INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS QUANTUM MONDAY

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1. March 3: Junehyuk Jung: Spectral geometry with application to number theory

Thank you for the introduction and I'm grateful to have the opportunity to talk on quantum Monday. I want to talk about nodal domains of eigenfunctions on negatively curved surfaces. Take a look at my picture. The Chladni pattern was drawn more than a hundred years ago. If you spread out sand on a metal plate and play the violin next to it. This is the typical patterns you'll be getting. Surprisingly, these are related to the zero set of eigenfunctions on the square.

As the picture gets more complicated you'll see you get more domains that the lines divide things into.

The structure of the talk, I'll give a general setup for the theory. Since this arises from spectral geometry, I'll give a review of that field. Then I'll give my most recent theorem with Steve Zelditch, which is a theorem about these figures. After that, I'll connect these techniques to number theory. I'll explain how these can be applied to number theory. So this gives a way to understanding GL_2 automorphic forms in general.

[Why GL_2?] Because there are Laplacian eigenfunctions on hyperbolic surfaces. In these figures we see curves on surfaces. You can think of nodal domains and sets in higher dimensions, but we don't have techniques to attack this.

So let (M, g) be a smooth compact Riemannian surface. I'll consider two cases. In the first you don't have boundary, and we consider $-\Delta_g\phi = \lambda\phi$, where Δ_g is the Laplace-Beltrami operator. When we have boundary, we consider either the Dirichlet or Neumann boundary problem. We have the same condition on the interior. On the boundary, we have $B\phi = 0$, where B is the boundary operator such that $B\phi = \phi|_{\partial M}$ in the Dirichlet case and $B\phi = \partial_\nu\phi|_{\partial M}$ in the Neumann case.

Spectral geometry is the study of these eigenfunctions with large λ. We're interested in what happens as λ goes to ∞.

The nodal set Z_ϕ is the zero set of the eigenfunction ϕ. The critical point set C_ϕ is $\{x \in M | \nabla(\phi)(x) = 0\}$. The singular point set Σ_ϕ has both of these vanishing.

Nodal domains are the connected components of the surface minus the nodal set. [Discussion of pictures]. I can raise some quantitative questions. How long is the nodal line? You might want to compute the total length. One can also ask what the total curvature of the nodal line is. What is the number of critical points (as λ goes to ∞)? How about the singular points?

Today's talk will be about how many nodal domains you get? What is $N(\phi)$?

There are some standard conjectures. I'm first going to talk about known results. Firstly, regarding the total length of the nodal line, we know that the total length is bounded below by some constant multiple of $\lambda^{\frac{1}{2}}$ and above by $\lambda^{\frac{3}{4}}$. The lower bound is due to Brunning and Yau in 1978; the upper bound by Donnelly and Fefferman in 1987 and Dong in 1992 (he was the first to mention). What is better about the total length, if M is analytic, then we have a sharper upper bound of $\lambda^{\frac{1}{2}}$.

In the case of higher dimensional manifolds, you get the same asymptotics when M is analytic. But as the dimension grows the smooth upper bound gets worse and worse. All of this is in the empty boundary case.

Secondly, the total number of singular points, they are points and you can count, and this number is less than $\lambda^{\frac{1}{2}}$. Surprisingly, for generic metrics, $|\Sigma_\phi| = 0$. In a generic manifold there are no singular points for any eigenfunctions. This is due to Uhlenbeck in 1976.

The number of nodal domains $N(\phi)$ is less than some constant times λ. This is due to Courant's general nodal domain theorem, which says that ordering your eigenfunctions by the size of the eigenvalue, the nth has fewer than n nodal domain. Combine this with Weyl's law which lets you count eigenfunctions.

I should mention that Weyl's law, which works for any manifold, says that the number of eigenvalues less than T is the volume of the manifold divided by some global constant c times $T^{\frac{d}{2}}$ pluss $O(T^{\frac{d}{2}-1})$. This lets you read off the volume of the manifold from the eigenfunctions.

The fourth example is that there exist some (M, g) and a sequence of eigenfunctions ϕ_j such that although $\lambda_j \to \infty$, one has uniformly bounded number of critical ponts or nodal domains. This is due to Lewy in 1977. Spherical harmonics having just three nodal domains with large eigenvalue is in my picture. There's no general lower bound for the number of nodal domains. You can find a sequence in the square torus that have just two nodal domains. This was done in 1925, 1953, etc. by [a number of people].

Conjecture 1.1. *For any given M and g, can we find a sequence of eigenfunctions such that the number of critical points goes to ∞? This is one of Yau's problems. Can we find a sequence so that the number of nodal domains goes to ∞? This was Hoffmann-Ostenot.*

The second conjecture is a little stronger.

Conjecture 1.2. *If (M, g) is negatively curved (bounded away from zero) then for any sequence of eigenfunctions, we must have $N(\phi_j) \to \infty$.*

Conjecture 1.3. *This is Bogomolny and Schmit, published in Physical Review Letters. They predicted that if $M = SL(2, \mathbb{Z})\backslash\mathbb{H}$, then $N(\phi) \sim \lambda$.*

Here are some known cases. In 2007, Nazarov-Sodin proved that on S^2, for random spherical harmonics $N(\phi) = c\lambda + o(\lambda)$ "almost surely." By random spherical harmonics, the sphere you have high multiplicity of the eigenvalues. For eigenvalue approximately m you have approximately m eigenfunctions. Say you pick an orthonormal basis on that eigenspace $\phi_1, \ldots \phi_m$ for $\lambda = m$, then consider $\sum \alpha_j \phi_j$. If α_j is chosen randomly, following a Gaussian distribution, then with probability 1 (given by these Gaussian variables) one has this property. This tells you that the top right figure is quite rare and only happens with probability 0. One interesting thing is that they couldn't compute c. Sarnak conjectured that c must be the same for any manifold.

Now some number theory. If we think about the Maass-Hecke cusp forms (that I'll talk about later) then assuming the Lindelof hypothesis, the second conjecture is true, in fact that $N(\phi_j)$ is greater than a constant times $\lambda_j^{\frac{1}{12}-\epsilon}$. This is Ghosh-Reznikov-Sarnak last year.

Now without assuming anything, I can prove that $N(\phi_j)$ is more than a constant times $\lambda_j^{\frac{1}{8}-\epsilon}$ for almost all ϕ_j.

We can introduce this kind of picture on the sphere. We can consider Hecke operators on S^2 but then what one can do, all Hecke eigenforms under the Lindelof hypothesis one has $N(\phi_j) \to \infty$ via Magee in 2013.

You can see that the assumption on the surface was exploited for these papers. Here I'll state my theorem with Zelditch.

Theorem 1.1. *Assume that (M, g) is compact negatively curved smooth surface without boundary and I'll assume there's an isometric involution, orientation reversing isometric involution. Then if we think of joint eigenfunctions of the Laplacian and this involution σ, if the fixed point set is seperating (so you get two pieces with boundary on removing it) then there exists a density 1 subset A in M such that the number of nodal domains goes to ∞ inside A. So the first conjecture is true if you add this symmetry.*

We could find a lot of examples without any symmetry assumptions. So recently we showed that

Theorem 1.2. *If (M, g) is a generalized Sinai billiard. Then for any eigenbasis of the Dirichlet or Neumann boundary problem, one can find [missed]*

Let me explain the local structure of the nodal set. The nodal set could look arbitrary. You can find functions vanishing on any specified set. But there's nice structure. Around regular points, where the derivative doesn't vanish, we can find some small neighborhood so that it looks like a curve passing through x. More interestingly, around singular points, where the derivative also vanishes, then, due to various people (Bers 1955, Cheng 1976), then it's always some number of curves passing through that point. This is true for any solutions for elliptic partial differential equations. This plays a significant role in the proof. For example, from this you can look at Z_ϕ when ∂M is empty looks like a bunch of closed curves possibly passing through each other. Then when there is boundary, it might be a finite union of closed curves and segments with endpoints on the boundary. So from this observation, you can put a graph structure on the nodal set. You let the nodal set be a graph. Each intersection point is a vertex. Then the nodal set becomes an embedded graph on a surface. The critical observation here is that the graph structure, the nodal domains are the face of the graph. What can we do with graphs on the surface? Euler's inequality. This says if v is the number of vertices e edges, f faces, and c components, then $v - e + f - c \geq 1 - 2g$. Since f is the number of nodal domains, I get $f \geq 2 - 2g + e - v$. This comes from the discrepancy between edges and vertices. Every vertex has more than two edges emanating from it, you can say $2e \geq \sum_V degx$ which is more than $2(v - a)$ where a is the number of vertices having degree greater than or equal to 3 plus $3a$. So I get $2(e - v)$ is greater than or equal to three, so $f \geq 2 - 2g + \frac{1}{2}\#x$. So if you can count the number of these things on the boundary you get a growing lower bound coming through this.

