J \cong \mathcal{X}_J(w_0) \cong \mathcal{X}_J(w_d)$.*

So let's understand this. The first isomorphism, we want the Schubert variety corresponding to this longest word w_0. If you know about Grassmanians, the Schubert variety corresponding to the longest Grassmanian is the Grassmanian itself. So we can then think of this in the affine world. There's a complete symmetry of the Dynkin diagram, and now you could the same thing deleting s_d and look at the affine Schubert variety there and you get the same thing.

Define $y = w_0 w_d$. It's nice to notice that y is reduced, is in W^J, and $\mathcal{X}_J(y)$ is stable under left multiplication by $\mathcal{G}$.

What I'll do now is try to give a description of what this map looks like. I know it has, I've said what y is, it would be nice to see how this embeds the cotangent bundle into this Schubert variety.

Let's look at $Gr(2,4)$. First of all, very quickly, what we'll do here, specifically, we're taking the cotangent bundle, we take the base, let's call the map μ, we'll take X_J isomorphically to $\mathcal{X}_J(w_0)$ and will take T_0^*X as a dense open to $\mathcal{X}_J(w_d)$. It will embed the tangent space of the identity as the biggest Schubert cell of the Grassmannian. That's a little bit of motivation.

The cotangent bundle itself, if you recall that any parabolic subgroup has a decomposition $M \rtimes U_0$ into a [missed] and a unipotent radical and at the level of Lie algebras this becomes a direct sum $\mathfrak{m}_0 \oplus \mathfrak{u}_0$. So for $Gr_{2,4}$, then $J = \{s_1, s_3\}$ and in that case the parabolic subgroup is matrices with a 0 two by two block in the lower left. The decomposition is given by a block diagonal matrix made of two 2×2 blocks and the unipotent radical, which is a 2×2 block in the upper right plus the identity.

At the Lie algebra level, you get the weight spaces from looking at weights that contain excluded node. In this case we're looking at the weights that involve the node s_2. So there are two quick identifications that show us how the cotangent bundle can be embedded at all. The basic fact is that T^*X_J is the fiber product of G with the unipotent radical $\mathfrak{u}_0$ over P_J. In this case $\mathfrak{u}_0$ is blocks in the upper right. One way to see this, looking first at the tangent space of the identity, this can be thought of as $(\mathfrak{g}/\mathfrak{p}_J)^*$. This is a subset of $\mathfrak{g}^*$ which maps via the Killing form to $\mathfrak{g}$ and lands in $\mathfrak{u}_0$ inside it.

It follows from that that the entire cotangent bundle is $G \times \mathfrak{u}_0/(g, X) \sim (gp, p^{-1}Xp)$ for all $p \in P_J$. The cotangent bundle is this fiber product.

There's one more identification I want to mention quickly, the opposite unipotent radical U_0^- sits inside $SL(n,4)$ but it can naturally be identified with the big cell of the Grassmanian, when we mod out by P_J we map to this U_0^-. So we can think of the big cell as sitting inside $SL(n,4)$ itself instead of its quotient. Similarly, just as we can think of G sitting inside of $\mathcal{G}$, when we look at constant term matrices G_0 it's the Levi subgroup of the parabolic subgroup $\mathcal{P}_0$. Similarly, $\mathcal{G}_d$ is the Levi subgroup of $\mathcal{P}_d$. This gives another embedding of $SL(n, \mathbb{C})$ into $SL(n, F)$. In the lower left you have arbitrary numbers time t and in the upper right times t^{-1}, with arbitrary constant numbers on the diagonal blocks. Similarly we can think of this other unipotent radical U_d^-, using the symmetry of the Dynkin diagram here, and in this case as matrices it gives us matrices with t^{-1} terms in the upper right. We have $\mathfrak{u}_0 \mapsto U_d^-$ by taking $X \mapsto exp(Xt^{-1})$. This construction works for all Grassmanians and cominiscule Grassmanians of all Lie types as well. This kind of, it becomes harder to generalize to noncominiscule Grassmanians. We've embedded the unipotent radical, and just as we can translate around by the action of G, we can have G act on U_d^- which we will send to U_d^- mod $\mathcal{P}_J$ and look at the action of G_0 on that. It turns out that when you act by G_0 that fills out the entire Schubert variety. It also gives us the cotangent bundle.

That's a very handwavy explanation for how that works.

8. Michael Usher: Persistent homology and Floer theory

This work is joint with Jun Zhang. Thank you for the invitation. It's good to be in Pohang again. I'll assume that most people know what Floer theory is. Probably more people in the world but fewer people in this room know what persistent homology is. This comes out of applied topology. Can you infer the topology of some object in a high dimensional Euclidean space by sampling points from it? This developed in the early 2000s. Separately, Floer theory has continued on its way. A key structure in both has been filtrations on complexes. Polterovich, notably, has started thinking that persistent homology is a good way to think about what is happening in Floer theory. This is supposed to show as well that notions from persistent homology can be extended from Morse to Novikov things. We have theorems that parallel the standard theorems of persistent homology but with very different proofs.

I'll begin by introducing the basic algebraic object in persistent homology.

Definition 8.1. An (R)-*persistence module* over a field K is a collection of vector spaces $\mathbb{V} = V_t$ for $t \in \mathbb{R}$ with linear maps $V_s \to V_t$ when $s \leq t$ such that $V_t \to V_t$ is the identity and $V_r \to V_t = V_r \to V_s \to V_t$. As concisely as possible, this a functor from $\mathbb{R}$ as a poset to vector spaces.

What are some examples. There's a very simple algebraic example, or family of examples. Say I have an interval $I \in \mathbb{R}$. What I can do is define $(K_I)_t$ as K if $t \in I$ and 0 otherwise. I should tell you the maps. The maps are the identity when you can and 0 otherwise. We call these interval modules.

Why do we talk about these in topology? You can consider the following. Take a topological space X and a function f, not necessarily continuous, to $\mathbb{R}$. Then let $V_t = H_k(\{x | f(x) \leq t\})$. There are natural maps that come along with this. The s version is contained in the t-version for $s \leq t$, and I can take the map induced by this inclusion. This should remind you of Morse theory.

So part of why I mention the algebraic example is the following classification theorem, due to Vomorodian–Carlsson 2004, and Crowley-Boevey.

Theorem 8.1. *Assume that you have a persistent module with V_t finite dimensional. Then there is a direct sum decomposition $(V_t)_{t\in\mathbb{R}}$ as $\bigoplus_I (K_I)_t$ for a uniquely determined collection of intervals I, called the barcode of (V_t).*

To find the dimension of V_t, what is the dimension? For each interval, is t in that interval? If it is, it contributes 1; otherwise it contributes 0. More generally, the rank $V_s \to V_t$, that's the number of intervals containing both s and t.

Let's see what you get for a certain function on the circle. Consider the height function for this picture. [picture] I could think about a persistence module for every k. I could do that sublevel set construction for various values of k. If k is not 0 or 1 then you get the 0 group, and the empty barcode. For $k = 1$, V_t is K when $t \geq 5$ and 0 otherwise, and by itself that doesn't tell me the bar code because in principle you have to know what the maps are. It's true that above 5 these are always the identity map. This whole persistence module is $K_{[5,\infty)}$.

The more interesting value of k is 0. If I just tell you the V_t inmididually that doesn't give you the whole bar code, but it's useful. So I'll look at the 0th homology. I have 0 if $t < 0$. Crossing 0 I get 1 connected component. Then I continue until 1 and get 2 connected components. Then from 1 to 2 I'm two dimensional, then three dimensional until height 3 I have two and then crossing 4 I get one and that persists.

[pictures.]

Now I want to take a more Floer theory style view of this. The bridge is Morse theory. Milnor's book is largely about homology of sublevel sets. In Schwarz it's ODEs. I'll transition from sublevel sets to ODEs.

Suppose I have a Morse function f on a compact manifold X. Then one can construct a Morse complex $CM_*(f)$ which is $\bigoplus K\langle p\rangle$ where the sum is over critical points, graded by the Morse index, and the boundary operator goes from $CM_{k+1}(f) \to CM_k(f)$. I should write the boundary of p and that's some matrix elements, ∂p is the sum of $n(p,q)q$, and that's the signed count of negative gradient flow lines from p to q. It's a well-understood fact that this boundary operator squares to zero and when you take homology gives you something isotopic to the homology of the manifold. This behaves nicely with respect to sublevel sets and I'll explain how.

So there's a "*filtration function*" ℓ on $CM_*(f)$ which takes $\sum a_i p_i$ to, well, asks the highest critical value, yields $max\{f(p_i)|a_i \neq 0\}$. This gives me an $\mathbb{R}$-valued filtration on the complex.

Part of the point of negative gradient flow is that it goes down, and so this filtration is stable under the differential, it's a subcomplex and not just a subspace. The homology is the same as, $HM_*^t(f) \cong H_*(\{f \leq t\}; K)$. I get an inclusion induced map compatible with the isomorphisms. If you know a proof that the homology of the whole thing is the homology of M, you can easily get to these statements. I get an algebraization that gives a persistence module, and according to the classification theorem I get a bar code.

Floer complexes are different from Morse complexes. You can't do something direct like this, or if you do, you get a different thing. The decomposition you can follow and ask what happens. So you get a persistence module $\{HF^t(H)\}$ for Hamiltonian Floer theory on *monotone* symplectic manifolds. There is not an obvious way of expressing this as a sublevel set of some space, but algebraically it is that.

This works straightforwardly, you get a bar code. [some comments]. For non-monotone symplectic manifolds, $HF(H)$ is an analogue of Novikov homology, not Morse homology. I won't define Floer things, but I'll say a few words about Novikov homology. Instead of starting with a Morse function, you start with a closed one-form (thought of as the derivative of that function) which is transverse to the zero section of T^*X but not necessarily exact. Flow lines for the closed 1-form, those might close up. Keep track of extra information by taking a cover. Let Γ be the image of the homomorphism $\pi_1(M) \to \mathbb{R}$ given by integration around loops. I can form a covering space $\tilde{X}$ over X whose deck transformation group is Γ. Then $\pi^*\theta = d\tilde{f}$ for some $\tilde{f} : \tilde{X} \to \mathbb{R}$. I have $\widetilde{f}(gp) = \tilde{f}(p) + g$.

I can think about trying Morse theory but now I'm on a non-compact manifold. If I try to write down the formula for the Morse boundary operator, this is likely going to be an infinite sum. This is infinite dimensional as a chain complex over K. I can work over a Novikov ring (field) and then get finite dimensional complexes. The actions of field elements changes the filtration level, so this is not a persistence module over the Novikov ring.

I take $\Lambda = \Lambda^{K,\Gamma} = \sum a_i g_i$ where $a_i \in K$ and $g_i \in \Gamma$. Either the sum is finite or the g_i diverge to $-\infty$. The Novikov complex is $\bigoplus_p \langle \Lambda^{K,\Gamma}\tilde{p}\rangle$. [Some description]

Define $\ell(\sum a_i\tilde{p}_i) = \max \tilde{f}(\tilde{p})$ and $CN_*^t(\widetilde{f})$, that is, $\{c|\ell(t) \leq t\}$ is a subcomplex of K-vector spaces but not over the Novikov ring.

So we have some sort of persistence module kind of thing. This is not arbitrarily bad, there is a sort of coherent behavior under actions of the group elements. My results are basically, they show that things work out as well as one could hope given that issue.

I'll borrow a notion from non-Archimedean normed vector spaces, that of orthogonality, where $\ell(v + w) \leq \max\{\ell(v), \ell(w)\}$.

Definition 8.2. We say that $v_1, \ldots, v_n$ in $CN_*(\widetilde{f})$ are *orthogonal* if for all λ_i in Λ, we have $\ell(\sum \lambda_i v_i) = max \ell(\lambda_i v_i)$.

Then if $p_1, \ldots, p_n$ are the premiage of 0 under θ and $\tilde{p}_i$ are arbitrary lifts, then they are orthogonal. Consider $\tilde{p}_1$ and $\tilde{p}_1 + \tilde{p}_2$. These *might* be orthogonal. They'll be orthogonal if $\ell(\tilde{p}_1) \leq \ell(\tilde{p}_2)$

Theorem 8.2. *(U.–Zhang)*

- A *The Novikov complex (and algebraically similar complexes) $CN_*(\tilde{f})$ decomposes as a direct sum of very simple kinds of complexes, $0 \to 0 \to \cdots \to \langle y_i\rangle \to \langle \partial y_i\rangle \to 0 \to \cdots$ and $0 \to \ldots \to \langle x_i\rangle \to 0 \to \cdots$ where all of these guys are orthogonal.*
- B *Suppose you have a decomposition like this. Then elements of $\mathbb{R}/\Gamma \times [0, \infty]$, denoted $([a], L)$, given by $a = \ell(\partial y_i)$ and $L = \ell(y_i) - \ell(\partial y_i)$. For the second type I'll take $[\ell(x_j)], \infty$, this is independent of the decomposition you choose. We call this multiset the "verbose barcode" and the submultiset with $L > 0$ the "concise barcode."*
- C *If $\Gamma = \{0\}$, in that case $\Lambda = K$ and $CN_*(\widetilde{f})$ is the Morse complex, then there is a bijection between our concise barcode element $([a], L)$ and the barcode element $[a, a + L]$.*
- D *Two "Floer-type complexes" are filtered chain homotopy equivalent if and only if they have the same concise barcode.*
- E *There's a continuity result, there's the "bottleneck stability theorem." If you C^0-perturb your function, having thought about Floer theory for some period of time, this theorem existed in that literature and that was very striking, if you perturb your function f, what happens to the barcode? There's a*

partial bijection between the barcodes? Some things can be in bijection with the empty interval. You can match things up either similar looking intervals in the other one or the empty interval. We prove an analogue of that, but it's too complicated, so I think I'll end here.

