INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS QUANTUM MONDAY

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1. November 17: Theo Johnson-Freyd: Heisenberg-picture quantum field theory

I don't have, I normally have a pad of paper and I don't even have that. My goal, for what I would like to tell you, I'd like to propose a definition for what I call a kind of Heisenberg picture in quantum field theory. There are various types, most of my talk is trying to propose a definition, so it will mostly be motivation.

There are various things you could mean, what I want it to mean is something in the style of Atiyah-Segal axioms. Then of course, no definition is very interesting if there aren't any examples, so maybe I'll mention examples, including examples that you wouldn't have if you didn't have slightly more general axioms.

I've been here for five hours and spent the time doing a computation with Gabriel so I don't know what you know.

I don't have any examples that are not topological.

All I mean by "in the style of Atiyah-Segal" is something like functors on a cobordism category or something like that.

All of the functional analysis has to do with some kind of fine-tuning the target, and I'm not going to touch it, because I don't know how to do it.

The motivation I want to begin with is something that I hope is very pendantic, which is to remind you what you learned in your first course on quantum mechanics. This gives some picture of why you might want to write down Atiyah's axioms.

The Schrödinger picture is probably on page one of a quantu mechanics textbook. So a Schrödinger picture quantum mechanical system conistis of:

(1) a Hilbert space $\mathcal{H}$ and perhaps
(2) some unitary operators, for each $t \in \mathbb{R}_{>0}$ a unitary operator $U_t : \mathcal{H} \to \mathcal{H}$ which satisfy a group law, and maybe
(3) some other distinguished states, I don't know, in $\mathcal{H}$, which you know how to prepare, and maybe som costates $\mathcal{H} \to \mathbb{C}$ which you know how to postpare. some distinguished "observables" in $End(\mathcal{H})$.

This doesn't tell you the interesting questions.

This is the data of a quantum mechanical system, and you have a specific Hilbert space in nature and you develop functional analysis and you ask questions and get numbers and already this tells you what to do, why you would care about the Atiyah-Segal axioms, because I've said let's picture intervals of length t, well, a quantum mechanical system in the Schrödinger picture is a functor from the category of spacetimes, which in this case is Riemannian, maybe I should say Lorentzian one-dimensional cobordisms, intervals, to some sort of category of Hilbert spaces or vector spaces or something. There's a subcategory of metric intervals, but I want some distinguished end, if my distinguished states are labelled by some set X, then I can picture a state as being, a label of a state, as being a length 0 morphism to the Hilbert space. I can incorporate any types of states and costates and observables, If I have an observable $\mathcal{O}$ I can picture it as a special type of morphism of length zero but with some data. So this is packaged together as a functor.

What you next read in a quantum mechanical textbook is about a Heisenberg picture. Let me try to write this down and fail. So a Heisenberg picture, still in quantum mechanics, is:

(1) Some algebra $\mathcal{A}$, maybe a C^*-algebra (and the example, you should be able to translate $\mathcal{H}$ to a Heisenberg picture, you let $\mathcal{A} = End(\mathcal{H})$,
(2) coming from the conjugation action by U_t, a family of parameterized algebra automorphisms. Another example to keep in mind, it's either a feature or a bug, this also accomodates classical mechanics, so you should think that $\mathcal{A}$ can be functions on a phase space, $C^\infty(M)$, and the family of maps is something like Hamiltonian flow, thought of as an operation on functions.

    So another example is perturbative deformation quantization.
(3) ?

Let me come back to the third point.

There's a deep problem with the Schrödinger picture. This isn't quite the physically meaningful thing, the Hilbert space $\mathcal{H}$ is not physical. The true states of the system are not vectors here but the projectivization, $\mathbb{P}\mathcal{H}$. That suggests that instead of talking about a Hilbert space you should talk about its projectivization. That suggests that $Vect$ should be replaced with a projectivization. But superposition, additivity, is important and I've thrown that away.

If you have two systems modeled by a pair of Hilbert spaces, you can produce new systems from these. You get things like the system where you have a particle either in system $\mathcal{H}$ or system $\mathbb{K}$, and this is modeled by $\mathcal{H} \oplus \mathbb{K}$, and you also have the thing where you have one guy in each one and they don't talk to one another, this is the tensor product. It's very hard to projectivize the direct sum or the tensor product from the projectivization of the factors.

So the Heisenberg picture fixes this problem too. Maybe this is the best reason to abandon the Schrödinger picture. You can ask about recovering the Hilbert space from its endomorphisms. The data, the algebra of endomorphisms of $\mathcal{H}$ encodes projectivization of $\mathcal{H}$ exactly and not $\mathcal{H}$ itself. The automorphisms, algebra automorphisms, of $End(\mathcal{H})$ is $PGL(\mathcal{H})$. a projective bundle is not a bundle of vector spaces, each projectivized, it's the same thing, rather, as a bundle that looks locally like a matrix algebra. You should think about talking about families of quantum mechanical systems. Already if you take, maybe, $\mathbb{RP}^2$ and stabilize once, this is the smallest manifold with a projective bundle that doesn't come from a vector bundle.