In the boundaryless case I need the involution because when you have a nodal curve passing through the fixed point set, well you can treat that like the boundary.

Here's a question. Say f is continuous, real valued, on $[0, 1]$. How can you tell if f has a 0 on the interior. My answer will be trivial but I want any others. If

$f(0)f(1) < 0$ then there should be a zero. The method I've used in this paper is if $\int_0^1 |f|dx \geq |\int f|$. Then I found out that this condition is somewhat not very useful, we can't count the number of zeros. There have to be some other techniques.

So I'll fix $\beta \subset \partial M$ and compare $\int_\beta |\phi|$ against $|\int_\beta \phi|$. I'll apply the Kuznecov sum formula to give this. This becomes the relative trace formula eventually. This was proved by Zelditch in 1992. The trace formula says, say ϕ_i is an orthonormal eigenbasis for the Laplace Beltrami operator, ordered by eigenvalues. Fix a submanifold H of M and fix a smooth function f on H. Look at

$$\sum_{\lambda<T} |\int_H \phi_i f dV_H|^2 = c\int f^2 dV_H \sqrt{T} + O(1).$$

[An example spoken but not written].

From this, I know the average size. I'll take f to be a bump function on β to make it look like the square of the right hand side. This tells you that $|\int_\beta \phi_j|$ is approximately $\frac{1}{\lambda_j^{\frac{1}{4}}}$ on average. To be precise you can actually prove using Chebyshev's inequality that it's less that $\frac{log^{\frac{1}{2}}\lambda_j}{\lambda_j^{\frac{1}{4}}}$. So now we need a lower bound for the left hand side. Surprisingly, most PDE techniques are only concerned with L^2 or L^∞ norms. So there's no direct way to estimate the L_1-norm. So what I'll do is, we can say $\int_\beta |\phi_j| dz \sup|\phi(z)| \geq \int |\phi|^2 dz$. This is Holder's inequality.

Regarding this, we actually know that $\sup|\phi(z)|$, when M is negatively curved is bounded above by a contsant times $\frac{\lambda_j^{\frac{1}{4}}}{\log \lambda_j}$ which is due to Berard, exploiting the error term of Weyl's law. He divided that thing by a log term. All that is left is to find a nice enough L^2 bound to get a nice L_1 bound. The theorem called quantum ergodic restriction gives you $\int_\beta |\phi|^2 dz \to c_\beta$ for a density one subsequence of the eigenfunctions. Therefore if you combine all these ingredients, then for a lower bound of the L^1 norm you will be getting $\frac{\log \lambda_j}{\lambda_j^{\frac{1}{4}}}$. For the upper bound you had a smaller numerator.

All that's left is to say what QER is.

If you fix $\beta_1, \ldots, \beta_r$ are disjoint subsets of the boundary, then a density one subsequence of eigenfunctions, if you apply this argument, $N(\phi_j) > R$. Then if you look at this statement closely, it has nothing to do with the choice of the $\beta_1, \ldots, \beta_R$. This allows you to find a density one subsequence with a growing number of nodal domains. This is heavily used in the theorem I'm going to say right now. All I'm going to say right now is what QER is and before that I need to say what quantum ergodic theory is.

This is concerned about, for a given manifold, say that you weight your volume form by the eigenfunction $|\phi_j(z)|^2 dV_g$ and want to see what happens when the energy goes to ∞. This quantum ergodic theory deals with the limit of that distribution function as the energy goes to ∞. If you look locally the energy spreads out evenly. The quantum ergodicity theorem supports this heuristic but not completely.

For any compact smooth manifold without boundary (when you add boundary you need more conditions) and for any orthonormal eigenbasis $\{\phi_j\}$ there exists a density one subsequence of $\mathbb{N}$ such that you approach dV_g. They do spread out evenly on the manifold. This was proved by Zelditch in 1987 and some other people in the late 80s.

When they proved this, they used the kind of diagonalization argument again and again. The computation itself was about integration of a function, a special function, against these. Then they approximate anything using the diagonalization argument. It might be different j. The subsequence can depend on f but you can throw away that dependency. This is the theorem. Then here I'd like to mention the quantum unique ergodicity conjecture. This theorem, one can ask what happens when, well, can you take the whole sequence? The answer is no, not even on the sphere. There are eigenfunctions concentrated basically on the equator. If you normalize your surface measure then it gets concentrated more and more on the equator, and the limiting measure is a singular measure supported on the equator. The theorem is wrong if you don't say density one. The quantum unique ergodicity conjecture says that if M is negatively curved, then $A = \mathbb{N}$. So the reason why people expect there are no exceptional eigenfunctions is when you think about measures that arise as weak limits of the weighted measure, it was proved these have to be invariant under geodesic flow. That's chaotic in the negatively curved case. It's hard to find even a singular but stable-under-flow example. For example, Anantharaman said it has to have entropy greater than zero.

So now let me explain the quantum ergodic restriction theorem. What do we get if we restrict the measure on hypersurfaces? This theorem tells you that if H is "not very symmetric," then $|\phi_j(z)|^2 dV_H$ approaches dV_H along a density one subset. So again this is wrong if you remove the density one condition. You have to allow the density zero exceptions. I'm not going to talk very much about the not very symmetric condition. Then there's a theorem by Toth-Zelditch last year, if H is β and you take some smooth function supported on β you get the result.

So this is the end of the proof of my theorem. Let's have some break and then I'll talk about number theory.

So you can, here is one comment about nodal domains. You can think about the vector field with gradient of the eigenfunction and can triangulate with respect to this vector field and each vertex corresponds to points that are local minima and maxima of the eigenfunction. For each nodal domain you find one, and that corresponds to the point where every vector points to or from. For instance, my argument, the previous proof gives you, it tells you that there are a growing number of minima and maxima.

Can we find an upper bound for this number almost everywhere? If we triangulize the manifold with respect to that flow we'll get domains that are not the same as the nodal domain. We can also study this triangulation. Zelditch was very interested in this.

[discussion degenerates a little bit]

So when $M = SL(2,\mathbb{Z})\backslash\mathbb{H}$, then $-\Delta\phi = (\frac{1}{4} + t^2)\phi$, and Maass-Hecke cusp forms are parameterized like this. And then $T_n\phi = \lambda_\phi(n)\phi$, where T_n is the Hecke operator

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum 0 \leq b \leq d-1 f(\frac{az+b}{d})$$

These are key to studying automorphic forms on GL_2. Because these operators commute with the Laplacian, you can assume that you've picked things that are joint for these and the Laplacian. So that's what Maass Hecke cusp forms are. Now the isometric involution is the center line, the purely imaginary axis. I'll call this η. It's $Fix(\sigma)$. I'll assume ϕ is even or odd, assume $\phi(x+iy) = \pm\phi(-x+iy)$. Call these even and odd.

Now I'm going to fix a curve here, γ, and then see how many sign changes i can find here. That will give you the number of domains. If I want to compute the number of sign changes on γ, it's slightly different, we need to look at

$$\sup|\phi(z)|, ||\phi||_{L^2(\gamma)}, \sup_{|} \int_{[\alpha,\beta]} \phi(s)ds|$$

The goal is to prove that for *all* eigenfunctions you have a growing number of nodal domains.

If we know the three quantities, if ϕ changes sign m times on γ, then I'll call the points a_i. You can bound $||\phi||_{L^1\gamma} = \sum|\int_{a_j}^{a_{j+1}} \phi ds|$ Then this is bounded above by my third quantity M_ϕ times $(m+1)$.

So then I can also get $||\phi||_{L^1\gamma} \geq \frac{||\phi||_{L^2(\gamma)}}{\sup|\phi|}$.