[some discussion]

9. August 13: Leonardo Constantin Mihalcea: An affine quantum cohomology ring and periodic Toda lattice

There was an unexpected result in 1999 about how the relations of the quantum cohomology of flag manifolds are related to the conserved quantities of the Toda lattice of the dual. People immediately wanted to generalize this, and one idea was that this should apply to affine flag manifolds, which are infinite dimensional and thus we cannot do Gromov–Witten theory there. I will describe an intermediate ring where all of this can be done.

I'll state the problem and then give an answer. In the third hour of my talk I'll give an idea of my proof. This is joint with Mare, and builds on work with Buch, Chaput, and Perrim.

In the first part I'll talk about flag manifolds. I'll introduce flag manifolds in all types and give you an example to think about. Pick G to be a complex simple Lie group, for example $SL_n(\mathbb{C})$ and inside it let's fix a Borel subgroup B, for example the subgroup of upper triangular matrices. The flag manifold is G/B. In the case when G is SL_n and B is the upper triangulars, you get the space of flags $Fl(n)$, which is $F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n$. In types A through D, you can realize these as submanifolds of this flag manifold $Fl(n)$.

I'll talk about the Weyl group W, for $Fl(n)$ the Weyl group is S_n. Now let's talk about Schubert varieties. These are certain distinguished subvarieties, this has an action of G which is transitive above, and of the opposite Borel, the lower triangular matrices in our example. The Schubert varieties are closures of the opposite Borel's orbits. So $Y(W) = \overline{BwB/B}$ and $Y(W) = \overline{B^{-}wB/B}$.

The Schubert varieties are important if you want to study the cohomology of G/B, it turns out that $H^*(G/B)$ is free on $[X(w)]$ or $[Y(w)]$. So the length of w is the smallest number of simple transpositions necessary to write w, and $[X(w)] \in H_{2\ell(w)}(G/B)$ or $[Y(w)]$ has $2\ell(w)$ as *codimension*. So anyway, $[Y(u)] \cdot [Y(v)] = \sum c_{u,v}^{w}[Y(w)]$. These numbers are nonnegative integers and calculating them is an important question. We have algorithms but no formula where they are manifestly positive.

Now let me talk about the quantum cohomology QH^* of the flag variety. This is a ring; as a module it's very easy. It's the cohomology module tensored with $\mathbb{Z}[q]$, where q is not a single parameter but a multiparameter indexed by a basis of $H_2(G/B)$. If you believe the description I gave, you'll see that the second homology group, and this will be important, is generated by $[X(s_i)]$ where s_i is a simple reflection in W. This is just $(i, i+1)$ in my example.

As a $\mathbb{Z}[q]$ module, this has a basis given by the Schubert classes. What's interesting is the multiplication. If I multiply $[Y(u)] * [Y(v)]$ I get $c_{u,v}^{w;d} q^d [Y(w)]$ where d is a multiindex, and q^d is $q_i^{d_i}$. This is a graded ring, The grading, let me spell this out concretely, it says that $\ell(u) + \ell(v) = \deg q^d + \ell(w)$. So what is the degree of q^d? The degree of q_i is 2, and this is the integral of the first Chern class of the Tangent bundle of th flag manifold intersected with the Schubert curve, and that's two. The degree of q^d is $2(d_1+\cdots+d_n)$. I didn't tell you what are $c_{u,v}^{w,d}$. These are Gromov–Witten invariants.

Some people are less familiar, so let me give a quick decription of these. But first, here's an example of some easy Schubert varieties., let me have a flag $F_1 \subset F_2 \subset \mathbb{C}^3$, and then $w = s_1 = (12)$. Then $X(s_1) = \{F_1 \subset \mathbb{C}^2 \subset \mathbb{C}^3\}$. Then $X(s_2) = \{\mathbb{C} \subset F_2 \subset \mathbb{C}^3\}$.

Okay, Gromov–Witten, so $c_{u,v}^{w,d} = \langle [Y(u)], [Y(v)], [X(w)]\rangle_d$, this is the Gromov–Witten invariant that counts rational curves, maps $\mathbb{P}^1 \xrightarrow{f} G/B$, where I have three marked points, $f(0) \in g_1 Y(u)$, $f(1) \in g_2 Y(v)$, and $f(\infty) \in g_3 X(w)$, where g_i are general in G in a certain sense. If $d = 0$, then a rational map of degree zero has image a point, then $f(i)$ is a point. I want to count intersections of $X(w)$, $Y(u)$, and $Y(v)$. So specialized to $d = 0$ I get the regular multiplication.

We have formulas. We don't have positive formulas. There are algorithms out there.

Just from this description you can deduce some basic facts. You can deduce the fact that as a ring, we have that the cohomology, the usual cohomology, has generators and relations, we have $\mathbb{Q}[p_1, \ldots, p_n]/\langle \text{RELATIONS}\rangle$. These are called Borel relations. These are elementary symmetric functions in the p_i for the A_n case. From this presentation, you can deduce the following fact. The presentation is similar, it's got the ground ring $\mathbb{Q}[q]$, and I have the same number of relations, but they're deformed, $\tilde{R}_i$ is a deformation of the ordinary relation R_i. These are not so hard to find in type A, so let me write [unintelligible], Fomim–Gelfand–Postnikov, Astashkevich–Sadol. It was a bit of a surprise when the solution for all the G were given. The relations are the same as the conserved quantities for the ordinary Toda lattice in type A. Let me state the type A version of this, being a little bit vague.

A digression on integrable systems. This is a mechanical system that describes the motion of n charged particles on a line. You know the positions and charges and the point is to describe the dynamics of the system. There are Hamilton equations. This is described by a Hamiltonian, $H(p,r) = \frac{1}{2}\sum p_i^2 - e^{2(r_i - r_{i+1})}$. The Hamilton equation is, for $q_i = e2(r_i - r_{i+1})$, is

$$\frac{\partial q_j}{\partial t} = \frac{\partial H}{\partial p_j}; \qquad \frac{\partial p_j}{\partial t} = -\frac{\partial H}{\partial q_j}.$$

What we want is certain conserved quantities. There is a lax matrix which looks like

$$A = \begin{pmatrix} p_1 & q_1 & & & & & \\ -1 & p_2 - p_1 & q_2 & & & & \\ & -1 & p_3 - p_2 & q_3 & & & \\ & & & \ddots & & & \\ & & & & -1 & p_n - p_{n-1} & q_n \\ & & & & & -1 & -p_n \end{pmatrix}$$

Then the determinant of $A + \lambda I = \lambda^{n+1} + \sum \lambda^{n-1}\tilde{R}(p,q)$. So this is in a totally different area!

Theorem 9.1. *(Givental–Kim, Kim)*

$$\langle \tilde{R}_i(p,q)\rangle$$

is the ideal of relations for $QH^*(G/B)$.

There is a Toda lattice for any Dynkin diagram. You get exactly the relations in quantum cohomology.

What is the statement of my problem? Immediately after this was given, you had people like Guest–Otofuji who made the following

Conjecture 9.1. *If there is a quantum cohomology for* affine *flag manifolds which satisfies certain natural properties, then I'll have a description like this*

$$QH^*(\mathcal{G}/\mathcal{B}) = \frac{\mathbb{Q}[q][\tilde{p}_1, \dots \tilde{p}_n, \tilde{p}_0]}{\textit{conserved quantities for the affine Toda lattice}}.$$

The affine Toda lattice describes the movement on the circle, the affinization of the line.

The idea is that first of all, you consider the quantum Toda lattice. This is a quantum integrable system. So you observe that you start with something that is a relation, you just see that. Then you establish that a differential operator that commutes with the Hamiltonian gives a relation in quantum cohomology.

So this was the conjecture for type A affine flag manifolds. So you'd like several things. You'd like to have a quantum cohomology ring. Then you'd want to be able to compute, and say that these correspond to the affine Toda lattice.

My story has two parts, a good part and a bad part. We can do something like this but not quite for $\mathcal{G}/\mathcal{B}$. We define an intermediary quantum cohomology ring. With respect to that ring, we can prove that it has a presentation with relations given by the affine Toda lattice.

I'll give a proposal for how to define certain Gromov–Witten invariants for affine flag manifolds. To understand how to define these, let me recall how you solve the similar problem in the finite case. Because this is infinite dimensional, there is no moduli space of stable maps. I want an alternate definition that is inspired by what happens in the finite case.

So let me take a digression on Gromov–Witten invariants on G/B. How do you describe the quantum cohomology ring. The generators correspond to divisor classes. It's enough to know the quantum Chevalley formula, $[Y(s_i)][Y(u)] = c^{w,d}_{s_i,u} q^d [Y(w)]$.

To make this precise, I have to tell you the Gromov–Witten invariants $c^{w,d}_{s_i,u}$. Just by definition, let's do the computation. This is

$$\langle [Y(s_i)], [Y(u)], [X(w)] \rangle_d.$$

I don't want to recall the moduli space of stable maps. This number is obtained by interesction theory on the moduli space of stable maps of degree d with three marked points

$$\int_{\overline{M}_{0,3}(G/B,d)} ev_1^*[Y(s_i)] ev_2^*[Y(u)] ev_3^*[X(w)]$$

The space changes from being a $\mathbb{P}^1$ to being the union of rational curves. There is an evaluation map for each marked point. The map takes f and sends it to $f(p_i)$, the ith marked point.

When a divisor shows up it can be taken out of the Gromov–Witten invariant, and then we have to intersect the divisor with the curve, so you get

$$([Y(s_i)] \cap d) \int_{\overline{M}_{0,2}(G/B,d)} ev_2^*[Y(u)] ev_3^*[X(w)].$$

But this is

$$([Y(s_i)] \cap d) \int_{G/B} (ev_3)_* ev_2^* [Y(u)] \cdot [X(w)].$$

It's easy to intersect a B-stable variety with a B^--stable variety. So once you can calculate this pushforward you're basically done. So this is where the notion of curve neighborhoods comes up.

Definition 9.1. Let $\Omega \subset G/B$ a subvariety and $d \in H_2(G/B)$. The curve neighborhood of Ω is the union of all rational curves of degree d passing through Ω. A scheme-theoretic way of saying this is $ev_1(ev_2^{-1}\Omega)$. This is a subvariety of G/B.

For example, for $\mathbb{P}^2$, I'll give a couple of examples. Say $X = \mathbb{P}^2$, with $d = 1$, and Ω a point. Then $\Gamma_1(pt)$ is the union of all lines passing through a point. So then you get all of $\mathbb{P}^2$. But for G/B, this turns out to be nontrivial for certain degrees. Maybe I'll stop here and explain after the break what curve neighborhoods have to do with this.

So I'll be supported on the curve neighborhood of $Y(u)$. We can say the following fact,

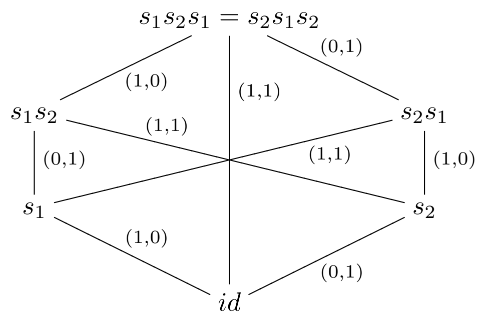
Theorem 9.2. *(Buch–Chaput–M.–Perrim, Buch–M.) The curve neighborhood of the Schubert variety $Y(u)$ is again a Schubert variety $Y(u(d))$. The curve neighborhood of the irreducible variety is itself irreducible. We have an algorithmic description of $u(d)$, which is an opposite Hecke product between u and z_d, and I'll explain what the Hecke product is, coming from the Lie combinatorics, and z_d is defined by the property that $Y(z_d)$ is the curve neighborhood of a point. This Weyl group element is used to define what is z_d.*

The opposite Hecke product is, $u \bullet s_i$ is us_i when this drops the length of u and u otherwise. You take your Weyl group element, write it in terms of simple reflections, and iterate. It turns out not to depend on decomposition or order of parentheses. You can write $z_d = z_{d'} \bullet s$.

Say I want to multiply this by s_3, then I have to multiply by s_3 and see whether the length decreases or increases. So what you get is u. If you multiply by s_1 you get $s_1 s_2$.

Let's take again the flag variety $Fl(3)$. A degree is d_1, d_2. There are two divisors, and the Weyl group is S_3 generated by s_1 and s_2. I'm interested for example, let's compute $\Gamma_{(1,0)}(pt)$, $\Gamma_{(0,1)}(pt)$, and $\Gamma_{(1,1)}(pt)$. Because the point is a Schubert variety, I'll have to give you Weyl group elements. You have to analyze the geometry of the Toda-stable curves. This being a homogeneous space it has a lot of structure. There is this moment graph, which encodes T-fixed points (vertices) and T-stable curves (edges) between two T-fixed points. Here T is a maximal torus. In this case it's a maximal torus in SL_3.