I don't know Hilbert spaces with parameters ranging over this, but you could write things down, bigger examples, that just don't globally come from something like this.

So this is clean for the Heisenberg picture, $End(\mathcal{H} \otimes \mathbb{K})$ is $End(\mathcal{H}) \otimes End(\mathbb{K})$ up to functional analysis.

I've been going very slow.

Anyways, so, how do you incorporate a line in a vector space? this should give you its annihilation ideal in $\mathcal{A}$. To handle number three I'll say

3 some (left or right) ideals

Does this make sense as a functor? I could write down algebras and algebra isomorphisms. This makes sense modulo the ideals, those maps are not isomorphisms.

There are reasons this is not the right answer, purely topological, in a higher dimensional field theory, there's no reason that a manifold in higher dimensions, a generic cobordism, will give an isomorphism of Hilbert spaces. I have a Hilbert space of states on a circle, and the value of the map is the matrix whose entry is the amplitude to have those inputs and that output, that data doesn't package into an isomorphism in general.

Before I wrote down that I had a vector space with some unitary operators, really I meant automorphisms, I should have just said linear maps. Then I no longer have conjugation. Problem two is the ideals.

So what can you do? Whatever a Heisenberg picture is, it should involve the two examples I wrote down. You could say, what can you do? Let me just give the answer.

Let me just give you the answer for where you can do this. You could ask yourself what happens, let me say it this way, here's what you can do, and it will solve the problem of ideals and the problem of unitary maps.

Talk about the category of algebras and bimodules, the Morita category. This is not the place I wanted to start. I'm getting out of order.

You could work with algebras and bimodules which won't fix the—let me say why I would think to do this. In classical field theory, basically PDEs, you have something well understood in terms of functors on cobordisms,. There you can think of the data in a cobordism setting. What you do is to every cobordism you associate the space of solutions of that PDE, and on the boundary you put Cauchy data, germs of solutions near the boundary. You have some map from X (over the cobordism) to Y and Z. So classical mechanics is well-described by algebras of functions. What does a map like this look like in spaces? It's a map like

$$C^\infty Y \to C^\infty X \leftarrow C^\infty Z$$

and this looks like a bimodule picture. So you could try to work with algebras and bimodules. It almast works.

[some discussion]

You can turn a homomorphism $f : A \to B$ into a bimodule, its modulation, $Modulation(f) = B$ with A acting by f and B by itself.

If you try to apply this, when you do modulation of conjugation, you get the identity bimodule, you can untwist by a bimodule automorphism.

You can break this by talking about a pointed bimodule. You can break the conjugation by remembering where 1 goes. If you look at A acting on itself on the right by multiplication and on the left by conjugated multiplication (by U_t) then this is isomorphic to the identity bimodule with normal multiplications. The isomorphism is multiplication by U_t. These are not isomorphic if I have 1. It's not hard to show that this is faithful.

I can say it in general, I'm ready to make a definition.

Definition 1.1. *A* Heisenberg-picture *quantum field theory is a symmetric monoidal functor from some category of spacetimes to algebras, pointed bimodules, and pointed homomoorphisms.*

Let me mention a few generalizations that you can write down quickly.

The first generalization I want to flag is that you can work derived on the target, use derived world with A_∞ algebras, pointed A_∞ bimodules and so-on. This is already an $\infty, 2$-category.

There are some field theories where you have to do derived geometry. Derived stuff was introduced in physicists earlier than in mathematics or at the same time, people were doing all this BRST stuff.

You could do higher codimension boundaries, "extended" things, the other picture, you might want higher dimensional things, then you have to do some category theory. You might want to, if your spacetimes have objects of codimension 0 through k, then you want the top dimensional things, you extend pointed bimodules, associative algebras (which act on vector spaces), 2-algebras, k-algebras (or E_k-algebras) and let me mention the first real theorem. This is not due to me but I should mention it.

Theorem 1.1. *(Calaque-Scheimbauer)*
I'd prefer to list her first, it's almost all her PhD thesis but she insists he get equal credit.
Given a symmetric monoidal $\infty, 1$-category S, there exists a symmetric monoidal ∞, k-category $Alg_k(S)$, where k-algebras are the objects, and so on up to dimension k where you have pointed things.

Now you can eyeball things to be categories, but we have to do hard checking to make sure whether things are ∞, k-categories.

Let me mention one more generalization beyond, a failure of all of this, this was all a great story but when I talked about classical mechanics, I wanted to use an algebra of functions. Not all spaces are affine. Not every space is faithfully encoded, space means something like scheme or stack, if you do locally compact Hausdorff spaces, they're all affine, but if you want schemes, those aren't affine, for gauge theory you're using stacks and those aren't affine. Many spaces are what I'd call "2-affine." Gaitsgory calls it 1-affine but I think he's wrong. A space, scheme, stack, is 2-affine if X, well, you can write quasicoherent sheaves on X, and you ca look at the spectrum, $Spec(Qcoh(X))$, so if $\mathcal{C}$ is a (adjectives) symmetric monoidal category, then $Spec(\mathcal{C})$ has points the symmetric monoidal functors from $\mathcal{C}$ to R-modules, this is probably the fpqc topology. If you do quasicoherent sheaves on another, well, you get an equivalence of stacks $X \to Spec(Qcoh(X))$.