It's technical but I can expand $\phi(z)$ out, look at the Fourier expansion with respect to the cusp to get the expression

$$\phi(z) = \sum \rho_\phi(n)\sqrt{y}K_{it}(2\pi|n|y)e^{2\pi inx}$$

Then quantum unique ergodicity is equivalent to $\sum \rho_\phi(n)\rho_\phi(m+n)\psi(\frac{n}{t_\phi})$ converging to zero. This was a Fields medal.

Consider the triple product L-function, $L(\frac{1}{2}, \phi x\phi x\phi_0)$. This being 0 (with some other condition) is equivalent to QUE.

[I stop taking notes].

2. March 10: Hiro Tanaka, Factorization homology for stratified manifolds II

Like I said last time, this is joint with David Ayala and John Francis. Last time, what did we do? We chose a symmetry monoidal category $(\mathcal{C}, \otimes)$ like the category of spaces with direct product or the category of chains or vector spaces with tensor product. Today we'll also use the category of chains with direct sum for fun.

We built E_n algebras in these categories. We saw last time that for $n = 1$, an E_n algebra in vector spaces is an associative algebra. Given our E_n algebra we constructed an invariant "factorization homology" of framed n-manifolds. For instance, when $n = 1$, this gives an invariant of 1-manifolds. Given A, it associates to a circle the Hochschild homology of A.

Today what I want to do (it might be ambitious) is classify all homology theories, give some examples, and then move on to the stratified case. Does anyone have qeustions about the reminder I gave?

Let's give a classification in part one. I'll classify what John Francis originally called homology theories for manifolds.

Given an E_n algebra A, which is a functor from framed n-disks with disjoint union to $(\mathcal{C}, \otimes)$, the factorization homology functor $\int A : Mfld_n^{fr} \to \mathcal{C}$ satisfies the following properties.

- It respects the topological structure of hom spaces. Isotopies go to homotopies. So for instance in chain, an isotopy gives us a chain homotopy.
- It can be lifted to a symmetric monoidal functor (using II)

- It satisfies excision. If I have a manifold M which is given as a decomposition of two manifolds glued along a product manifold $Y \times \mathbb{R}$, then we have the formula for factorization homology $\int_M A = \int_{M_0} A \otimes_{\int_{Y\times\mathbb{R}} A} \int_{M_1} A$.

This is a satisfying kind of invariant. Let me give a name to functors with nice local to global properties.

Definition 2.1. *A functor H from framed manifolds to $\mathcal{C}$ is called a homology theory for framed n-manifolds. Let $\mathcal{H}_n^{fr}(\mathcal{C})$ denote the category of such homology theories.*

We have functors. Given E_n-algebras in $\mathcal{C}$ I get a functor to homology in $\mathcal{C}$ via factorization homology. We have a functor the other way called restriction. You can work with just disks if you have a homology theory that evaluates on all manifolds.

Theorem 2.1. *(Francis) This functor, factorization homology, is an equivalence of ∞-categories, and the restriction is an inverse.*

Studying manifold theory in a black box, these three natural conditions, every one of these things comes from an E_n-algebra. To study things for spaces that satisfy Eilenberg Steenrod axioms, you just check what they do on a point. This is a generalization of that.

What's a sketch of a proof. How do you prove this equivalence for ordinary homology theories? You start with a point, then move to a sphere, and then think of spaces as CW-complexes and you know the effect from Mayer-Vietoris. At the end of the day I get a homology, determined completely by what I did to a point.

Given a manifold you can put a Morse function on it and build your manifold with Morse functions. You can recover the value on your manifold from what it does to handlebodies and that you can get from the sphere which you can get from $\mathbb{R}^n$.

Now let's do the example of homology in the usual sense. Take our target category to be chain complexes with direct sum. This is a symmetric monoidal structure. Let me give a lemma.

Lemma 2.1. *Given $V \subset Chain$ there is a unique algebra structure on V with respect to direct sum.*

Let's give a proof sketch. Set V to be a vector space for convenience. An algebra structure on V is a lot of data but in particular has maps $m : V \oplus V \to V$ and a unit map $0 \to V$. The unit map is uniquely defined. This satisfies certain conditions. If we weren't in the world of vector spaces, the conditions are actually extra data. In vector spaces, it's some conditions like associativity.

A map m is just two maps $V \to V$. If $0 \to V$ is the unit, then the multiplication map should satisfy equations which imply that both maps are the identity and the map is addition. So E_n algebras are just chain complexes. Then chain complexes are in bijection with homology theories in this category.

Claim: a homotopy theory with target $Ch, \oplus$ is just singular homology with coefficients in some chain complex V. What does excision tell us? If we have the manifold X decomposed as usual, then excision tells us, first of all, what's a module structure? If M_0 is a module over V. I should give a map $M_0 \oplus V \to M_0$. Moreover, on M_0 it should be the identity. So it's just a linear map $V \to M_0$. Call it g_0. Likewise, a left module is the same.

Now how do I compute the tensor product? The factorization homology over X is $\int_{X_0} V \oplus_{\int_{Y\times\mathbb{R}} V} \int_{X_1} V$. How do I compute this? I take a colimit over a diagram that looks like

$$M_0 \oplus M_1 \leftarrow M_0 \oplus \int_{Y\times\mathbb{R}} V \oplus M_1 \leftarrow \cdots$$

Here M_i is $\int_{X_i} V$. It's easy enough to check that this is a pushout, that's just an exercise. When you have a pushout diagram of chain complexes, you get a long exact sequence in homology. With some homological algebra this is Mayer Vietoris.

That's more or less the sketch of a proof. You recover everything else from what you get on $\mathbb{R}^n$.

Now I'd like to go on to the stratified case if that's all right.

If you don't just study manifolds but also algebraic geometry or combinatorics, then you know about interesting singular spaces. It makes sense to try to find invariants for stratified spaces. When people say stratified spaces, they might mean a bunch of things, so let me give you what I mean.

For example, what's a singular 1-manifold? It should include one-manifolds with no singularities. It might also include real lines glued along a vertex. Anything glued out of these is a singular one-manifold. So graphs where you're allowed to have no vertices.

Let me generalize this. It might look like $\mathbb{R}$ or like some n-valent vertex. Locally every 1-manifold looks like a cone on a zero manifold (crossed with $\mathbb{R}$).

Given Z, the cone of Z is $Z \times [0,\infty)/(z,0) \sim (z',0)$. If z is a circle, you start out with the cylinder and end up with a cone.

The real line is the cone on the empty set (if my definition doesn't say that, it should) cross $\mathbb{R}$.

This is my model of singular manifolds.

Definition 2.2. *We say X is a singular n-manifold if X looks locally like a cone on Z times $\mathbb{R}^{n-k}$ where Z is a singular $k-1$ manifold.*

Some examples might help, say in the case $n=2$. If $n=2$, if we take $Z=S^1$, then the cone of Z is a cone. It's a copy of $\mathbb{R}^2$ with a marked point. If Z is some collection of points, then I get a graph cross $\mathbb{R}$. If Z were the empty manifold I'd get $\mathbb{R}^2$ with decoration.

Let me say the algebraic consequences and then take a break. What are some consequences? As a black box, one can define a category $Mfld_n^{\mathcal{S}}$ of singular manifolds. Let me say that you can decorate this category with all kinds of things. You can restrict kinds of singularities. You can get notions of atlases. You can put structure on the manifolds. I've been agnostic about framing. You can demand framings or not. In normal theories we demanded framings, we could drop that, whatever we want. This is singular manifolds with certain kinds of structure.

Why would we want to generalize? We can get different kinds of structure. Let me give some examples and then we'll take a break. For notation, let $Disk_n^{\mathcal{S}}$ denote the subcategory of $Mfld_n^{\mathcal{S}}$ which is local structures or shapes of $Mfld_n^{\mathcal{S}}$, closed under disjoint union.

Let me tell you what I mean. If I were looking at $n=2$, I'd get Y-shapes cross the real line and the marked point and so on. With no singularities I'd get $\mathbb{R}^2$ and disjoint copies of it.

Let me give some examples. I know it's abstract.

Definition 2.3. *A $Disk_n^{\mathcal{S}}$ algebra is a symmetric monoidal functor from this category to the target category.*

If there were no singularities this would be the E_n structure you're used to.

What are some examples?

Let $Mfld_1^{\mathcal{S}}$ be the category of framed 1-manifolds with boundary. Framing will just be orientation. You have a circle, you have $\mathbb{R}$, and you have at least two other objects, the positive and the negative half-open intervals. In particular, anything glued up of these three pieces, these three lines, is the category of disks.