The moment graph is



So from this $\Gamma_{(1,0)}(pt) = s_1$ and $\Gamma_{(0,1)}(pt) = s_2$. The neighborhood for $(1,1)$ is the whole flag manifold, you can go from the identity up to s_1s_2, or to s_2s_1 by two edges of appropriate total weight, or to $s_1s_2s_1$ by only one edge, and if you have any vertex, it contains all the vertices below it. And $(1,1)$ is the smallest degree with this property.

So how do I define curve neighborhoods for the affine flag manifolds? these are, let $\mathcal{G}$ be $G(\mathbb{C}[t,t^{-1}])$, and $\mathcal{B} = \{g \in G(\mathcal{C}[t])\}$ with the property that $g(0) \in B$. Then $\mathcal{G}/\mathcal{B}$ is an affine flag manifold. It has all the data I described before. The Weyl group is the affine Weyl group W_{aff}. Now $X(w) = \overline{\mathcal{B}w\mathcal{B}/\mathcal{B}}$ is a Schubert variety.

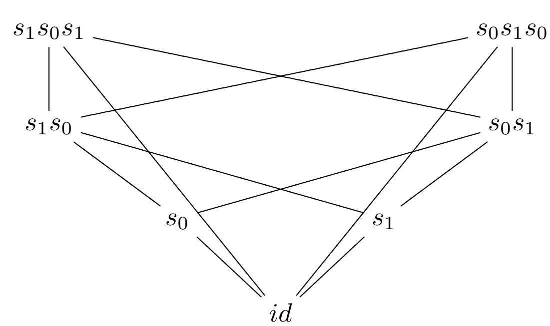
Note that $\mathcal{G}/\mathcal{B}$ is an ind-variety, it's a union $\mathcal{X}_1 \subset \mathcal{X}_2 \subset \cdots$ where all of these are finite dimensional.

You can prove that the $\mathcal{X}_i$ are unions of finite dimensional Schubert varieties. Now $H_*(\mathcal{G}/\mathcal{B})$ is $\bigoplus \mathbb{Q}[X(w)]$ where w varies over W_{aff}.

I need to understand the curve neighborhoods, and these had a set theoretic description as the union of curves that pass through the variety. Atiyah observed it still makes sense to consider the union of all rational curves that pass through a point, that's still finite dimensional. We also reobserved and proved that.

Fix $d \in H_2(\mathcal{G}/Bb)$, so this is an effective combination of the Schubert curves, in the affine Weyl group. To be on safe ground, let's fix $w \in W_{aff}$ and define $\Gamma_d(X(w))$ to be the union of all rational curves of degree d passing through, intersecting $X(w)$. You need to prove this is well-defined. I'll wave my hands. Your $X(w)$ lives in some stratum. Then the curve neighborhood there is well-defined. Then the curve neigborhood stabilizes as the strata increase. Then we prove that this definition makes sense. It's easy to see that this is a union of Schubert varieties, but identifying them is the tough part.

Take $\mathcal{G}$ of the type $A_{1,1}$, and the moment graph looks like this:



and you can get the degree of an edge by counting how many 0 and 1 it adds.

So now you get an integral of $[Y(u \bullet z_d)][X(w)]$. This is very easy to compute, it's either 0 or 1. Given that the curve neighborhoods exist in the affine case, you want to use this to define your coefficients.

Definition 9.2.

$$\langle [Y(s_i)], [Y(u)], [X(w)]\rangle_d = ([Y(s_i)] \cap d) \int_{\mathcal{G}/\mathcal{B}} [Y(u)] \cap [\Gamma_d(X(w))].$$

Since I only have ten minutes left, let me say, we can define a kind of Chevalley operator, $[Y(s_i)]*_{aff}[Y(u)]$ is the classical part plus $\sum \langle [Y(s_i)], [Y(u)][X(w)]\rangle_d q^d [Y(w)]$. I need to tell you the degree of the q_i this is something like $intc_i(T_{\mathcal{G}/\mathcal{B}}) \cap [X(s_i)]$. The problem is that the tangent bundle is a bit tricky. But Dan Freed made sense of this and found that this was 2. You get that the degree is 2. So it makes sense to talk about the degrees of the two sides. There is one more parameter from the one more reflection.

What can we say about this? Unfortunately, this is not what we were looking for.

Theorem 9.3.

$$[Y(s_i)] * ([Y(s_j)] * [Y(w)]) = [Y(s_j)] * ([Y(s_i)] * [Y(w)]) \pmod{q_0 q^{\theta^\vee}}$$

Here θ is the highest root. You expand in the simple roots and take the coefficients. This is the failure of a certain connection to be flat. Whatever the Dubrovin connection would be, it's not a flat connection.

The reason I took the flag manifold to be in Laurent polynomials and not Laurent power series is, instead of taking $\mathcal{G}/\mathcal{B}$, take $\mathcal{G}/\mathcal{T}$. Then the fiber here is contractible. These two spaces are the same homotopically. The advantage of working with $\mathcal{T}$ is that now I can evaluate at 1. This will go into G/T, where T is the maximal torus in the Borel. So what does this give us. This gives an algebraic map from G/T to G/B. This buys us a map between the cohomology of the finite flag manifold and the cohomology of the affine flag manifold $H^*(G/B) \to H^*(\mathcal{G}/\mathcal{B})$, and this map is injective. In other words, you have a copy of the finite flag manifold inside the copy of the affine flag manifold. I can try to translate, not considering all the classes of the affine case but only those in the image of this map.

What I'm going to do is to consider an operator obtained as $(\quad) *_{af} e_1^*([Y(s_i)])$. I can pull back and multiply, call these T_i. The previous operators are not commutative. It turns out somehow magically that these operators T_i will be commutative on $H^*(\mathcal{G}/\mathcal{B}) \otimes \mathbb{Q}[q]$.

Then this defines a quantum product that deforms the product on the finite flag manifold. Let me state the theorem that this has something to do with the affine Toda lattice.

Theorem 9.4.

(1) *There is a well-defined product on $H^*(\mathcal{G}/\mathcal{B}) \otimes \mathbb{Q}[q]$. This is identified with $e_1^*\mathcal{H}^*(G/B) \otimes \mathbb{Q}[q]$.*

(2) *This product is closed and associative, and commutative. Mod q_0 you get the usual quantum cohomology ring. It also has the property that the generators and relations and it's*

$$\frac{\mathbb{Q}[q][p_1, \ldots, p_n]}{\substack{\textit{conserved quantities for the twisted affine Toda lattice}\\ \textit{associated to the Langlands dual group of } G}}$$

This is a suggestion as well that the quantum cohomology ring cannot be defined in this way.

10. Daniel Massart: Overview of Mañé's conjecture

So thanks a lot for inviting me to this wonderful place. So I'm going to speak about Lagrangian dynamics. I assume that none of you know what this is but at least the symplectic people know what a Hamiltonian is. This is Hamiltonian dynamics seen through duality on the tangent bundle. The first four items, Tonelli Lagrangians, minimizing measures, minimization and cohomology, and minimization and homology should be accessible to everyone. Then the remainder in the second half should get worse.

10.1. **Tonelli Lagrangian.** We live in a closed smooth compact manifold without boundary M. A Tonelli Lagrangian is a function on the tangent bundle of this manifold $L : TM \times \mathbb{T} \to \mathbb{R}$ (here $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) that satsifies some nice hypotheses. There is a periodic dependence on time. So the Lagrangian might be the kinetic plus potential energy of the pendulum, and you switch on and off a magnetic field and the pendulum is a magnet.

This Lagrangian should be C^2, strictly convex, so that for (x, v, t) in $TM \times \mathbb{T}$, we have $\frac{\partial^2 L}{\partial v}^2 > 0$, this is a positive definite bilinear form. Then I should have $\frac{L(x,v,t)}{|v|} \to \infty$ as $|v| \to \infty$. Put some Riemannian metric on the manifold, and it doesn't matter which you choose since M is compact. There is a well-defined notion of superlinear function even though I haven't chosen a particular Riemannian metric.

Let me give the canonical example.

$$L(x, v, t) = \frac{1}{2}|v|^2 + f(x, t).$$

When the Lagrangian does not depend on t, I call it autonomous and omit the vector t.

Now I'll view the Lagrangian as a kind of cost function. To go from a point x to y in my manifold, I should pay the integral of L along my path. That's the action functional. The action of a path $\gamma : [a, b] \to M$ is $\mathbb{L}(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t), h)dt$. The hypotheses insure that the Legendre transform is a diffeomorphism between the tangent and cotangent bundle.

What is the point of the hypotheses? It's the following theorem, due to Tonelli (whence the name)

Theorem 10.1. *Given two points x and y in M and a time interval $[a, b]$, there is a way to go from one to the other in the given time paying as little as possible, there exists a minimizing path $\gamma : [a, b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = y$*

The *Euler–Lagrange flow* on $TM \times \mathbb{T}$ consists of following the minimizing path issuing from point x in direction v. The hypotheses ensure that this minimizing path is unique so this definition makes sense and defines a flow locally. You don't know if your minimizing path will exist for all time, but at least it will exist for a short period of time.

I'm going to assume this flow is complete. This is unphysical but not so stupid, this is supposed to show the existence of diffusion, flows that go to infinity in TM, so if there is a lack of completeness, there is already diffusion.

If the flow is not complete, the story is finished. The main problem in this theory is Arnold's conjecture, which says that for a generic Lagrangian, whatever that means, (I'll state the most ambitious version of Arnold's conjecture) there exists an orbit of the Euler–Lagrange flow which is dense in TM. This is a weak version of the Boltzmann ergodic hypothesis. The trajectory of a particle in a box is equidistributed in the box. We do not speak of equidistribution here but it should be dense. The autonomous version is dense in its energy level.

This conjecture is already rather weak because you only want to know about one orbit. Finding any invariant subset of the flow is already an achievement. When some guy finds a new periodic orbit of the three-body problem, this is fantastic and you publish it in the Annals of Mathematics. This has to be an interesting invariant set, so it should be small (the whole of TM is invariant). It has to have information about the dynamics. You can take the orbit at a point x in a direction v. But if I know that something is a periodic orbit, that's good. Let me define Mather's method for finding interesting invariant sets.

10.2. **Minimizing measures.** The basic idea is to look for invariant measures of this flow. Then look at their support. Measure spaces are compact. So if you want to use variational methods, measures are well-adapted to this.

How did I define the Euler–Lagrange flow? I minimize the action for some path. I can also integrate the Lagrangian against some measure. I try to minimize the action against some measure. If I'm lucky, maybe the minimizers will be invariant. It works, although it's not so easy.

What you want to do is minimize the action of the Lagrangian on $TM \times \mathbb{T}$,

$$\min_{\mu \in ?} \int_{TM \times \mathbb{T}} L(x, v, t) d\mu(x, v, t).$$

You can't take all measures, or it won't be invariant. For instance, for $L(v, x, t) = \frac{1}{2}v^2 + cos(2\pi x)$ then the minimizing measure is the dirac measure at $x = \frac{1}{2}$ and $v = 0$. So we need a smaller set of measures.

I won't give you the definition where the Lagrangian depends on time, only the autonomous version.

Definition 10.1. I say that a compactly supported Borel probability measure on TM is closed if it satisfies the following condition. For any C^1 function $f : M \to \mathbb{R}$, the integral $\int df d\mu = 0$. Note that this integral makes sense because μ is a measure on TM and df is a function on TM, compact support means I don't have to worry about convergence.

So closed measures are dual to closed one-forms in some sense.

The nice thing about closed measures, is first you have this lemma

Lemma 10.1. *(Mather '91) Invariant measures are closed.*

[some explanation]

Theorem 10.2. *(Mañé, Bangert, Fathi, Massart) Among closed measures you can minimize the action, and the miinimizers are invariant.*

The theorem is hard but the idea is simple. Take a Riemannian manifold. Closed curves that minimize the length are geodesic, so they minimize the geodesic flow.

It's nice to know that minimizing measures exist, but what do they look like? Mañé proposed the following conjecture.

Conjecture 10.1. *(Mañé '97) For a generic Lagrangian, there exists a unique minimizing measure supported on a periodic orbit.*

From now on, "for a generic Lagrangian property P" means that for any Lagrangian, there exists a residual set of potentials (which may depend on the Lagrangian) in the C^∞ topology, such that for any potential in this set, property P holds for $L + f$.

You may think that this doesn't support Arnold's conjecture. This seems to be about finding invariant sets with *small* support, whereas Arnold's conjecture is about finding invariant sets with *large* support. The invariant sets from the minimizing measures will not be part of Arnold's orbit; it's a more complicated relationship.

First, Mather wants to construct some kind of ladder whose steps are the supports of invariant measures. Between the supports of invariant measures, he wants to build heteroclinic orbits, which are asymptotic at $\pm\infty$ to one or the other orbit. Then Arnold's orbit will be some kind of vine which winds around the support of an invariant measure, catch on a heteroclinic, go to the next invariant measure, and so on. For this to work, you need the support of the minimizing measures to be small. That's why it's good to have periodic orbits in Mañé's conjecture. You need sufficiently many steps to you ladder. Mañé's conjecture says you have a unique minimizing orbit. We'll need another idea for other invariant orbits.