If you have a geometric point with non-affine stabilizer, that fails. There is some active work to find things that, find what is 2-affine in that sense. My coauthors are pushing this. Probably all schemes are. All schemes with some mild conditions are. Probably all schemes you've ever used are.

If you want to incorporate, we have this framework that says that non-commutative spaces, affine non-commutative spaces, these are associative algebras. I'd like to suggest that 2-affine non-commutative spaces are, if you think of modules of A and remember which module is the rank one free module, the functor that takes an algebra to its modules along with the rank one free module, that's faithful. So 2-affine non-commutative spaces is pointed categories.

[discussion of pointed categories versus dg algebras]

In any case, the point is,

Definition 1.2. *A* 2-affine Heisenberg picture field theory *is a functor to pointed categories, pointed functors, and pointed natural transformations.*

Let me tell you what pointed functors are. If you have pointed categories $(\mathcal{C}, C)$ and $(\mathcal{D}, D)$, then a pointed functor should have a map $f : D \to F(C)$. A pointed natural transformation should commute. The generalization we have with Claudia is

Theorem 1.2. *(JF-Scheimbauer)*
Given a symmetric monoidal (∞, n)-category (think (locally) presentable categories) there exists a symmetric monoidal $(\infty, n+k)$-category $Alg_k(S)$ with objects k-algebras, up to pointed things at the end.

Example 1.1.

- *There is this well-known but not rigorously proven theorem of Lurie's that says these are determined by their value on a point. This is joint with Martin Brandenburg and Chirvasitu,*

  Theorem 1.3. *The category $QCoh(X)$, in one dimension, I should give you something dualizable, let's say X is a stack over $\mathbf{k}$. This is dualizable in presentable categories if X is affine, if X is BG if and only if $Rep(G)$ has eneugh projectives, and is not dualizable if X is a scheme containing a projective subscheme of positive dimension. Most schemes that aren't affine contain a projective subscheme.*

  These guys become dualizable if you work with derived versions.

  What does this tell you? G-bundles for an algebraic group won't work in this setup unless you do a derived version.

- *The example I'm interested in is three dimensional. It's a version of Chern-Simons theory. The target I want to take is $Alg_2(Pres - cat)$ which is a 4-category whose objects are braided monoidal categories. The one-morphisms are monoidal categories (with appropriate actions). The two-morphisms are pointed categories (with appropriate actions). The three-morphisms are pointed functors (with appropriate compatibility) and four-morphisms are pointed transformations.*

  An object $\mathcal{C}$ in here defines a quantum field theory, using the other theorem of Calaque-Scheimbauer, is that every object of $Alg_k(S)$ is k-dualizable. We need

  Theorem 1.4. *(JF) Only the trivial object is $k+n$-dualizable.*

  So it's interesting in this case to ask when you are a three-dimensional field theory. I only know this in the $2-2$ case. I don't know the derived version. If $\mathcal{C}$ a braided monoidal category (presentable) has the following properties:

  1. *The monoidal unit is compact-projective (over some fixed field), meaning that $Hom(x, \quad)$ is cocontinuous, and*
  2. *every object is a colimit of dualizable objects*

  then $\mathcal{C}$ defines a framed 3-dimensional TQFT, and I can actually construct this. If in addition you have what I might call a ribbon structure then $\mathcal{C}$ defines an oriented TQFT, fully extended.

  In presentable world you need the correct version of ribbon and so on.

  The example I like the best is the Temperley-Lieb category over $\mathbb{Z}[q, q^{-1}]$, or its cocompletion, the representations of it, this works. This is the free monoidal category generated by a self-dual object of dimension $-q^2 - q^{-2}$. Let me describe the TQFT. I should stop soon, but any time that you wander *across quantum topology and see a skein-theoretic presentation, but if you work with cocompletions, arbitrary cokernels and so on, then you get one of these things. I don't know other examples. So to a point, well, let me tell you the example in top dimension, this version of Chern-Simons theory, to a surface you get the category whose objects are framed configurations points in the surface, well, it'll be the cocompletion of this category, so you get framed points in the surface, and the morphisms are framed tangles in* $\Sigma \times I$ *modulo the skein relation. This is pointed by the empty configuration.*

In dimension three, you get a functor to this pointed category from the trivial category. This should give you an object of that category, of $Z(\Sigma)$ *and a point on that object. That's a map from the empty thing to* $Z(M)$*. So I need an object and a point.*

Free cocompletions are the contravariant functors. So some version of the Yoneda theorem says that for any linear category $\mathcal{C}$ *the free cocompletion of* $\mathcal{C}$ *is the category of contavariant functors to whatever I'm enriched over, and I need to give you a functor* $\mathcal{C}^{op}, Vect$*. So what does it do to some configuration? It's the relative skein module of that configuration. That's the vector space of all tangles in* M *modulo the skein relations. I need to give you a map, and that's the empty tangle.*

The theorem is that these objects that have been around package together in this way. I'm running out of steam and it's 7PM so I'm going to stop.