What kind of algebraic structure does this give us? In case you're keeping track this is the cone on a point.

What is a $Disk_1^{\mathcal{S}}$-algebra? It contains all the disjoint unions of $\mathbb{R}$ so whatever you associate to $\mathbb{R}$, that should be an E_1-algebra. It has an associative algebra. What about to the positive interval. Call the thing M_0, the object $\mathcal{C}$. What does this have, what structure? It's a functor so if I have a map $\mathbb{R}_{\geq 0} \sqcup \mathbb{R} \to \mathbb{R}_{\geq 0}$? This should go to a map $A \otimes M_0 \to M_0$. If I have a symmetric monoidal functor, I'll get a right module. Now maybe you see the punchline. How about the other manifold with boundary? You get a left module. This fancy notation gives you an E_1-algebra with two modules.

What is the invariant you get from factorization homology. What do you get from the interval? You get the tensor product over A of M_0 and M_1. That's the end of part one.

Shall I start again? Now we're really getting into something fun and exciting. I'll state two theorems that are generalizations of earlier theorems. First let me give a definition.

Definition 2.4. *Fix an algebra A, a functor $Disk_n^{\mathcal{S}} \to \mathcal{C}$, symmetric monoidal. Then factorization homology is the left Kan extension*

$$\begin{tikzcd} Mfld_n^{\mathcal{S}} \arrow[r, "\int A"] & \mathcal{C} \\ Disk_n^{\mathcal{S}} \arrow[u] \arrow[ur] & \end{tikzcd}$$

Theorem 2.2. *(Ayala-Francis-Tanaka) If $X = X_0 \cup_{Y\times\mathbb{R}} X_1$ then $\int_X A \cong \int_{X_0} A \otimes_{\int_{Y\times\mathbb{R}} A} \int_{X_1} A$*

Theorem 2.3. *(Ayala-Francis-Tanaka) There's an equivalence of categories from $Disk_n^{\mathcal{S}}$-algebras to homology theories for stratified manifolds*

Here's a cheater's way of saying this. You build your manifolds out of handlebody decompositions. We characterize our finite-enough manifolds. In spirit the proof is the same.

The tensor product I discussed is an example of the theorem.

You can also put marked points on your one-manifolds. So now let's take 1-manifolds with marked points (and no boundary). Locally it looks like a copy of $\mathbb{R}$ with an embedding of $\mathbb{R}^0$ inside. So now what is a disk algebra in this setting? As before, I get an E_1 algebra A fro the real line. A real line with a marked point gets M. There are two interesting embeddings from the real line. If I get one marked point I get $\mathbb{R} \sqcup \mathbb{R}^0 \subset \mathbb{R}$ and two maps of this to the marked point. I claim that this corresponds to actions of the algebra on M from the left and the right. Drawing the obvious diagram, this is a bimodule action. For this choice of $\mathcal{S}$ and this choice of singularity, what is an algebra? An E_1 algebra and a bimodule.

Now what's factorization homology of a circle with a single marked point? Doing the same application of excision as last time, it's the factorization homology of the real line tensored over two lines with the factorization homology of the bimodule. So what is this?

We know factorization homology on all these intervals. I get a tensor product $A \otimes_{A\otimes A^{op}} M$. So those in the know in algebra, this is Hochschild homology with coefficients in the bimodule M. This is a definition for Hochschild homology with coefficients in any category.

Of course, that's not the only manifold with marked points. If I put k marked points, it's the Hochschild homology with coefficients in $M^{\otimes k}$.

Let me make one more comment about $\mathcal{S}$. Here I took framed one manifolds with marked points. What if I included marked points and colors? This could be a game where every marked point has either red or blue. There would be two different colors. Red and blue. So to each of these you'd associate different objects. An algebra is an algebra with two bimodules. Now if you evaluate you get Hochschild homology with coefficients in tensor products. You can also give intervals colors, and these are different algebras. You can glue together different manifolds with different labels in some physical settings so this gives a model for those physical systems.

Now let $Mfld_2^{\mathcal{S}}$ be framed 2-manifolds with marked points. I have a framed two manifold with marked points. It locally looks like $\mathbb{R}^2$ or $\mathbb{R}^2$ with a marked point. What do we get? To $\mathbb{R}^2$ we associate an E_2 algebra, and to the marked $\mathbb{R}^2$ we'll get M, and now the space of module actions is interesting. We can find embeddings, a circle's worth.

So we get an S^1 worth of module actions. There's very non-trivial geometry going on.

[Some discussion]

As an example, in the three dimensional setting, you can take framed three manifolds with framed sub-one-manifolds. These are the kind of singularities we allow. If you're keeping track, we are allowed a three manifold with a link inside it. Locally, this looks like a copy of $\mathbb{R}^3$ or near the link, $\mathbb{R}^3$ with a standard $\mathbb{R}^1$ inside. If you're keeping score, this corresponds to $c(\emptyset)\times\mathbb{R}^3$ or $c(S^1)\times\mathbb{R}$. What are these two things? In the case that $\mathcal{S}$ stands for this kind of singularity, you get an E_3 algebra for $\mathbb{R}^3$, call it A, and what do we associate to $\mathbb{R}^1$? Call it M and see what kind of structure it has. The $\mathbb{R}^1$ parts have an E_1 algebra structure. I can shrink down and embed my E_1, chicken on a shish kebab, I had some in Busan this weekend. So an algebra, we get a map $M\otimes M\to M$. This is an E_1 algebra. So now we get an E_3 algebra and an E_1 algebra. There's also an action of the E_3 guy on the E_1 guy. There's an interesting family of embeddings that I can find.

This is a bunch of pictures. This is the higher Deligne conjecture.

Theorem 2.4. *I apologize if I miss some names. Getzler-Jones, Voronov, McClure-Smith, Thomas, Francis, Lurie, et cetera*
In short, the above data is equivalent to giving a map $\int_{S^1\times\mathbb{R}^2} A \to HH^*(M)$ *of* E_2 *algebras.*

Now the question is, what are the link invariants you get? It's hard to find E_3 actions on E_1 algebras?

Some simple computations that you can do, this part may be a little category theoretic. Can we detect something? The biggest question is where you can get examples.

So here's an example. Where might you get such an example? If someone gives you an algebraic structure, can you give an example? A cheater's answer is a free algebra. You can create the free algebra, that's a cheater's answer but let's do it. Fix X. Then I claim, well, $Disk_n - Alg(Spaces)$ forgets to spaces. This has a left adjoint called free. So here I've gotten rid of "framing" to make things easier. What is $Free_{Disk_n}(X)$? it's a disjoint union of configuration spaces $Conf^k(\mathbb{R}^n) \times_{\Sigma_j} X^j$.

I claim this is a $Disk_n$ algebra. If you have two configuration like this, you can shove the configurations in, that's the E_n algebra structure.

Here's a theorem.

Theorem 2.5. *(Ayala-Francis-Tanaka) We can take factorization homology of this $Disk_n$-algebra. We get $\amalg Conf^j(M) \times_{\Sigma_j} X^j$.*

If you like, X is a point, then you just get $Conf^j(M)$. Hopefully that's understandable. If you now have a link inside of M, then you get, evaluating on $Free_{Disk_3^S}(X, Y)$, a souped up version of this, $\amalg Conf^j(M\backslash L) \times Conf^k(L)$ (with points in X and Y). Let's take Y to be empty.

Then we get the configuration space of j points in the complement of the link. You can detect π_1, which is a knot invariant. That was the last example I wanted to end with.

3. March 24: Bruno Vallette: Givental action and trivialization of circle action II

Let me tell you briefly what I'm going to do. I have two operadic resolutions I want to do totally explicitly. This will give a good definition of homotopy BV algebra. You get a good notion of homotopy BV. Then from that I will derive two things, address some questions. Inside we have the BV operator, Δ, and we relax, we think of that as a circle. When we trivialize we get one thing, when we go to the higher things, we get Givental.

Last time we had a homotopy retract, and if we had an operator which squares to zero. We got a first δ_1, δ_2, et cetera. The relation $\Delta^2 = 0$ is broken and instead we get $\partial\delta_2 = \delta_1^2$. Higher up, we get that $\partial\delta_n = \sum \delta_i\delta_{n-i}$. This is a mixed complex becoming a multicomplex. This is $T(\Delta)/\Delta^2$, a representation of this algebra, and I call this a circle because it's $H^*(S^1)$.