10.3. **Minimization and cohomology.** The summary of the last episode, we want to solve Arnold's conjecture. We need to build a ladder whose steps are periodic orbits. Let's take closed one-forms on the manifold M. That's a function on the tangent bundle. I'm going to assume my Lagrangian is autonomous. I can consider $L + \omega$, which is linear in each fiber. The sum is strictly convex and superlinear. This is again a Tonelli Lagrangian. Furthermore, this has the same Euler-Lagrange flow of M. This is because Euler–Lagrange flow is defined to be minimizers of the action. The integral of ω over a small disk between two pieces of path is the same with respect to ω. Then $L + \omega$ and L have the same behavior.

Now you can do the same thing we just did for $L+\omega$, which will give you a new minimizing closed measure.

Now I do the same thing and get a new minimizing measure. This is another invariant set of the same flow. If ω is exact, we get the same measure, because adding df doesn't change things since μ is closed. For each cohomology class we get a new minimizing measure.

To construct Mather's ladder, we'll try to minimize simultaneously in many cohomology classes. The minimimizing measures should have small supports.

The version of Mañé's conjecture adapted to this is the following, Mañé's conjecture with cohomology. Both this and the last conjecture are posthumous, although this one predates the other one.

Conjecture 10.2. *For a generic Lagrangian, there exists a dense open subset $U(L) \subset H^1(M, \mathbb{R})$ such that for any cohomology class c in $U(L)$ there exists a unique $(L + c)$-minimizing measure, where this means $(L + \omega)$ where $[\omega] = c$, supported on a periodic orbit.*

10.4. **Minimization and homology.** Gabriel has repeatedly asked whether these periodic orbits are different and we'll see how different they can be.

Here, what did I do to get many invariant measures? I modified the cost, I found optimal measures for a different kind of cost. Now I'll do something different, keep the same cost and minimize under some kind of constraint. I'll define the homology class of a measure. First, an observation. If μ is a closed measure on TM, then consider the following map $\omega \mapsto \int \omega d\mu$ from the space of close 1-forms to $\mathbb{R}$. This is linear and depends only on the cohomology class of ω, so this defines an element of the dual of the cohomology $(H^1(M,\mathbb{R}))^*$, but this is a finite dimensional vector space, so this is $H_1(M,\mathbb{R})$. So this map defines a homology class for the measure μ. Now I can ask about minimizing the action in a given homology class.

If I have two minimizing measures with different homology classes, then they cannot be the same. For each homology class h, there exists a cohomology class c such that any (L,h)-minimizing measure is also $L+c$-minimizing. This process of minimizing in homology classes lets me distinguish among minimizing measures that I already had. To see why this is true, I'll introduce the α and β functions of the Lagrangian L.

First, take a closed curve, take measure equidistributed on this, this is the homology class of the curve divided by the time period of the curve.

If the dimension of M is 2 and the homology class is rational (a real multiple of an integral class), then it's supported on a periodic orbit.

Proposition 10.1. *If M has dimension 2 and L is autonomous and $h \in \mathbb{R}H_1(M,\mathbb{Z})$“ $\subset''$ $H_1(M,\mathbb{R})$ (this inclusion is true modulo torsion) And μ is (L,h)-minimizing, then μ is supported on periodic orbits.*

Then I need to find a dense open set of cohomology such that those correspond to these rational homology classes.

11. September 10: Kei Irie: Dense existence of periodic Reeb orbits and ECH spectral invariants

ECH stands for embedded contact homology. Today I want to explain a new application to the dynamics of Reeb orbits. We start from recalling very basic notions. Let (Y,λ) be a contact manifold. That means that Y is $2n+1$-dimensional manifold and λ is a 1-form on it. $\Omega^1(Y)$ denotes the space of C^∞ 1-forms on Y. It is called contact if and only if $\lambda\wedge(d\lambda)^n$ is nonzero for all y in Y. Given such a λ, we can define the Reeb vector field R_λ by the formula $d\lambda(R_\lambda,*)=0$ and $\lambda(R_\lambda)=1$.

Now $P(Y,\lambda)$ consists of the periodic orbits $\gamma:\mathbb{R}/T_\gamma\mathbb{Z}\to Y$ such that $\dot\gamma = R_\lambda(\gamma)$.

So for example, consider a Riemannian manifold (M,g). Let S^*_gM the pairs (q,p) where $q\in M$ and $p\in T^*_qM$ with $||p||_g=1$, the unit cotangent bundle. We have a projection to M and we can define λ by the formula $\lambda(v)=p(\pi_*(v))$. Then R_λ is the geodesic flow and the periodic orbits correspond to geodesics on the manifold.

Theorem 11.1. *Let Y be a closed 3-manifold and suppose that the the set of contact 1-forms is nonempty. Call this $\mathcal{C}(Y)$, this is an open set within the 1-forms on Y in the C^∞ topology.*

Consider $\{\lambda\in\mathcal{C}(Y)|\bigcup \mathrm{im}\gamma \text{ is dense in} Y\}$. This space of contact forms satisfying this property is residual*, that is, it contains an intersection of countably many open dense sets. In particular it is dense.*

Recall the result of Herman, who shows that there exists ω a symplectic form in $T^4=(\mathbb{R}/\mathbb{Z})^4$ and S a closed hypersurface in T^4. In general a hypersurface in a symplectic manifold lets us define f^ω_S, the kernel of ω restricted to S. He shows that if S' is sufficiently close to S in C^∞ norm, then $f^\omega_{S'}$ has *no* closed leaf.

Ekeland showed something about having infinitely many in some cases.

This says that C^∞-closing lemma cannot be true for Hamiltonian systems.

Theorem 11.2. *Let Σ be a closed surface and let $\mathcal{G}(\Sigma)$ be the set of Riemannian metrics on Σ with C^∞ topology. Then the set of $g \in \mathcal{G}(\Sigma)$ such that $\bigcup \mathrm{im}\gamma$ is dense in Σ, where γ varies over nonconstant closed geodesics, is residual in $\mathcal{G}(\Sigma)$.*

This doesn't follow from the first theorem because it involves perturbing only the metric, not the contact form.

So let me remark, form M closed and simply connected and generic g in $\mathcal{G}(M)$ there are infinitely many primitive closed geodesics on (M, g). This is due to Hingston and Rademacher (and others).

As far as I know, this statement is new. This follows from a recent development in contact homology. Let me make a brief review, but first let me talk about non-degeneracy. Let γ be a periodic orbit in $P(Y, \lambda)$. We take p on the embedded orbit, and take ξ_p, the kernel of λ, a codimension 1 subspace of T_pY. We can find a ρ, a *linearized return map* along γ. We say γ is nondegenerate if and only if 1 is not an eigenvalue of ρ.

This does not depend on the choice of p.

Theorem 11.3. *For C^∞-generic contact form λ, any γ in $P(Y, \lambda)$ is nondegenerate.*

Now we give a very quick review of embedded contact homology. The basic theory was developed by Hutchings and Taubes. We consider only 3-dimensional contact manifolds. We consider injective periodic Reeb orbits γ. This has a natural free S^1-action. We take a quotient by S^1 and denote this set $P_0(Y, \lambda)$.

Definition 11.1. Suppose that λ is nondegenerate in the sense above. Then an *ECH generator* is a finite set $\{(m_i, \alpha_i)\}$ where m_i is in $\mathbb{Z}_+$ and $\alpha_i \in P_0$, satisfying the conditions

- for $i \neq j$ we have $\alpha_i \neq \alpha_j$.
- If α_i is hyperbolic, then $m_i = 1$.

Note that $\emptyset$ is an ECH generator.

Given an ECH generator α, then we can define $[\alpha]$ as $\sum m_i[\alpha_i]$ in $H_1(Y, \mathbb{Z})$. Then $\mathcal{A}(\alpha)$ is $\sum_i m_i \int_{\alpha_i} \lambda$, which is a nonnegative real number. Note that $[\emptyset] = 0$ and $\mathcal{A}(\emptyset) = 0$.

Let Γ be a homology class of Y. Then $ECC(Y, \lambda, \Gamma)$ is the free $\mathbb{Z}/2$-module generated by ECH generators α with $[\alpha] = \Gamma$. Then for $L > 0$, the space ECC^L is the subspace of ECC generated by α with $\mathcal{A}(\alpha) < L$.

To define the differential, we take an almost complex structure J on $Y \times \mathbb{R}$ satisfying several conditions. Let me omit them. It's invariant by $\mathbb{R}$ (whose coordinate is s, and $J(\partial s) = R_\lambda$ and j acts on ξ_λ, positive with respect to $d\lambda$)

For ECH generators α and β, consider the set of J-holomorphic currents u in $Y \times \mathbb{R}$ satsifying the conditions

(1) The restriction $u|_{Y \times \{s\}}$ goes to α as $s \to \infty$ and to β as $s \to -\infty$. This lets us define the so-called ECH index, which is an integer, and the second condition is

(2) The ECH index is 1.

Call this $M_J(\alpha, \beta)$, and this is generic.

Now define ∂_J on ECC with $\partial_J \alpha = \sum_\beta \#_2 M_j(\alpha,\beta)\beta$.

Remark 11.1. When $c_1(\xi_\lambda)+2PD(\Gamma) \in H^2(Y,\mathbb{Z})$ is torsion, then ECC has a relative $\mathbb{Z}$-grading with $|\partial_J| = 1$.

Theorem 11.4. (1) *For generic J, $\partial^2 j = 0$. Note that* $\ker\ \partial_J \Im \partial j$ *depends only on Y, Γ, and ξ_λ.*
(2) *For $L > 0$, the boundary ∂_J preserves ECC^L. We also have the same kind of invariance as before.*

Let me move to spectral invariants. Take any nonzero σ in $ECH(Y,\Gamma,\xi)$; to it we can assign a positive real number $C_\sigma(Y,\lambda)$. This is done first by Hutchings, in the paper "quantitative ECH."

When λ is nondegenerate, then $C_\sigma(Y,\lambda)$ is the infimum of $L > 0$ where $\sigma \in \mathrm{im} i^L$.

In the degenerate case, take $(h_j)_j$ such that $||h_j||_{C^0} \to 0$ as $j \to \infty$ and $(1+h_j)$ nondegenerate for all j. Then $C_\sigma(Y,\lambda) \coloneqq lim_{j\to\infty} C_\sigma((Y, 1+h_j)\lambda)$.

Proposition 11.1. (1) *For all $h \in C^\infty(Y,\mathbb{R}_{\geq 0})$, we have $C_\sigma(Y,(1+h)\lambda) \geq C_\sigma(Y,\lambda)$.*
(2) *for all $a \in \mathbb{R}+$, $C_\sigma(Y, a\lambda) = aC_\sigma(Y,X)$.*
(3) *C_σ is in the set $\{\sum m_i \int_{\alpha_i} \lambda | m_i \in \mathbb{Z}_+$ and $a_i \in P_0(Y,\lambda)\}$.*

Lemma 11.1. *$A(Y,\lambda)_+$ is a closed set $\mathbb{R} > 0$ of measure 0.*

Now we can state the following remarkable result by

Theorem 11.5. *(Cristofaro, Gardiner, Hutchings, Ramos) Let (Y^3,λ) be closed and connected. Suppose our class $c(\xi_\lambda) + 2PD(\Gamma)$ is torsion, then we have the relative $\mathbb{Z}$-grading. Let $\{\sigma_k\}$ be a sequence of nonzero homogeneous classes in ECH such that $|\sigma_k| \to \infty$ with k.*

Then the following converges:

$$\frac{c_{\sigma_k}(Y,\lambda)^2}{|\sigma_k|} \to \int_Y \lambda \wedge d\lambda = vol(Y,\lambda).$$

Here's a remark. A sequence $\{\sigma_k\}$ satisfying this growth property always exists for any (Y,λ). This comes from the following result. The $ECH(Y,\Gamma,\xi)$ is ismorphic to $\widehat{HM}^{-*}(Y, S_\xi + PD(\Gamma))$ and you know this is unbounded.

Let me give the key lemma, the C^∞ *closing lemma* for Reeb dynamics.

Lemma 11.2. *Let (Y,λ) be a closed contact manifold. The for every nonempty open U in Y and $\epsilon > 0$, that there is an $f \in C^\infty(Y,\mathbb{R}_+)$ with $||f-1||_{C^\infty} < \epsilon$ and $\gamma \in P(Y, f\lambda)$ with $\mathrm{im}\gamma \cap U$ nonempty.*

Let me prove this. Let Y be connected. Take h in $C^\infty(Y,\mathbb{R})$ supported in U, not identically zero, and $||h||_{C^\infty} < \epsilon$.

Now I claim that there exists $t \in [0,1]$ and $\gamma \in P(Y,(1+th)\lambda)$ such that $\mathrm{im}\gamma \cap U \neq \emptyset$. Then we'll take $f = 1 + th$ and be done. Call this $f\lambda$ by λ_t.

Suppose not. Then for any $t \in [0,1]$ and any $\gamma \in P(Y,\lambda_t)$, this image of γ is disjoint from U. This means for any t, $P(Y,\lambda_t) = P(Y,\lambda_0)$ because their Reeb vector fields coincide on $Y\backslash U$. Then for any t, $\mathcal{A}(Y\lambda_t)+ = A(Y\lambda_0)_+$.