We saw that an algebra over $D = T(\Delta)/(\Delta^2)$ does not have good properties. I should replace this by something projective. We replace with something quasifree, meaning it's free except for the differential.

What do we have on the right hand side? We have $\mathbf{k}1$, $\mathbf{k}\Delta$, and nothing else. We start with a free algebra on δ_1, and we get δ_1, $\delta_1^{\otimes 2}$, et cetera.

I'm saying this should split with respect to some weight. This splitting will simplify my life. Say Δ has weight 1, so 1 is weight 0, and so on. This is not the homological degree. It's bigraded. If you want the homological degree, I will write it for you. Chopin agrees with the bigrading. The homological degree of Δ was 1, the homological degree of δ sholud be as well. So now we're good to weight 1. So we introduce a first syzygy to kill this guy. So I put δ_2 in weight 2. It's the homotopy for $\delta_1^{\otimes 2}$. So it must live in homological degree 3. I have nothing else in weight two. It's perfectly acyclic. I'm doing what you're doing with modules and syzygies. I'm not working with free modules, I'm working with algebras. Now what does δ_1 and δ_2 generate? Well, we have $\delta_1 \otimes \delta_2$ and $\delta_2 \otimes \delta_1$ in weight three. I have a non-trivial kernel of the differential which is the sum of the two. Let me introduce a generator δ_3 which should be of degree 5 and weight 3. The boundary of δ_3 should be this sum, $\delta_1 \otimes \delta_2 + \delta_2 \otimes \delta_1$.

Higher up I hope you see the same pattern, where $boundary(\delta_n) = \sum \delta_k \otimes \delta_{n-k}$. This is a step-by-step Koszul-Tate resolution. This is the transfered structure. Let me call this algebra D_∞. What is a D_∞ algebra? It's a dg algebra map from D_∞ to $End(A)$. This is a morphism of dg algebras. First let's see that it's a morphism of algebras. The underlying guy is a free algebra. So it's free on $\delta_1 \oplus \delta_2 \oplus \cdots$. To be a morphism of associative algebra, you need to give the image of the generators. So δ_1 should give you a map, give you $\bar{\delta}_1$ in $Hom(A, A)_1$. The image of δ_2 should be $\bar{\delta}_2 \in Hom(A, A)_3$, et cetera. Now these must commute with the differential. Each time I have a generator, I can apply the differential to get $\sum \delta_k \otimes \delta_{n-k}$ and then look at the image in $End(A)$ and get the composite $\sum \bar{\delta}_k \circ \bar{\delta}_{n-k}$. If I do it the other way around, I get $\partial\bar{\delta}_n$. We've proved that to have a well-behaved algebra structure, we should replace by a free algebra. Now I've solved the problem.

What we do, if we have algebras of type P, the idea is to encode that with an operad $\mathcal{P}$ which encodes the category but is not well-behaved up to homotopy. You'd like to be able to find a homotopy stable category, like homotopy P-algebras. That's the kind of guy you have on the right hand side. You resolve $\mathcal{P}$ to $\mathcal{P}_\infty$. Let's say $\mathcal{P}_\infty$ is quasifree, then because it's quasifree, we get a category that's well-behaved in homotopy.

So we can view Δ as an operator with one input and one output, like $+$. We should compose them in many ways, like $\mp$, but I write this linearly because there is only one output. But I quotient by Δ^2 and then everything vanishes except $+$.

What about As? An associative operator has one product, if you look at the free operad on that, I get planar binary trees. If I want this to be associative, I identify left combs with right combs. So what do I have? I have the trivial tree, I have my product, and in arity three, let me see what I have. For me, the product also has the homotopy type of a point. But in three we have two points but we have to identify them. Let me try to find a resolution of this?

What does it mean? We are in the world of operads. We want a free operad with a differential so that the underlying complex is equivalent to As. I need a generating product. I take the free operad on that one binary generator. I will recover A_∞ algebra by hand. In arity five you have five planar trees. Do you agree? They are all identified with the associative relation. It's just one point. If I consider the free operad generated by a single product I get the same kind of picture. Let's postpone that. I'll put that in degree zero, drawing a topological picture. I have the right thing in arity two. In arity three I have total degree zero, I have two points on the left and one on the right. These cannot be quasiisomorphic. I have to introduce a homotopy to relate them. I should introduce one generator in three inputs of degree 1. I want to attach the interval on these two products. So d of the 3 to one corolla is the associator of the product.

This is exactly what you do with a module. You get, passing to chains, a quasi-iso. You're perfectly happy. Let me do the next step. I've introduced the corolla, it will appear in the next step. I'll get a tree, and d of that tree will be one of the five vertices. So I continue and draw in, I think I have the five trees. Here's what I have in arity four. I have loops. I must introduce one new corolla of arity four and its boundary is the sum with appropriate signs of the boundary of this pentagon.

You introduce one corolla for each n whose boundaries are stacks of two corollas. I have resolved the operad. This is a quasifree resolution of As. Let me call that A_∞. So A_∞ is a free resolution of As. What is a morphism of dg-operads to $End_A = End(A^{\otimes n}, A)$. This only needs to be defined on the generators since A_∞ is quasifree. I'll get a degree zero product in $Hom(A^{\otimes 2}, A)$, a degree 1 operation of arity 3 μ_3, and so on. If you look at what's going on with the corolla, you get $\partial\mu_n = \sum \mu_k \circ_i \mu_\ell$.

You can give that example to any freshman undergraduate. Then you can tell me, Bruno okay, that's cool, you did two very trivial examples. One generator with simple relations. What if now we do a more complicated example.

Let me take a more difficult example, the operad encoding BV algebras. It's made up of a commutative product and a Δ operator. The relations are associativity for the product, we should have that $\Delta^2 = 0$, and we should have the order two relation, [picture] if I do Δ of the product, this should be the same as doing Δ of two of them, and multiplying by the third, there are three of these, and then there are three places to do Δ of one of them.

This is the operad BV. The product comes with a symmetric group action. We take the symmetric action for granted. If you can say what to put there, by hand, you should get the fields medal. Solving that problem will give me a good notion of homotopy Batalin-Vilkovisky by hand.

We were starting from a Δ and a commutative product. Remember the two examples we did before. Relaxing Δ up to homotopy, we should have δ_1, δ_2, and so on, surely this has to sit there. The associativity should relax up to homotopy too, a product, a triple product, and so on. I've relaxed the two relations independently. I should introduce a guy for their interrelation. I should have d of that guy is the seven term relation. You need to introduce a first homotopy for that relation. That will fill your space. This is the number of operations you have.

If you have associative operad, you resolve to A_∞, for commutative you resolve to C_∞. A C_∞ algebra is an A_∞ which vanishes on shuffles.

Now Koszul duality theory. Here's a machine. What do you mean by it? If you give me an operad, if I want simple models, I'm taking extra data. I assume it comes equipped with a presentation with generators and relations. If R is quadratic inside the free operad, meaning that there are at most two vertices, then you tell me, I can't apply Koszul duality theory to the BV operad. We can use Koszul duality theory to resolve $\mathbf{k}[\Delta]$ and As but not BV. Koszul duality was extended from homogeneous quadratic to linear and quadratic by me and my collaborators for this purpose and then to constant linear and quadratic in Millès' thesis.

We make something, a "cooperad" which has a cocomposition that is coassociative in a certain way. We have a candidate in a quadratic case for the syzygies $\mathcal{P}^{¡}$. Here we have trees in $\mathcal{P}^{¡}$ and we apply the decomposition everpwhere we can. Your d_2 amounts to doing the sum over all vertices by splitting each vertex in two. This gives you a candidate. It also comes equipped with a morphism of *dg* morphisms to P. You can check whether a certain complex is acyclic.

I've been talking for more or less an hour, do you want a break now or do you want the BV case first?

How do we define morphisms? What is a BV_∞ algebra structure? There is what I call the Rosetta stone of operads. There are at least four equivalent ways to say what a P_∞ structure is. The first answer to describe P_∞ algebra structures on a chain complex. The first definition is $Hom_{\mathrm{dgOp}}((\mathcal{T}(P^{¡}), d_2), End_A)$. So what is this? I have a map $P^{¡}, End(A)$, but it should agree with the differential. If I were taking maps from a coalgebra to an algebra this is a convolution algebra. This is a differential graded preLie algebra. Just trust me, this is a dg Lie algebra. What is your favorite equation in a dg Lie algebra? It should be the Maurer Cartan equation. $\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0$. Guys which commute with the differential are exactly Maurer Cartan elements of this Lie algebra. If you want, this the definition and it gives a deformation theoretical interpretation. This is where you can see deformation theory. I could prove homotopy transfer with this guy. I could give this here, but let me do it in a third place. A third is, what is the Hom? What does it mean that we're looking for maps for any n that go $P^{¡}(n) \to Hom(A^{\otimes n}, A)$, which is the same as having maps $P^{¡}(n) \otimes_{\mathbf{k}[S_n]} A^{\otimes n} \to A$. This is another data. I should have a map $\alpha : \bigoplus P^{¡}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \to A$. This is the cofree $P^{¡}$ coalgebra on A. Let me take As as an example? The dual is As (up to duality). So $As^{¡}(A)$ is $\bar{T}^c(A)$. A morphism of dg operads is equivalent to this sequence of maps on the cofree coalgebra $Coder(P^{¡}(A))$, and the compatibility is that it squares to zero. A $\mathcal{P}_\infty$ structure on A is the same as a square zero derivation on $P^{¡}(A)$. So now a morphism is a $P^{¡}$-algebra map commuting with the square zero derivations.

So we can look at maps. I can define an ∞-morphism to be a morphism of $P^{¡}$ algebras. The map is a morphism of coalgebras. It's characterized by a certain complicated condition.

[In a usual cochain complex, you can put chain maps, homotopies, et cetera, into one category. Is this a defect?]

We have to try again to define homotopy, tensoring, building cylinders, and so on.

[This doesn't have a nice presentation. It doesn't have a nice presentation.]

You have an exponential, infinitely many terms.

In non-symmetric operads you can take $C_*[0, 1]$. You get something nice and cute. They could do this because they have a nice interval.

[I'm still objecting, it is difficult to find an explicit formula of when two morphisms are equivalent. It was not even clear until recently that it forms a homotopy category.]

There's definitely work that needs to be done to prove this, Lefevre-Hasegawa did this in his PhD thesis. It's definitely not trivial.

I think I'll use this later on but shall I define it now? I did the case of As but what I should do is to do this for the algebra of dual numbers and get a notion of ∞-morphisms for those. Here $D^{¡}$ is the cofree coalgebra on one generator δ, and now, if I continue I can recast a D_∞ module, it's a square zero codifferential inside the comodule $T^(\delta)(A)$.

Let me explain to you two things.

I'm going to do something operadically speaking, tell me if it corresponds to what you did. We have quadratic and cubical, I can't apply Koszul duality theory. You add an extra generator, in this case the Lie bracket induced by the product and the differential. You introduce an extra generator and you introduce a relation, which is that the bracket is the default of the product and delta having compatibility. Instead I get the Leibniz relation between the product and the bracket. So what is a BV algebra? It's a product and bracket and an operator, I have a Gerstenhaber algebra and then get an operator. If you step back, you see that this is quadratic. I'll definitely put this $BV^{¡}$, and that's how you cook it. I did this with Galvez-Carrillo and Tonks.

Theorem 3.1.

$$(T(BV^{i}), d_1 + d_2) \to BV$$

is a quasiisomorphism.

I've introduced an extra generator. I took three generators. I have a Lie bracket. Now I had a Δ operator, I have a commutative product, but I won't have this by hand, it will be in a big something, I have the Lie bracket and I'll relax that. I'll have homotopies for the Leibniz relation.

This produces a notion of BV_∞ algebra. What I'm explaining, what is the size? It's Gerstenhaber, δ Gerstenhaber, δ^2 Gerstenhaber, and so on.

The idea we had with Gabriel, and he has the right to complain even more, is this guy isn't minimal, in Sullivan's homotopy theory, here I have trees, I explode into two and some places we do something internal. Instead of taking $BV^{¡}$ with a differential, but with Gabriel, we take the homology and take the free operad on that.

Theorem 3.2 (Getzler)**.** *$H_*(BV^{i}, d_\phi)$, well, it eats a δ and a commutative product and gives you one bracket. All the homology dies except in some places. I need the resolution of the circle action $T^c(\delta)$. The rest, how does it organize? It's nothing but the cohomology of the open moduli space of curves $H^{*+1}(\mathcal{M}_{0,n+1})$.*

Let me interpret this as BV. What is the rest of the statement?

Theorem 3.3 (Drummond-Cole-Vallette)**.** *$T^c(\delta) \oplus H^{\bullet+1}(\mathcal{M}_{0,n+1}), d)$ is a minimal resolution of BV*

What is a BV_∞ algebra? You can split into two, three, four, and higher up.

Now what is a map to End_A? The first bit on the left hand side is δ_1, δ_2, and higher up. This is a coalgebra. On this guy, you only have a d_2. This is nothing but a multicomplex. Inside a BV_∞ algebra you have a multicomplex. Then you get operations that are labelled by the cohomology of your open moduli space. Getzler proved that this guy is the Koszul dual of $H_*(\overline{\mathcal{M}}_{0,n+1})$. I considered this last week, and this operad was encoding hypercommutative algebras. Moreover, this operad is Koszul. If you take trees on this with d_2, you get homotopy hypercommutative algebra. This bit is homotopy hypercommutative, maybe? Not really. Why not really? It has as many operations as one but, if I have no d_3 and higher, it would be so. Inside this combinatorics, you might be able to split into three, taking guys from the left hand side. It satisfies some relations with the left hand side. Alone, they are perfect, either one, but together they must mix.

Let me give you one or two examples. What can we say? Let me make that precise? If inside a BV-algebra, the $T(\delta)$ is zero, you get a hypercommutative algebra up to homotopy. Let me write this on the other side.

Example 3.1. *If all δ_i are zero, then this notion is equivalent to a hypercommutative algebra up to homotopy. This amounts to the vanishing of the $T^c(\delta)$ part.*

What if everything vanishes except the product and the δ_i? The higher homotopies for μ are zero, it's commutative associative. We get a multicomplex. It strictly commutes with the product. That is, δ_n is order n with respect to the product.

I wanted to conclude with something nice.

We can prove a nice theorem. Let me half-prove the conjecture I mentioned the other time. If you take $fD_2/S^1 \cong \bar{\mathcal{M}}_{0,n}$. Let me prove this with the model. We did this over the rationals. Resolve the whole picture. Over the rationals, the homology of the framed little disks is the operad BV, so I take BV_∞/D_∞. If I take this quotient, we get $T(H^{\bullet+1}(\mathcal{M}_{0,n+1}))$ which is $H_*(\overline{\mathcal{M}}_{0,n+1})$. You can prove this on a topological level, you'll have to ask Gabriel in the Journal of Topology, right?

I keep saying from the beginning that the motivation is the homotopy transfer theorem. The motivation is the homotopy transfer theorem.

Theorem 3.4 (Drummond-Cole–Vallette)**.** *Starting with a BV_∞ algebra and a deformation retract, for any such structure on A, there exists an equivalent BV_∞ structure on H such that I can rectify to produce a strict BV algebra, perhaps not precisely the one I began with but an equivalent one.*

We encode good invariance. I'll use that later on. In the particular case, when we transfer we get the structure on the right hand side, if δ_i are zero on the right hand side we get homotopy hypercommutative on the right hand side. If the transferred δ_i are zero, the structure is a $Hypercomm_\infty$ algebra. I'll give you several examples of that, like the de Rham cohomology of a Poisson manifold. When is this transfer zero is the question of homotopy trivialization of the circle, but how do you detect that? I'll give you conditions for that and if we have time I'll talk about the Givental action.