Now for any Γ and σ, $c_\sigma(Y, \lambda_t)$ is continuous in t. So it's constant. Then $c_\sigma(Y\lambda_0) = c_\sigma(Y\lambda_1)$. Then the volume of these two manifolds are equal. This cannot be true, since h is nonnegative and not zero.

This argument is very standard in sympletic topology. If you combine this standard argument with something else, you get the closing lemma.

This argument works for all dimensions. The machinery is only in three dimensions.

Let me prove the first theorem. You get directly that γ is nondegenerate by perturbing. Then take (U_i), a countable base of open sets in Y. Consider contact forms in Y such that there exists nondegenerate γ in $P(Y, \lambda)$ where $\mathrm{im}\gamma$ meeets U_i. This set of contact forms, call in $\Lambda(U_i)$ is open and dense in $\mathcal{C}(Y)$. Then $\lambda \in \bigcap \Lambda(U_i)$. So the union of $\mathrm{im}\gamma$ is dense in Y. This is a standard argument.

Let me conclude with the statement of the closing lemma for the second theorem.

Theorem 11.6. *Let (Σ, g) be a closed 2-dimensional Riemannian manifold. For all U nonempty and open in Σ and ϵ, there is $f \in C^\infty(\Sigma, \mathbb{R}_+)$ which satisfies $||f-1||_{C^\infty} < \epsilon$ and γ a nonconstant closed geodesic of Σ, fg with $\mathrm{im}\gamma \cap U$ nonempty.*

12. September 24: Dmitry Kaledin: Cyclic homology of a different kind

I used to give a series of talks. This time, it's basically just one talk, it's a simple thing, but it's interesting. It should have been noticed ten years ago.

Let me first remind you of cyclic homology. You start with an associative algebra, flat over k, and one defines the periodic cyclic homology of A, $HP(A)$, as the homology of a certain bicomplex. One starts with the Hochschild complex $A^{\otimes 3} \to A^{\otimes 2}A \to A$ and one observes that one can repeat this many times, and then one can put another thing between with the same terms but a different differential b', let's call it the bar differential, and another description is that this is an A-bimodule, the whole thing is then acyclic. One can put horizontal differentials in to make the thing a bicomplex. There's a cyclic group of order n acting by permutation. So σ acts on $A^{\otimes n}$ by $(-1)^{n+1}$, you have differentials $1-\sigma$ and $1+\sigma+\cdots+\sigma^{n-1}$. Then there is the issue, how do you take the total complex? If you look at the diagonal, there are an infinite number of terms, so what do you do? There are two options. For $HP(A)$, one takes the product total complex and then takes the homology.

What if instead I take a different totalization, the sum?

Definition 12.1. $\overline{HP}(A)$ is the homology of the sum total complex.

If you look at Loday's book on cyclic homology, or anywhere, it says that if the base field contains $\mathbb{Q}$ then this thing is 0. For a bicomplex you have two spectral sequences; if they're nice then they converge to the same thing. In this case one converges to the sum and the other to the product.

This one first computes the horizontal and then the vertical homology. The homology of the rows is $\vee H(\mathbb{Z}/n\mathbb{Z}, V)$, and that's torsion (eventually) so in the inverse limit, this is torsion in all degrees. Then in characteristic zero, this is just 0 so there's nothing to talk about.

Then, it was suggested by Kontsevich around ten years ago that in positive characteristic, this is actually an interesting thing to consider. I don't think anyone took it seriously. I was working on this at that time, but I didn't get the point of it.

Recently there was another development, in algebraic geometry, where something similar happens. This led me to revisit the subject and see that something interesting happens here. Let me tell you the story. This was not joint work but some kind of ping-pong between Beilinson and Bhatt. The setting is the following. Assume for the moment that A is commutative. If it's commutative and smooth, then periodic cyclic homology is intimately related to de Rham cohomology of the associated algebraic variety. They considered instead the case when A is commutative but not smooth. They wanted some version of de Rham cohomology.

One standard way that goes back to Illusie is derived de Rham cohomology. You take a resolution, which is smooth. Since we're interested eventually in positive characteristic, it's better to do this in simplicial rings. So what's a resolution? It's some simplicial ring but under Dold–Kan you can think of it as a complex, you want $DK(A) \cong A$. You can do this in a standard way. Take generators and relations and you can do a completely stupid standard procedure. Termwise this is smooth and even more, a polynomial algebra. Now you can take Ω_{DR} of (A) termwise. What you end up with is a certain bicomplex. The rows are just $A_0 \to \Omega^1 A_0 \to \cdots$, $A_1 \to \Omega^1 A_1 \to \cdots$ and so on. The vertical differentials come from the simplicial structure.

Again, this has possibly an infinite number of terms on the diagonal. Even if every guy is finitely generated, there's no way to control the length of varying rows. So there is this issue again of two kinds of convergence.

The standard answer, going back to Illusie, $H_{DR}(A)$ is the homology of the product total complex. This means the spectral sequence that converges is the one where first you compute the vertical homology and then the horizontal. In degree 0 it's just A. Illusie proved that this is independent of the resolution, and later it's derived exterior powers. Then there's a spectral sequence from $\wedge\Omega(A) \to H_{DR}(A)$, and the notation is correct, this does not depend on the choice of resolution of A.

The non-standard answer, $\overline{H}_{DR}(A)$, is the homology of the sum total complex. Exactly like in the other story, this at first looks stupid to consider. Suppose we're in characteristic zero. The spectral sequence that converges starts with horizontal homology. These are de Rham homologies of affine spaces, and it's well known that there is no homology. The result after one term is k in every degree and then after the next differential it's just k.

It turns out that in characteristic p, the solution is different, what's different is de Rham cohomology of an affine space. Let me remind you how this goes. Then this is nottrivial. For a perfect field k, you have $H^i_{DR}(B) \cong \Omega^i(B)$. There are a couple of spectral sequences you can use and one is the "conjugacy spectral sequence." One of the funny things is that the first term is the same as the Hodge spectral sequence. Now if you look what happens for the sum total complex, it converges exactly, starting in this other direction, to $\overline{H}_{DR}$. We have the conjugate spectral sequence as well, and it converges too to $\overline{H}_{DR}(A)$. Beilinson had a specific example in mind, and it's the following. The take p-adic numbers, and then algebraic closure. I am doing characteristic p, so then mod out by p. That's A, and $H_{DR}(A)$ is nontrivial only in degree 0, and there it's the so-called B^*_{dR}, something that shows up in p-adic Hodge theory and is very difficult to define. In that theory, there's a smaller ring, B_{cris}, which maps to B_{DR}. So a question is how to recover B_{cris} using a similar construction, and that's the version using $\overline{H}_{DR}(A)$.

So what did I do then? I came back to the periodic cyclic homology. Let me state right away, that what I come up with cannot be a generalization of this de Rham thing. This de Rham thing required characteristic 0 and then choices are necessary for what little can be done in characteristic p. So what I'm doing is motivated by but independent of the example.

Let me state the theorems that I can prove about cyclic homology. I start with the situation I just erased, for cyclic homology algebras. I start with A which is associative, unital, and flat over some k a commutative ring. I have this observation, that $\overline{HP}(A)$ is always torsion. There are basically two results. Let me think how I should. The first one is a comparison result. Assume that A has finite homological dimension (the diagonal module over A has a finite resolution). In this case, we have a canonical long exact sequence

$$\overline{HP}(A) \to HP(A) \to HP(A) \otimes \mathbb{Q} \to \cdots$$

If k contains $\mathbb{Q}$ then the latter two are the same and the first one vanishes. If k is torsion then the third one vanishes and the first two are the same. If k is $\mathbb{Z}$, then, well, this thing is periodic. There are odd terms and even terms. The even degree terms are 0 and the odd degree terms (of $\overline{HP}$) are $\mathbb{Q}/\mathbb{Z}$. In the long exact sequence this is $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$.

What's the most important thing about periodic cyclic homology? There's the Hodge to de Rham spectral sequence. There's a spectral sequence that starts with $HH(A)((u))$, formal Laurent power series in u with coefficients in $HH(A)$. Now if the characteristic is $p \neq 2$ and k is perfect, you have something analogous to the conjugate spectral sequence which starts with $HH(A)$ and now has a trick. The product total is naturally in Laurent power series. In the sum you can only do Laurent polynomials, but I want to do a trick and do Laurent power series in u^{-1} which is the same because you're bounded above. And this converges to $\overline{HP}(A)$. A nice property of Hochschild is that it's Morita invariant. So the cyclic homologies will be the same. Then $\overline{HP}(A)$ is also Morita invariant.

In order to get applications, I should move to dg algebras and *derived* Morita equivalence. This is what I want to do now. Let me give definitions and then we'll have a break?

For dg algebras. Here the story becomes more involved. One always considers it only up to quasi-isomorphism. How does one define Hochschild homology? One writes down the Hochschild complex and now it becomes a bicomplex, there's also the differential of A, so it's a bicomplex. The prescription is to always take the sum-total complex for this guy. If you do this, then you immediately see that this definition is invariant under quasi-isomorphism. Positselski has developed the theory where you take the product but that's a little bit funny.

We denote this complex by CH. It's convenient to add the other acyclic column. So I split into pieces with one odd and one even column. The differential that is horizontal is $1 - \sigma$. Then the other term survives as map B that goes from one side to the other. So I have $\widetilde{CH}$ the extended guy, which has the same cohomology, and then there's this map B. In the literature, there's a canonical contracting homotopy on the acyclic complex. I could never remember the definition of B defined directly on the CH complex, it's ugly, so for exposition it's easier to do it this way. So we have the map B, and it shifts the degree by 1. It squares to zero (which in this picture is obvious) and then the usual definition of $CP(A)$ is that it's $\langle \widetilde{CH}(A)((u)), d + Bu\rangle$.

So we want a version for the other direction. We define $\bar{CP}(A)$ to be $\langle \widetilde{CH}(A)((u^{-1})), d+ Bu\rangle$. Here there is no map. So for comparison we introduce $cp(A) = \langle \widetilde{CH}(A)[u, u^{-1}], d+ Bu\rangle$.

By definition, we have maps $\overline{CP}(A) \xleftarrow{R} cp(A) \xrightarrow{L} CP(A)$. Let me state the theorem and then we'll take the break.

Lemma 12.1. *The new things $\overline{CP}(A) \otimes \mathbb{Q} = cp(A) \otimes \mathbb{Q} = 0$.*

There is a fine point. You have the diagonal, you have finite, arbitrary linear combinations, and then you could take half in one direction and half in the other. In a tricomplex, you can draw any shape on the plane and bound by that shape. I have no idea which ones are reasonable or not. I always take sum-total for CH and then i play with the others.

Theorem 12.1. *Assume A is bounded and smooth. Bounded means something slightly subtle in the case where the ground ring has infinite homological dimension. Smooth means the diagonal is perfect. Then R is a quasiisomorphism and L fits into a distinguished triangle $cp(A) \xrightarrow{L} CP(A) \to CP(A) \otimes \mathbb{Q} \to$*

Theorem 12.2. *Let k be perfect and the characteristic be p then $\overline{CP}(A)$ is derived Morita invariant and if the characteristic is not 2, there's a spectral sequence $HH(A)((u^{-1})) \to \overline{HP}(A)$.*

Let's have a break, and then I'll explain about how this works.

So what I want to do is indicate the spectral sequences, the rest is boring. So there is some small categoristic technology that allows you to reduce everything to $\mathbb{Z}/p\mathbb{Z}$. So first the conjugate spectral sequence (along with its Cartier isomorphism).

As a brief reminder of something you probably know, there's Connes' small category Λ. Objects are configurations of points on a circle, maps are homotopy classes of maps that take marked points to marked points, et cetera. So $[n]$ is a wheel with n vertices. So Λ includes Δ^{op} via j. You have $A^\flat : \Delta^{op} \to k - mod$. Connes introduced this to package cyclic homology in a way that doesn't require you to write down the whole complex. So $HH(A) \cong H(\Delta^{op}, A^\flat)$. Conne's observation is that $A^\flat$ extends to the bigger category and then $HC(A) \cong H(\Lambda, A^\flat)$. Things like this spectral sequence are pure linear algebra.

For periodic things, $HP(A) = \lim_u^d HC(A)$, the inverse limit.

For the conjugate thing, we need Λ_p. If you have a wheel in Λ and the number of vertices is divisible by p, say np, then of course you have an endomorphism of this guy. The endomorphism is an element of the cyclic group of order n. Then we have morphisms the maps that commute with this. So you hav $i : \Lambda_p \to \Lambda$. It also projects by π by taking the quotient. The projection is not an equivalence but it's close. The fibers are a point modulo $\mathbb{Z}/p\mathbb{Z}$.

Lemma 12.2. *(edgewise subdivision) For any $E : \Lambda \to k - mod$, the map i does not change homology at all, $H(\Lambda_p, i^*E) \to H(\Lambda, E)$ is an isomorphism.*

So now make an observation $H(\Lambda, E) \cong H(\Lambda L\pi_* i^*E)$, this is totally tautological.