Under which conditions do you have, let's say you start from a BV structure. You have a Δ and a product. What kind of structure on A could make the Δ vanish on homology? We have three examples of that. One is the $d - \Delta$ condition.

$$ker d \cap ker \Delta \cap (Im d + Im \Delta)$$

This is Deligne–Griffiths–Morgan–Sullivan. This means that $A \cong H \oplus S \oplus \Delta S + dS + \Delta dS$, and Δ and d are isomorphisms on this S square. This proves that $\Delta = 0$. All these conditions imply that $\delta_1 = 0$. All of these imply that $\delta_1 = 0$ and actually that $\delta_2 = 0$ et cetera. But what is the implication for the BV algebra structure? When you transfer the δ_1 is killed. This example was treated by Barannikov–Kontsevich and treated in Manin's book. On $H(A, d)$ they have an action of $H_*(\overline{\mathcal{M}}_{0,n+1})$. There is another condition that Jae-Suk coined which is called semi-classical. So for this any element in $H(A, d)$ admits a representative in $Ker\ \Delta$. Then you can get formality of A as a Lie algebra with d. The $d\Delta$ lemma implies it. The third condition, implied by all of these, is that $\delta_i = 0$, and then you get H carries a $Hypercomm_\infty$ structure with trivial differential.

So from the weakest condition we get the strongest result.

With the strongest result I can reconstruct the BV.

[Some discussion.] This allows you to understand the mixture. Now let me look at the conditions on Δ to see when these things vanish. We have this general framework for P_∞ structures. I'll restrict to mixed complexes, D-algebras and D_∞, multicomplexes. What is the Lie algebra which controls D-algebra structures? I have maps $\bar{T}^c(\delta), Hom(A, A))$. I have $\delta, \delta^2, \delta^3$, and this is isomorphic to $zEnd(A)[[z]]$. So z has degree -2. Here you know the Lie structure. What is the differential? The differential of $r_i z^i$ is $\partial(r_i)z^i$. I have a Lie algebra. My multicomplex are just Maurer-Cartan guys. This is $MC(\ggg_{\Delta,A})$. If I have some α, the classical deformation theory for dg algebras, I have $\lambda \in \ggg_0$ induces $d\lambda[\lambda, —]$ in $\Gamma(MC(\ggg))$. We say two structures are gauge equivalent if we have γ with $\dot\gamma(t) = d\lambda + [\lambda, \gamma(t)]$ along with $\gamma(0) = \alpha$ and $\gamma(1) = \beta$.

So the trick is to take $\ggg \oplus \mathbf{k}\tilde{d}$, a new Lie algebra with $d\tilde{d} = 0$ and $[\tilde{d}, \alpha] = d\alpha$. Now take α to $\tilde{d} + \alpha$ and the Maurer Cartan equation becomes $[\tilde\alpha, \tilde\alpha] = 0$. This I can solve. If we have $\lambda \in \ggg_0^+ = \ggg_0$, we can solve $\dot\gamma(t) = [\lambda, \gamma(t)] = ad_\lambda(\gamma(t))$. We know how to solve that. The solution is easy. It's $\gamma(t) = \exp(t\ ad_\lambda)(\alpha)$ (provided things are sufficiently nilpotent).

I could definitely make a $\ggg^+$, but instead I'll look in the counital version, $Hom(T^c(\delta), End(A)) = End(A)[[z]]$. Then $\bar\delta \cdot 1 \mapsto \partial_A$. Here I have a differential graded associative algebra $\mathcal{A}_{\Delta,A}$. So now working there, my equation is easy to solve. I have $\exp(r_{t\lambda} - \ell_{t\lambda})$ where $r_{t\lambda}$ takes x to $x \cdot t\lambda$. But these two operators commute and I'm done, I get $e^{-t\lambda}\alpha e^{t\lambda}$. I should have been putting tildes on things. The gauge action is now exactly that formula. So $\bar{d} + \beta = e^{-\lambda}(\bar{d} + \alpha)e^{\lambda}$.

Let's call a guy gauge trivial if there is a λ such that...

Theorem 3.5 (Dotsenko–Shadrin–Vallette)**.** *α is gauge trivial if and only if it is homotopy trivial.*

A multicomplex is homotopy trivial if and only if it's gauge trivial. So if I want to do the proof, this equation is the equation for morphisms of multicomplex. So $e^\lambda(\bar{d} + \beta) = (\bar{d} + \alpha)e^\lambda$. So e^λ is a map which is the identity on the first level.

Let me do the proof. If α is gauge trivial, then it satisfies this equation. There is a morphism between the trivial structure and α. Then you can see that α is zero on homology very easily. You have a gauge interpretation and a homotopy interpretation and they agree very well.

Now let me tell you, if you take the de Rham cohomology of a symplectic manifold which satisfies the hard Lefschetz condition, then you get a dg BV algebra structure with this sort of degeneration on the de Rham cohomology. You have $[\iota_\omega, d_{dR}]$ and this is BV. This is a theorem by Merkulov and Olivia Mathieu. Then you use Barannikov-Kontsevich-Manin and get the hypercommutative structure.

Let me take something simpler. You just need Poisson. Remove the hard Lefschetz condition and you still get this, and we have much more, you get hypercommutative up to homotopy. The proof is now one line. I need to prove that α is gauge trivial. Here, I take $A = \Omega M$ and do Cartan calculus. I have a structure α it's a series in z, it's $d_{dR} + \Delta z$, that's it.

Now let me put that in here. I want to find a λ so that $e^\lambda \circ d_{dR} \circ e^{-\lambda} = d_{dR} + \Delta z$. I'll take $\lambda = i_\omega z$ and I claim that's enough. So this is, using the same trick as before, $e^{ad_{i_\omega z}} d_{dR}$ which is $d_{dR} + [i\omega, d_{DR}]z + [i_\omega, [i_\omega, d_{dR}]]z^2 + \cdots$ but all terms after the first vanish since $i_[\omega, \omega] = 0$. If you consider Jordan manifolds, which include contact manifolds, you get a bivector field $w \in \Gamma(\wedge^2 TM)$ and $E \subset \Gamma(TM)$, you cook up a Δ_2 which is order three. So this is a homotopy BV algebra. The contact manifold's forms have a homotopy BV algebra structure. This is in my paper, it's very readable. You can prove the same result. Then you can refine.

I hope you like this because I like this very much. This was, I'm inside this big world of homotopy BV algebras. We are $T(\bar{T}^c(\delta) \oplus H^{\bullet+1}(\mathcal{M}_{0,n+1}))$ I'd like an interpretation with a dg Lie algebra. It was enough for $Hom(\bar{T}^c(\delta), End(A))$ to look at $zEnd(A,A)[[z]]$. So we should have two pieces, $\ggg_{\Delta,A}$, and another part, $Hom(H^{\bullet+1}(\mathcal{M}_{0,n+1}), End(A))$. These are both Koszul duals. This is again a convolution algebra and Maurer-Cartan guys correspond precisely to $hypercom_\infty$ structures an A. If I have a cooperad up to homotopy, this endomorphism thing is an L_∞ algebra.

The most accurate way to say this mathematically is the following, $\bar{T}^c(\delta) \subset \bar{T}^c(\delta) \oplus H^{\bullet+1}\mathcal{M}_{0,n+1} \to H^{\bullet+1}\mathcal{M}_{0,n+1}$. When I go higher, I get the following exact sequence $\gg_{\Delta,A} \hookrightarrow \ell_{BV,A} \to \ggg_{hypercomm,A}$. We looked before when $\bar{T}^c(\delta)$ vanishes. Now we'll get Givental, when it doesn't.

Now I have to work with classical deformation theory in any L_∞ algebra. I want to do deformation theory there. I consider the Maurer-Cartan equation $d\alpha + \frac{1}{2}\ell_2(\alpha,\alpha) + \frac{1}{6}\ell_3(\alpha,\alpha,\alpha) + \cdots$. Like this it would not exist but applied to one α it works because of *local* nilpotence. The gauge action is a bit more subtle. Let me take $\lambda \in \ell_0$, then I can look at $d\lambda + \ell_2(\lambda,\alpha) + \frac{1}{2}\ell_3(\lambda,\lambda,\alpha) + \cdots \in T_\alpha MC_\infty(\ell)$.

This defines a vector field at point α. Now we can try to integrate. Let's hide that under the carpet. Now we can conclude. So [c'est bon, ça]. So let my take a hypercommutative structure α. That's a Maurer-Cartan guy here. Let me take for λ in the D_∞ side the $r_i z^i$. I said this acts on hypercommutative algebras.