So let me say what you can do for HP. You have $\lim H(\Lambda, E)$, and up to now p is an integer, but now assume that the characteristic is p. Then when you pull back the generator is zero. The result is that the endomorphism u is actually induced by an endomorphisms of the other guy, so you can take the inverse limit locally before taking homology of the category, $H(\Lambda, limL\pi_* i^* E)$. But homology doesn't commute with inverse limits so this thing accepts a *map* from $\lim H(\Lambda, E)$. So if E corresponds to some algebra, $E = A^\flat$, then one can show that $\overline{HP}(A) = H(\Lambda, \lim(\pi_* i^* A^\flat))$. I don't want to discuss the proof, but this is useful, because we can just compute these smaller parts.

If we have $[n] \in \Lambda$, well, let me first evaluate $L\pi_* i^* A^\flat([n])$. So this is A^{pn}, not A^n¡ and then this is the homology of the cyclic group, $H(\mathbb{Z}/p\mathbb{Z}, A^{\otimes pn})$. When you do the inverse limit, what you get is Tate homology, which is by definition, if you just compute by the standard complex.

The main lemma to relate this back is the following.

Lemma 12.3. *For any vector space V (in applications $A^{\otimes n}$) and an intteger i we have a canonical identification between $H * \mathbb{Z}/pzZ, V^{\otimes p}, V\rangle$*

I start with something, I compute $\overline{HP}$, and it turns out the homology objects of this complex are just $A^\flat$.

You can check that this is the same isomorphism you get by the usual Cartier isomorhphism.

So why is this true? Let me explain the proof. First of all, the map. The trick is to take a map that is not linear, $\varphi(v) = v^{\otimes p}$. Modulo the image of the differential it becomes additive and indeed becomes an ismorphism.

To prove this, it's a construction, choose a basis in V, take the basis in V induced by it. I decompose the whole thing into its diagonal and the complement. The important thing is to start with an arbitrary map, not one which is linear.

The lemma shows that the homology objects of this $L\pi_* i^* A^\flat$ are $A^\flat$ and this induces the conjugate spectral sequence.

I want to explain how to generalize this to dg algebras. Right away it seems hopeless because the degrees are wrong. If we had a grading, if v had degree 0 then this would be degree 0, and that's wrong; if v had degree 1 then that would have degree p, and that's still wrong.

What's wrong? You can take homology of $\mathbb{Z}/p$ with ceefficients in the pth tensor power of V and that's not v. We get a bicomplex, and it does not seem to be related to V itself. A bicomplex is like this [picture].

I think this is enough but I wanted to advertise this lemma.

13. Nov. 18: Vasile Brinzanescu: Algebraic complete integrability of some Hamiltonian systems

Just to fix notation I'll give some small introduction. So M will be a manifold. By this I mean a real smooth manifold or a non-compact complex manifold. By $\mathcal{F}(M)$ I denote the algebra of functions on M. In the first case we take smooth functions and in the second case holomorphic functions.

I believe it is not necessary to give the definition of a Poisson manifold. So M is Poisson. Let me give just one definition, to fix notation. We should have:

Definition 13.1. I take $(M, \{\ ,\ \})$ of rank $2r$, and then $\mathbb{F}$ is an s-tuple $(F_1, \dots, F_s)$ of functions $F_i : M \to \mathbb{C}$ (or to $\mathbb{R}$) such that this is involutive and independent. This means functionally independent. This $\mathbb{F}$ is *completely integrable* if s is dim $M - r$.

What is hidden here? We do not see the equation here, but the system, what's hidden is this Poisson bracket.

Now, in fact, the $\mathbb{F}$ is a momentum map. Practically every function here, [unintelligible], first integral, that means conservation laws. It means you have s first integrals. That means complete.

This integrability is also called Liouville integrability.

Of course, everybody knows, but let me recall the theorem of Liouville, or let me say Arnold.

Theorem 13.1. *Let $(M, \{\ ,\ \}, \mathbb{F})$ be as above be a completely integrable system. Let $m \in M$ be a point. Denote by $\mathbb{F}'_m$ the connected component of $\mathbb{F}_m \cap \mathcal{U}_{\mathbb{F}} \cap M_{(r)}$ which contain m. Here $\mathbb{F}_m$ is $\mathbb{F}^{-1}\mathbb{F}(m)$. Then $\mathcal{U}_{\mathbb{F}}$ is $\{m \in M : dF_1 \wedge \cdots dF_s(m) \neq 0\}$ (this is the open dense set where dF_i are independent). Then $M_{(r)}$ is the set of points in m where the rank at m of $\{\ ,\ \}$ is at least $2r$.*

Then

1. *if $\mathbb{F}'_m$ is compact then it is diffeomorphic to a torus $(\mathbb{R}/\mathbb{Z})^r$.*
2. *if $\mathbb{F}'_m$ is not compact but the flow (given by an integrable curve) is complete (defined for any t) then there is a diffeomorphism to $\mathbb{R}^{r-q} \times T^q$ for some $0 \leq q < r$.*

What do I mean by an algebraically complete system.

Now M will be a complex (algebraic) manifold. As I said, this should be non-compact. I change the notation a little bit. Let me put h instead of $\mathbb{F}$ for holomorphicity. Now this is to $\mathbb{C}^s$, so let h be a complete integrable system, complete in the sense above.

The, well, M should be a non-singular affine manifold. The function $h = h_1, \ldots, h_s$ should be regular functions, algebraic functions defined overall.

Now

Definition 13.2. h is an algebraic completely integrable system (a. c. i. system) if each generic fiber of h is a Zariski open subset of an Abelian variety and the Hamiltonian vector fields generated by h_i are translation-invariant (we call this linear).

Why do we have this Abelian variety? If you have a system given by polynomial equations, then you complexify it and think in the complex case. What happens? In the Liouville theorem, in the first case, it's a torus. This Abelian variety is that torus (complex) of course. The second part is that the Hamiltonians are the translations on the torus. That's the idea, when it's a complex, we know the topology is a torus, so we ask this to be an Abelian variety, the vector field through h should go to an invariant vector field on the torus, a linear one. It should be linear in time.

Definition 13.3. We say h is a generalized algebraically completely integrable system (gen. a. c. i. system) if each generic fiber of h is a Zariski open subset of a commutative algebraic group on which the Hamiltonian vector fields generated by h_i are translation-invariant.

So what is an algebraic group? It's an algebraic manifold, with the structure of a group, but I ask for commutativitiy. We will see by examples. The second one corresponds in some sense to the second case in the Liouville theorem.

This was some kind of introduction. In the second part I'll speak about the problem and some results.

We fixe $N \in so(n)$, it's the Lie algebra of $SO(n)$, just real skew symmetric. By $Sym(n)$ we denoe $n \times n$ symmetric matrices of order n. So our space M will be this space of symmetric matrices. We begin with the system. Now differential equations appear. If $n = 2p$ we choose $N = \begin{bmatrix} 0 & V \\ -V & 0 \end{bmatrix}$ and if n is odd, we choose N to be $N = \begin{bmatrix} 0 & V & 0 \\ -V & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ where V is a diagonal matrix.

Consider the system

$$\dot{X} = [X^2, N]; \quad X(o) = x_0 \in Sym(n) \tag{1}$$

It's clear that $X(t)$ will be in $Sym(n)$ at any time, and it's clear that the solution is defined for any t.

This system was introduced by Bloch and Iserles. There was some problem with a smaller dimension system of this time. Almost all of them were on tori, and then you can visualize them and then that led to this study.

There are some problems. Is this system completely integrable? Is it an algebraically completely integrable system? The answer in the first case is yes and in the second case, no, but it's generalized algebraically completely integrable.

It's easy to see that the system is equivalent to

$$\dot{X} = [X, XN + NX] \tag{2}$$

This is the Lax form of the system.

Now, what shall we do? The first thing we do is to define an N-bracket on $M = Sym(n)$, which is defined by $[X, Y]_N \coloneqq XNY - YNX$. Then this set of matrices is an affine subset of matrices. I want to be sure we are in the condition I set. It's simple that we obtain a Lie algebra, and we'll denote the same thing by $Sym(n, N)$, the same set but a Lie algebra.

Now we have a little proposition, quite simple, but in some sense interesting.

Proposition 13.1. *Let N be invertible (this means automatically that n is even and none of the entries of V are zero). Then*

$$Q : (Sym(n, N), [\ ,\ ]_N) \to sp(n, N^{-1}, [\ ,\ ])$$

where $Q(X) = NX$ (here $sp(n, N^{-1})$ is the set of matrices Z such that $Z^t N^{-1} + N^{-1} Z = 0$) is an isomorphism of Lie algebras.

We make another remark related to this system.

Define the Lagrangian of the Lie algebra $Sym(n, N)$ to be

$$\ell(x) = \frac{1}{2}\text{trace}(X^2) = \frac{1}{2}\text{trace}X \cdot X^T) = \frac{1}{2} \ll X, X \gg \tag{3}$$

Then we have the easy proposition

Proposition 13.2. *The equations $\dot{X} = [X^2, N]$ are the Euler–Poincaré equations corresponding to the Lagrangian 3 on the Lia algebra $(SYm(n, N), [\ ,\ ]_N)$.*

Remark 13.1. We can also define the "frozen" Poisson structure. The N-bracket we have can be described, for f and g smooth on $Sym(n, N)$, you can define the Poisson structure as $\{f, g\}_N(x) \coloneqq -\text{trace}X[\nabla f(x), \nabla g(x)]_N)$. If you take this inner product $\ll \gg$, you get a Lie structure on the dual algebra. In relation with this, we have another, the so-called "frozen" structure, given in the following way. If f and g, as above are smooth, then

$$\{f, g\}_{FN} = -\mathrm{trace}(\nabla f(x) N \nabla g(x) - \nabla g(x) N \nabla f(x)). \tag{4}$$

So we have a bi-Hamiltonian system.

Let's take a break.

Remark 13.2. Another remark. If N is invertible, then the Lie isomorphism Q takes the system, you obtain a Mischenko–Fomenko system [some discussion of this]

Now it's not very complicated, it can be obtained by a general procedure, you have a Lax pair with a parameter.

Proposition 13.3. *Let λ be a real parameter. The systems 1 and 2 are equivalent ot the lax pair with parameter*

$$\frac{d}{dt}(X + \lambda N) = [X + \lambda N, NX + XN + \lambda N^2] \tag{5}$$

So we have the following theorem

Theorem 13.2. *(Bloch, B., Iserles, Marsden, Ratiu)*

1. *For N invertible with distinct eigenvalues, the system 1 is completely integrable.*
2. *For N odd dimensional with distinct eigenvalues the system 1 is completely integrable.*

I'll give a hint later how to find the first integral.
We'll need another form of the lax pair

$$\frac{d}{dt}(X + \lambda N) = [\frac{X^2}{\lambda}, X + \lambda N]$$

Before giving another result, let me skip some steps and come back.

Remark 13.3. Just denote $X(\lambda)$ to be $X + \lambda N$, just notation. More generally, X could be any matrix with complex coefficients.

Now if you take $Q(\lambda, z)$ to be $\det(zI_n - X(\lambda))$, the characteristic polynomial of $X(\lambda)$, then if you take the equation $Q(\lambda, z) = 0$, this is a polynomial equation in the plane. You obtain an algebraic curve in the plane. This is the so-called spectral curve, which you have always when you have a lax equation with parameter. Then I have just a polynomial and just some roots, that will not be relevant for the problem. It's an idea to use this kind of lax thing. You ca associate to some systems (not all), then you can take this spectral curve, and you'll do things like that. For notation let $\Gamma_{X(\lambda)}$ we the curve $\{\}(\lambda, z) \in \mathbb{C}^2 | Q(\lambda, z) = 0\}$. If I look at $\mathbb{C}^2$ as an open part of $\mathbb{P}^2(\mathbb{C})$ and take the completion $\bar{\Gamma}_{X(\lambda)}$. For a generic value this will be nonsingular. Then you can construct for the curve the Jacobian, which is an Abelian variety with dimension equal to the genus of the curve. So you get $J_{X(\lambda)}$.

Okay, and the hope is that maybe, maybe, this system will have the integral curve situated in an open part of the Jacobian. This does not work always. The first problem that you encounter is that usually the dimension of the Jacobian is not the s from before. Then what do you do? You have to find a smaller part of the Jacobian. Then what do you have to do? You use some method to obtain a commutative algebraic group with a non-compact part that is an extension of the Jacobian. Then maybe you will obtain this.

I need this Jacobian to present the last result.

Theorem 13.3. *(Bloch, B., Iserles, Marsden, Ratiu) For N invertible with distinct eigenvalues, the flow of the system is linear on the Jacobian $J_{\Gamma_{X(\lambda)}}$.*

I didn't prove the algebraic statement, but the vector fields are going in the Jacobian through some vector fields that are translation invariant.

These two theorems are in one paper. At that moment we could prove complete algebraic integrability. You could analyze the Fuchs or the Jacobian, [unintelligible]the right one, it took some time to get the complete result.

Let me go to the third part, which will be longer and more algebraic geometry.

13.1. **Algebraic complete integrability of 1.** Let me take the form 5. Then I have

$$\frac{d}{dt}(X + \lambda N) = [X + \lambda N, NX + XN + \lambda N^2].$$

In our problem X is a symmetric matrix. But for now let X be any matrix. We have real matrices, well, it doesn't matter, we just complexify everything. Now I introduce $Q(\lambda, z) = \det(I_n - X(\lambda))$, the characteristic polynomial as before.With M^N we will denote $\lambda N + gl_n(\mathbb{C})$, which is our Poisson manifold.

We'll denote M^N_Q the set $\{X(\lambda) \in M^N | \det zI_n - X(\lambda) - Q(\lambda, z) \text{is fixed}\}$. This is called the isospectral variety.

So $\Gamma_{X(\lambda)}$ is the spectral curve in $\mathbb{C}^2$. This spectral curve, if you fix it, then the isospectral variety is preserved. These are time-independent. This means that the coefficients in the isospectral variety, the coefficients of Q, are constants of motion. $Q = \sum c_{k\ell}\lambda^k z^\ell$

The coefficients are polynomial in the entries of the matrix X. These polynomials are the functions we are looking for. They give us the right number of functions that give us complete integrability. That's the idea, you have to work.

Let us denote $c = (c_{k\ell})$ as a vector, which should be constant. The values of this polynomial are in $\mathbb{C}^s$. So here comes the problem that I put there. For a generic point, for a generic point c, the curve Γ is nonsingular, $\bar{\Gamma}$ is nonsingular, but not for all. Thinking of the definition of the system, it means the application is just taking the coefficients, so this means just the fiber is just the isospectral variety.

Now I will say just a few words and then I will stop, how we attack the problem. Okay. I'll put this in this way. We define, I should use some result of Beouiville and Gavrilov. They take a more general system than this, but let me explain the idea. Let $\mathcal{U}$ be the affine space of polynomials $Q(\lambda, z) = z^n + s_1(\lambda)z^{n-1} + \cdots + s_n(\lambda)$. You can identify the affine space with $\mathbb{C}^n$ by taking coefficients. Now every s_i should have the following property: the degree of s_i in λ is smaller than i for all $i = 1, \ldots, n$.

Now we define the map $h : M^N \to \mathcal{U}$, we take a matrix here, take the matrix $X(\lambda)$ to the characteristic polynomial.

Now we introduce the following group. $G :=$ is the projective group of $GL_n(\mathbb{C}, N)$, it's formed by the matrices which commute with N. Then we have an action by conjugation on the system. All these elements are just symmetries of the system 5.

Geometrically, this is the reason why we consider. This will be some kind of Hamiltonian system, completely integrable, but if you do not take into account this group, you have to take the reduction modulo the symmetry of the system. Then what is the fact? The fact is, you take M^N, take $h : M^N \to \mathcal{U}$, and you take M^N/G. You have the quotient through M^N/G. Then you have $\tilde{h} : M^N/G \to \mathcal{U}$. The action is free, proper, whatever, this quotient is an affine space, and we call this the reduced system. If you know something about the reduced system then you can say something about the original system. This was nonsingular so the reduced system is also nonsingular, we're in the situation we described. I like this idea because it's clear that you have to take into account the symmetry of the problem. This is the way we, well, okay, I'll stop here. Only one remark. So it's so nice that M^N is actually a principal bundle with group G over M^N/G. So everything is so nice, so nonsingular. Moreover, take Q in $\mathcal{U}$. Then $h^{-1}(Q)$, a fiber in M^N, but if you take $\tilde{h}^{-1}$ that's a fiber in M^N/G. The idea is that you have to say things about the fiber in the reduced system. We obtain something on the Jacobian or a subvariety. Because it's a principal bundle, we'll get the fiber in the original case as an extension of the Jacobian using the group G. So you have to compute the group G. The fact that the reduced case is completely integrable is Beouiville. The fact that the original case is completely integrable is Gavrilov. Then [unintelligible]. In the next two hours, I'll give details about all this construction and how they all work.

This is the idea. I want to prove that these two, just in the case where they're symmetric [unintelligible].

14. Nov. 20: Vasile Brinzanescu: Algebraic complete integrability of some Hamiltonian systems

So let me remind you of the equation we are working with

$$\frac{d}{dt}(X + \lambda N) = [X + \lambda N, NX + XN + \lambda N^2] \tag{6}$$

I'll treat now the arbitrary case, X is any matrix, not necessarily symmetric. In fact the result I will need is for a polynomial with bigger powers of λ. The only point is that they fix a matrix to the highest power of λ. Instead of having one matrix of unknowns, you take some kind of $\lambda^d N + \lambda^{d-1} X_1 + \cdots + \lambda X_{d-1} + X_d$, so you have many unknowns. The N, you only have to suppose that it is regular. What is regular? Every eigenspace is one dimensional. So we take distinct eigenvalues. Then the spaces have dimension 1. You can do a similar system for much more complicated polynomials in λ but the computations are the same.

I denote by $Q(\lambda, z)$ the characteristic polynomial, $\det(zI_n - X(\lambda))$. We fix this characteristic polynomial. Then we denote by M^N_Q the isospectral variety, that is

$$M^N_Q = \{X(\lambda) | \det(zI_n - X(\lambda) = Q(\lambda, z)\}$$

We denote by Γ_X the spectral curve $Q(\lambda, z) = 0$ which will generically be nonsingular and $\tilde{\Gamma}_X \subset \mathbb{P}^2_{\mathbb{C}}$ its completion, also generically nonsingular.

So then all the matrices of this form $M^N = \{X + \lambda N | x \in gl_n(\mathbb{C})\}$ maps to $\mathcal{U} = \mathbb{C}^S$ which takes $X(\lambda)$ to its characteristic polynomial, taking its coefficients.

There are a few results, not many, about singular fibers. You have to do a lot of things in that case. It's complicated and you cannot do effective computations.

All of our definitions are for the generic fiber. Now, also, I introduce this group $G = \mathbb{P}GL_n(\mathbb{C}, N)$, which is the projective group for the matrices which commute with N. It's very simple to verify that this is invariant under conjugation by G, this system. Then I take the quotient by this group M^N/G. The action, G acts freely and properly. The map is a very nice map and then we get a map $\tilde{h} : M^N/G \to \mathcal{U}$. We have these two systems. We call the quotient the *reduction* of the big one by the symmetry group of the problem. Now, I need some preliminaries on the Jacobian, just a few words.

I'll talk about the generalized Jacobian. I'm not so sure, but I'm an older man, I read this in the book of Serre. Some people said that Serre introduced it, I'm not so sure. It could have been before. Half of the book is about the generalized Jacobian, half [unintelligible]some number theory. I do not know exactly.

Let's let Γ denote Γ_X, just notation, a nonsingular curve. Take it actually to be the projective completion. You fix a so-called modulus which is an effective divisor $m = \sum \eta_i p_i$ where $p_i \in \Gamma$ and $\eta_i > 0$. For any such pair (Γ, m) we associate a generalized Jacobian. For a fixed curve you have a Jacobian, but for any effective divisor you have a generalized Jacobian.

Firstly, we define a singular curve $\tilde{\Gamma}$ (not to be confused with the projectivization) as $\Gamma_{reg} \cup \{\infty\}$. The Γ_{reg} is $\Gamma - S$ where $S = \text{supp}(M) = \{p_1, \ldots, p_k\}$. We intuitively, we take the modulus, and make it one point.

It's maybe two chapters to define this, not so complicated, I read it as a student. The structure sheaf on the curve $\tilde{O}$, I'll describe just the fiber at p. It will be $\mathcal{O}_p$ if $p \in \Gamma_{reg}$. If $p = \infty$ it will be $\mathbb{C} + I_\infty$, where I_∞ is the ideal of functions, algebraic functions on Γ having a zero at p_i for all i of order at least n_i.

Already from this point, I can say what our modulus is.

Remark 14.1. For our integrable systems, m will be the divisor of poles, that is, the point at ∞, $\{p_1, \ldots, p_n\}$ of $\overline{\Gamma}_X$, the points at ∞ in the completion. You take homogenous coordinates and solve, only the highest degree remains, and the highest degree is [unintelligible], and this will be the point at ∞, this is a finite number of points. It's easy to see that the points will be distinct, so it will be $m = p_1 + \cdots + p_n$.

A line bundle $\tilde{L}$ on $\tilde{I}$ is an element in $Pic(\tilde{\Gamma})$ by which I mean $H^1(\tilde{\Gamma}, \tilde{\mathcal{O}}^*)$. It doesn't matter that this is nonsingular, this is the general Picard group.

Now I want to say that, let D be a divisor on Γ_{reg}, so I avoid this support. Then it's easy to see by definition that D has a local equation f_α and we take the function $g_{\alpha\beta}$ to be $\frac{f_\alpha}{f_\beta}$ which lives in $\tilde{\mathcal{O}}^*(U_\alpha \cap U_\beta)$. You get a one-cocycle and that's a line bundle, you turn a divisor into a line bundle in this way, $\tilde{L}_D$. You know that line bundles, divisors, and locally free sheaves of rank one are the same.

For example, what does it mean that $D_1 \sim D_2$ with respect to this modulus? They are equivalent if and only if there is a global meromorphic function f on Γ such that $(f) = D_1 - D_2$ and you have $\mathcal{O}_{p_i}(f - 1) \geq n_i$ for all i.

Up to now I just constructed the singular curve and explained the structure sheaf and how you take line bundles.

Now let $Pic^0(\tilde{\Gamma})$ be the subgroup of $Pic(\tilde{\Gamma})$ of degree zero line bundles. What is the degree? In the case of the curve it's a Chern class. With this equivalence, the degree of a divisor is just the sum of the coefficients.

Then this is a subgroup and we denote this by $J(\tilde{\Gamma})$, which should be $J(\tilde{\Gamma}, m)$. This will be the generalized Jacobian. For the, this is a commutative algebraic group.

If you like I can say a little bit more. I still have time. This is isomorphic to $H^0(\Gamma, \Omega^1(M))^*/H_1(\Gamma_{reg}, \mathbb{Z})$ [something about $(w) \geq -m$]

If g is the genus of Γ then this lattice $H_1(\Gamma_{reg}, \mathbb{Z})$ has rank $2g + k - 1$ and what else?

Now recall the usual Jacobian. If you remember, it's $H^0(\Gamma, \Omega^1)^*/H_1(\Gamma, \mathbb{Z})$. But this is just $\mathbb{C}^g/\Lambda$ where the (real) rank of Λ is $2g$. This is the Jacobian, an Abelian variety of dimension g, a complex torus.

This is the relation between the Jacobian and the generalized one.

We have $0 \to G \to J(\tilde{\Gamma}) \xrightarrow{\phi} J(\Gamma) \to 0$, we have a short exact sequence. The map, since every map comes from a line bundle of a divisor, $\tilde{L}_D \mapsto L(D)$. The divisor is only on $\Gamma - S$. It's simple to see, because you can move by equivalence you can always find something like that. There are many line bundles here going to 1.

The kernel of this map, $(G \cong \mathbb{C}^*)^{k-1} \times \mathbb{C}^{\deg m - k}$ where the degree of m is $m_1 + m_2 + \cdots m_k$.

Just a remark, in our case, the degree of m is n, I have distinct points of multiplicity 1. Then k is also n. So $G \cong (\mathbb{C}^*)^{n-1}$. It's also easy to see that the exact sequence up there is never a direct sum. It's a nontrivial extension. These are classified by an Ext^1.

For the statement I'll need, well, θ will be the canonical θ-divisor on Γ formed by the special line bundles L_D of degree $g + n - 1$ (in our case), where special means that $H^1(\Gamma, L(D)) \neq 0$. So special means they have nonzero H^1. It's a very old notion, more than 150 years. They didn't know what a line bundle was but they worked with divisors.

Okay, now you have to protest. This will be a theta divisor in $J(\Gamma)$, but I took the degree $g + n - 1$. It's usual to take degree 0 to define the Jacobian. But it's just a translate if you choose a different degree. You translate in the group with some point z.

There is a canonical θ divisor on the curve, $\phi^{-1}(\theta)$, special line bundles here, okay.

Now, so far so good, it's already too much. You should also be worried about the fact that when I pick a divisor, I mean a class of equivalent divisors. When you take the equivalence you obtain a class of divisors. This is why it's sometimes good to work with divisors and sometimes better to work with line bundles or sheaves, such as when you are pulling back.

I asked myself when I was first learning, why do we need line bundles, divisors, and sheafs. Sheafs for cohomology and the other two for various problems in geometry.

So back to integrable systems. I'll present you a result of Beauville and Gavrilov. Parts of the result are obtained by [unintelligible]and on the other side by the Russian school, especially [unintelligible]. Mumford worked for some years on this problem. In fact it's better to know, it's a very old problem, this problem. It was in some sense only fixed the right way as of the 90s. You obtain the solution directly by θ functions.

I want to make a picture that will remain here for the second hour. [picture]

$$M_Q^N \xrightarrow{q} M_Q^N/G$$

$$\begin{tikzcd}
 & & J(\tilde\Gamma_X)-\tilde\theta \arrow[r, "\phi"'] \arrow[d, hook, "\tilde\ell"] & J(\Gamma_X)-\theta \arrow[d, hook, "\ell"] & \\
0 \arrow[r] & G \arrow[r] & J(\tilde\Gamma_X) \arrow[r, "\phi"'] & J(\Gamma) \arrow[r] & 0
\end{tikzcd}$$

Up to this level, ℓ is a biholomorphism, this is algebraically completely irreducible. [some discussion, too fast] The maps ℓ and $\tilde\ell$ are eigenvalue maps. I'll take a break and then complete the discussion.