Theorem 3.6 (Dotsenko-Shadrin-Vallette)**.** $\widehat{r(z).\alpha} \in T_\alpha\{$*hypercom algebras*$\}$*, let me draw for you, here's the set of all* BV_∞ *structures. [picture of plane]. Among those I have homotopy hypercommutative algebras [line in the plane].*

First, $\ell_1^\alpha(r(z)) \in T_\alpha MC(\underline{\ggg})$ *is tangent to this. So they are in the same space and can be compared, and* $\widehat{r(z).\alpha}$*, the Givental action of* $r(z)$ *on* α*, is exactly the same as* $\ell_1^\alpha(r(z))$*.*

Here the model we proved, we used no intersection at all. He did some computations, combinatorics of trees plus homotopical algebra.

On the one hand you have a formula from intersection theory. On the other homotopical algebra.

Now I can maybe tell the sad story. We thought we found something nice, we submitted to Inventiones. The referee said, well, Kontsevich gave a talk on this ten years ago, saying that the Givental action should be a change of trivialization of the circle action. This is not that, this is the infinitessimal action. I have to integrate if I want that result. If I integrate, let me integrate to $t = 1$ and get Givental. So here I have Givental action. So how can I explain that side? I integrate that to $t = 1$, it's finite, no problem. Okay, what's the formula, it's the gauge action, so the formula is an exponential as before. So you have *something* like $e^{r(z)}\alpha e^{-r(z)}$, this is the infinity isotopy of a multicomplex. These are trivializations of the S^1 action. So now I can make this completely precise. The Givental action is precisely the same as the action of the trivialization of S^1. I have an operadic picture, maybe for Gabriel.

Maybe I want to tell you the action operadically.

[some questions about stabilizers, some comments from Jae-Suk]

Here is an interpretation. A Givental action amounts to an automorphism of operads. I'll act on $Hypercom_\infty \vee T(\bar{D}^{\mathrm{i}})$ with no suspension. This is trivialization of Δ-actions.

4. March 31: M5 branes, holography, and knots

Let M be a three manifold such as S^3. Let A be a connection on a principal G bundle over M. Then $[DA]$ is summed over all possible connections weighted by the Chern Simons functional of A.

$$\int [DA] e^{-\frac{1}{2\hbar} CS[A]}$$

The connection can be written as a $\ggg$-valued 1-form on M, where $\ggg$ is the Lie algebra of the gauge group G.

This connection is defined up to gauge equivalence $A \sim g^{-1}Ag + g^{-1}dg$.

Finally the Chern Simons $CS(A)$ is $\int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$. So this is $Z[M, G, \hbar]$.

This kind of invariants were first introduced by Witten more than twenty years ago. This should include the Jones polynomial. We will consider more structure and define a more complicated invariant, adding a knot K on this three-manifold and a representation R of the gauge group, a homomorphism $G \to GL(m)$. Here m is the dimension of the representation. So apply the representation to the holonomy of A along the knot and then take the trace.

We would call the representation a charge.

[Some discussion]

Witten found that these quantum invariants are related to knot invariants. $J_m(K, q)$ is one of these, it's $Z(S^3, SU(2), \hbar, K, R_m)/Z(S^3, SU(2), \hbar)$ where $q = e^\hbar$. Here R_m is the unique irreducible m-dimensional representation of $SU(2)$.

Before we turn, we can calculate the colored Jones polynomial by resolving crossings. There is no three-dimensional definition of the Jones polynomial until this.

Our main interest is for G a non-compact group, some $PGL(N) = SU(N)_{\mathbb{C}}$. So $PGL(2) = SL(2)$. We complexify our gauge group. The reason people are interested is because it has application to the volume conjecture (Kashaev, Murakami, Murakami)

Conjecture 4.1.

$$\lim_{m \to \infty} \frac{1}{m} \log |J_m(K, q = e^{\frac{2\pi i}{m}})| = \frac{1}{2\pi} vol(S^3 \backslash K).$$

So $S^3 \backslash K$ is a knot complement, removing a solid torus from S^3.

Let me explain the physical origin of this conjecture. For this, such a field theory interpretation is crucial. The charged particles are charged by m. We're considering m going to the infinite limit. The knot becomes very heavy and will deform the geometry. So S^3 becomes $S^3 \backslash K$, the knot becomes a black hole. This is a very naive physical argument. This improves to more general cases. Here we're taking m, doing analytic continuation, and give it complex values. We let $SU(2)$ become its analytic continuation $PGL(2)$. This m should be the same as the Chern Simons theory on $PGL(2)$. This is not rigorous but please believe it.

The asymptotic limit $J_m(K, q)$ for $m \to \infty$ and $q \to 1$ with q^m fixed becomes governed by $SL(2, \mathbb{C})$ Chern Simons theory on $S^3 \backslash K$.

I'll expand how the hyperbolic volume appeared here.

So what we'll do is perturbatively expand, for $PGL(2,\mathbb{C})$ Chern Simons on $S^3\backslash K$. We'll take the limit as $\hbar \to 0$. We'll use a method:

$$\int \lim_{\hbar\to 0} d\vec{z} e^{\frac{1}{\hbar}f(\vec{z})} = (\frac{2\pi}{\hbar})^{dim} \sum_{\alpha} e^{\frac{1}{\hbar}f_0+\hbar^0 f_1+\cdots}$$

The first term, $\frac{\partial f}{\partial \vec{z}}(\vec{z}(\alpha)) = 0$

$$f_0^{(\alpha)} = f(\vec{z}^{(\alpha)})$$

$$f_1^{(\alpha)} = -\frac{1}{2}\log\det(-\frac{\partial^2 f}{\partial z_i \partial z_j}(\vec{z}^{\alpha})).$$

The [missed] point of the Chern Simons functional are flat manifolds, $dA+A\wedge A = 0$. We have $\partial M = T^2$ if we are dealing with knot complements. We can pick eigenvalues for the holonomy along spanning pairs for this torus, those holonomies should be e^u and e^{-u}. You get a finite number once you impose these restrictions. The leading order is given by $-\frac{1}{2\hbar}CS(A^\alpha)$. This is the expansion $e^{\frac{1}{\hbar}S_0^\alpha + S_1^\alpha + \hbar S_1^{(\alpha)}}$.

One natural θ point, there is one natural hyperbolic metric, this is the definition of the hyperbolic manifold. Using this metric we'll give flat connections and then get the volume. We'll obtain a one-form so that $ds^2 = \sum e^\alpha \otimes e^\alpha$. So we construct $A^{geom} = -(w+ie)$

[getting tired, missing things.]

The second term of our expansion is the so-called rational torsion. We started from a poorly defined path integral, but the perturbative expansion has this well-defined form. The same will be true of the higher order terms.

So the volume conjecture is the first order part of this equation. This is Gukov, ten years ago. We'll extend this to the $SU(N)$ case, with $PGL(N)$. Here we'll use m which labels the representation, but here we need $(m_1,\ldots,m_{N-1})$ and K and $q = e^\hbar$. So we have $J_{(m_1,\ldots,m_{N-1})}(K, q = e^\hbar)$. In the limit as $\hbar \to 0$, with $m_i\hbar = 2\pi i$, we should get $\frac{1}{\hbar}S_0^{(geom)}(M = S^3\backslash K, N) + \hbar^0 S_1^{(geom)}(M,N) + \hbar^1 S_2^{(geom)}(M,N)\cdots$ The S_0, for $N=2$ comes from $S_0^{geom}(A) = CS(A^{geom})$. That's hyperbolic volume. In general it's $\frac{1}{6}N(N^2-1)vol(M)$.

The definition of S_0 is $maxIm(CS(A)^{flat})$, ranging over all flat connections.

For S_1 you get $\frac{1}{2}\log T[M, A^{geom}]$, where A^{geom} maximizes for S_0.

We say, use $R_N[A_{N=2}^{geom}] = A_N^{geom}$. That's our guess for A^{geom}. You could use it as a definition.

It's nontrivial to check that S_1^{geom} limits to $-\frac{vol(M)}{6\pi}N^3 + O(N^2)$. This is only valid asymptotically. This is another conjecture, we proved this with Chern Simons methods and Muller proved it in 2010.

Our conjecture is that the second order part $Im[S_2[N,M]]6\pi \overset{N\to\infty}{\to} -\frac{Vol(M)}{24\pi^2}N^3$.

We expect that the $n \geq 3$ leading term $\frac{1}{N^3}(S_{n\geq 3}) \to 0$.