Now I have to define for you these maps ℓ and $\tilde\ell$

Definition 14.1. The *eigenvector line bundle* of the spectral curve Γ_X, take a point (λ, z) in Γ_X, the completion of the curve, I mean, and now such a point, z here is a solution of the equation $Q(\lambda, z) = 0$. It's an eigenvalue for fixed λ. So we take an eigenvector f so that $X(\lambda)f = zf$, it can be written, has components $f(\lambda, z) = (f_1, \dots f_n)^T$, and we take this to be normalized.

What does this "normalized" mean? When you solve this equation, you have some denominator. These are meromorphic functions, but we can multiply, so we multiply and they have no common zeros. This vector will give us a map in $\mathbb{P}^1$.

The main idea is the following. Take the trivial vector bundle which is $\Gamma_X \times \mathbb{C}^n$ over Γ_X. Take here inside this something I'll call $L^\vee$ which, through a point (λ, z) the bundle, I take the line defined by this eigenvector, the normalized eigenvector. For the trivial vector bundle, I have $\mathbb{C}^n$. I take the line defined by this.

Then you take the lines thorugh the eigenvector f. Let L be the dual to $L^\vee$.

Then the map ℓ associates to the class of a matrix $X(\lambda)$ by conjugation (we take not just one matrix, rather a class), and this lands at L, the dual of this line subbundle, the dual eigenvalue map.

The $\tilde\ell$ is defined in a similar way, but you have to normalize in some other way, taking into account this modulus. It's more complicated but you can do that. You obtain the same thing, I'd have to explain it. Now, then, here, I'll put

Theorem 14.1. *(Beauville) The reduced $M_Q^N/G \to \mathcal{U}$ is algebraically completely integrable. What amounts to the same thing, ℓ is biholomorphic.*

Theorem 14.2. *(Gavrilov) The map $\tilde\ell$ is a biholomorphism, $M_Q^N \to \mathcal{U}$ is generalized algebraically completely integrable.*

Now we're finally coming to our problem. We have $M_Q^{N,sym}$, I take only symmetric matrices in M_Q^N.

Lemma 14.1. a *If $n = 2p$ and N is invertible then $GL_n(\mathbb{C}, N)$ is $P = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ where A and B are diagonal and $\det P \neq 0$ and so $G = GL_n(\mathbb{C}, N)/\mathbb{C}^* \cong (\mathbb{C}^*)^{n-1}$.*

b *It's just a simple computation but you should be careful. If* $N = 2p+1$ *and is nullity one (only one zero) then* $GL_n(\mathbb{C}, N)$ *is* $\begin{pmatrix} A & B & 0 \\ -B & A & 0 \\ 0 & 0 & \alpha \end{pmatrix}$ *with* $\alpha \neq 0$ *So* G *is again* $(\mathbb{C}^*)^{n-1}$

So G_1 is the subgroup of G generated by matrices where the a_i and b_1 are pairwise equal to each other. Take G_0 to be the quotient. It's interesting that this is isomorphic to the subgroup of $GL_n(\mathbb{C}, N)$ of matrices with the properties that $TT^t = I_n$. So you can take $M_Q^{N,sym}/G_0$. Then you get a natural map to M_Q^N/G.

Now the idea is the following. We have the key lemma.

—I should give a simple remark. If you take the transpose of $(X + \lambda N)$ you get $(X - \lambda N)$. So you obtain an involution on Γ, taking (λ, z) to $(-\lambda, z)$. Then c_1, the quotietn by this action, gives you a curve and a Prym variety Γ/c_1. The Prym is an Abelian variety included in the Jacobian of Γ. An involution on the curve gives a map between divisors, which gives a map on the Jacobian. This Prym is the antiinvariant part of the action. Then $J(\Gamma)$ is isogeneous (modulo some torsion is the same as) $J(G) \times Prym$. The Jacobian is *almost* a product.

Lemma 14.2. *$j : M_Q^{N,Sym}/G_0 \to M_Q^N/G$, induced by the inclusion, is injective. The map ℓ composed with j to Prym (Γ/c_1) is injective and biholomorphic onto an open subset of Prym.*

Now we prove what? The reduced system, the reduction by conjugation, which is G_0, is algebraically completely reducible.

Theorem 14.3. *(B, Ratiu) The reduced system $M_Q/G \to \mathcal{U}$ is algebraically completely integrable.*

This is the first result. For the reduced integrable system, this is algebraically completely integrable. If you want, you can compute the solution on the Prym. We should have assumed that N has distinct eigenvalues.

What about the unreduced case? We have Prym inside the Jacobian $J(\Gamma_X)$. Now the question is of algebra. In G we have a map to G_0, a surjective map. I have an injective map from Prym. So what kind of extension should I take in the middle?

We obtain the following exact sequence:

$$0 \to G \to \phi^{-1}(Prym(\Gamma/G)) \to Prym(\Gamma/G) \to 0$$

but the G perturbs us. It's an upstart. In my youth I was an algebraist, so let me play a little bit. Let us denote $G \to G_0$ by β. Then we get a map $\eta Ext^1(Prym(\Gamma/G), G) \to Ext^1(Prym(\Gamma/G), G_0)$. An extension is the same as the ext functor. So I take the thing in the middle as living in my ext group and I apply η and get this extension

$$0 \to G_0 \to E \to Prym(\Gamma/G) \to 0$$

where $E = G_0 \oplus \phi^{-1}(Prym(\Gamma/G))/K$. In any case, you're not allowed to take the direct sum, you have to divide, so this is not a trivial extension. It's natural. This is one construction of this morphim, in the laguage of extensions, from Eilenberg–MacLane. I'll say this K is $(-p(j), i(g))$.

So it's interesting, you cannot go down from M_Q^N directly. But you have these commutative algebraic groups and can prove indeed that this system is biholomorphic with an open part.

The last theorem is, the generic fiber of the system $M^{N,sym} \to \mathcal{U}_{[unintelligible]}$ is biholomorphic to an open subset of E. I cannot take dierctly what this goes to an open subset of what? What is the image? I have no other [unintelligible]. But algebraically it's quite natural.

Theorem 14.4. *This $M_Q^{N,sym}$ is a generalized completely integrable system.*

One minute. Gavrilov and [unintelligible]showed that [unintelligible]is not algebraically completely integrable. If you take the reduced, it is, but it's only generalized.

The system is not our system. So Γ in this case is an elliptic curve and m, if you write the characteristic polynomial, $m = p_1 + p_2$, two different points, and $\tilde{\Gamma}$ is a curve associated with the pair (Γ, m). You obtain the following

$$0 \to \mathbb{C}^* \to J(\tilde{\Gamma}) \to \underbrace{J(\Gamma)}_{\Gamma} \to 0$$

[Fast talking]

After that, Gavrilov realized that their computation can be done in a much more general way. Then he realized that he should use the general Jacobian.

A last remark, what is $\mathbb{C}^*$? It's nothing else but the complexified group of rotations around the axis of the Lagrange torus.

15. Dec. 21: Jun Ueki: Arithmetic topology on branched covers of 3-manifolds I

(1) Intro
(2) Idelic class field theory
(3) Iwasawa theory
(4) Galois deformations

Today I'll talk about the first parts.

- Arithmetic topology
- M^2KR dictionary
- Hilbert theory
- unbranched class field theory
- Iwasawa's old theorems and Galois cohomology of units

15.1. Arithmetic topology.

First Ill try to explain about arithmetic topology. This is a research field in number theory, an analogy to 3-dimensional topology. It is an analogy between number fields and 3-manifolds or between prime ideals and knots. It was first pointed out by B. Mazur in the 60s and developed by Reznikov and Kapranov systematically. Independently it was begun by Morishita.

The goal is to

- translate and find problems
- grow up the "dictionary", and
- explore the nature of the analogy.

There are various topics. For example, there is an analogy between Iwasawa theory and Alexander–Fox theory, pointed out by Mazur.

There is an analogy between $\left(\frac{p}{q}\right)$ Legendre symbol and $\ell k(K, K') \mod (2)$, and the Redei symbol for p_1, p_2, and p_3 with the Milnor invariant $\mu(K_1, K_2, K_3)$.

We didn't know why the Redei symbol was natural but looking at it as an analogue to the Milnor invariant makes it seem natural. Similarly, we can talk about the Milnor–Amono/Amono–Morishita invariant for quadruples of primes or knots. There is a conjecture of Deninger that there is a functor from number fields to 3-manifolds with foliations which takes prime ideals to closed orbits of dynamical systems.

We did only conceptual analysis in this talk.

15.2. **M2KR dictionary (basic).** A number field k is a finite extension of $\mathbb{Q}$. Actually, we think $\operatorname{Spec}\mathcal{O}_k \to \operatorname{Spec}\mathbb{Z}$, the integer ring of k. We assume M is a 3-manifold, connected, oriented, and closed. There is a finite branched covering over S^3, branched over a link (Alexander 1920).

For a prime ideal $\operatorname{Spec}\mathbb{F}_p \to \operatorname{Spec}\mathcal{O}_k$ corresponds to a knot $S^1 \to M$, an ideal $S = \{p_1, \dots p_r\}$ to an r-component link.

An unramified or ramified extension F/k corresponds to an unbranched or branched cover (branched over a link)

The étale fundamental group of $\operatorname{Spec}(\mathcal{O}_k - S)$ corresponds to $\pi_1(M - L)$.

I forgot to remark that an analogue of an infinite prime is an end of a 3-manifold. In this case it is the empty set. I'm not serious about the infinite primes in this talk.

15.3. **Hilbert ramification theory (branched Galois theory).** If F/k is Galois, then there is a natural injection of ideal groups in the opposite direction $I_k \hookrightarrow I_F$ which takes $\mathcal{P}$ to $\mathcal{P}\mathcal{O}_F$. If the degree is p then $\mathcal{P}\mathcal{O}_F$ is $\beta_1 \cdots \beta_p$ (decomposed) or β^p (branched) or β (inert).

If $h : N \to M$ is Galois, then we consider the 1-cycle group and there is a transfer map $h^! : Z_1(M) \to Z_1(N)$.

Fix a CW or PL structure on M and N, compatible with the covering map, so it admits a $Gal(h)$-action, including the branched set. Then we can define a natural map $C_*(M) \xrightarrow{h^!} C_*(N)$ by taking an open chain s to $\sum_{\sigma \in Gal(h)} \sigma S_1$,where S_1 is a component of $h^{-1}(S)$.

So $h^!(K) = K_1 + \cdots + K_p$ or $p\tilde{K}$ or $\tilde{K}$, as in the number theory. These behaviors are controlled by special subgroups of the Galois group. For F over T_i over Z_i over k or N over T_i over Z_i over N we have $\{1\} < I_i$ (the inertia group) which is itself in D_i, the decomposition group, which is in G.

15.4. **unbranched class field theory.** $Cl(k)$ is I_k/P_k, the ideal class group, the quotient of the ideal group by the principal ideals. It satisfies a reciprocity law $Cl(k) \cong Gal(k^{ur}_{Ab}/k) \cong \pi_1^{\text{ét}}(\overline{\operatorname{Spec}\mathcal{O}_k})^{Ab}$ A fact is that this group is always finite.

In topology, $H_1(M) = Z_1(M)/B_1(M)$, and the Hurewicz isomorphism says $H_1(M) \cong Gal(M_{Ab} \to M) \cong \pi_1(M)^{Ab}$. We sometimes fix the condition that $\#H_1(M) < \infty$, which is equivalent to $H_*(M) \cong H_*(S^3)$ if M is a rational homology three-sphere.

Now I'd like to present some example or exercise in translation.

15.5. **Iwasawa's old theorems.** Let F/k be finite Galois. If there is no non-Abelian sub-field extension, then

(1) $Nr : Cl(F) \to Cl(k)$ is surjective and $\#Cl(k)|\#Cl(F)$.
(2) If $(F : k) = p^V$, totally branched over a prime, then $p \nmid \#Cl(k)$ if and only if $p \nmid \#Cl(F)$.

A corollary is that for ζ_n a primitive root of 1, $p|\#Cl(\mathbb{Q}(\zeta_p))$ if and only if $p|\#Cl(\mathbb{Q}(\zeta_{p^n}))$ because of the cyclotomic fields $\mathbb{Q}(\zeta_{p^n})$ over $\mathbb{Q}(\zeta_p)$ over $\mathbb{Q}$.

Theorem 15.1 (U.). *Let $h : N \to M$ be a finite Galois branched cover.*

(1) *If there exists no nontrivial Abelian subcovers, then $h_* : H_1(N) \to H_1(M)$ is surjective, with $\#H_1(M)|\#H_1(N)$ and*
(2) *If the degree of h is p^n, totally branched over K a knot, then $p \nmid \#H_1(M)$ implies $p \nmid \#H_1(N)$.*
(3) *As a corollary, if $h_n : M_n \to M$ is a $\mathbb{Z}/n\mathbb{Z}$-cover branched over K, and $(m, p) = 1$ then $p|\#H_1(M_m)$ if and only if $p|\#H_1(M_{mp^n})$*

This comes from considering M_{mp^n} over M_m over M.