INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS QUANTUM MONDAY

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1. March 2: Ben Knudsen, raional homology of configuration spaces via factorization homology

Throughout M will be an n-manifold and we'll be talking about $Conf_k$ the configuration space $\{(x_1, \ldots, x_k) \in M^k | x_i \neq x_j\}$ and we'll mainly be interested in the quotient by Σ_k, called B_k.

There are many reasons to be interested; one is that they are surprisingly good invariants.

One is that $B_2(\mathbb{R}^n) \cong B_2(\mathbb{R}^m)$ if and only if $n = m$. Another is that $B_2(T^2 \backslash pt) \ncong B_2(\mathbb{R}^2 \backslash S^0)$. These spaces have different one-point compactifications, they are not proper homotopy equivalent.

A third example is that $B_2(L_{7,1}) \ncong B_2(L_{7,2})$, these are compact manifolds of the same dimension, homotopy equivalent, not homeomorphic, determined by the configuration spaces.

Anything that knows about homeomorphism type of compact manifolds is complicated. But if we work rationally this is much simpler.

One goal of this talk is to recover and remove hypotheses from classical results about these spaces $B_k(M)$ over the rational numbers. The second goal is to compute some Betti numbers. As an algebraic topologist you see an interesting space and you want to know its homology.

The first hour or so I'll recall classical results and give you some approaches for studying configuration spaces.

The fundamental calculation that is behind (implicitly or explicitly) ever result about configuration spaces is the computation of the homology of $Conf_k(\mathbb{R}^n)$. If I single out two points, I get a map π_{ij} to $Conf_2(\mathbb{R}^n)$. It's easy to see that $Conf_2(\mathbb{R}^n) \cong S^{n-1}$. I can pull back the standard volume form on the $n-1$ sphere $\omega_{ij} = \pi_{ij}^*(Vol_{S^{n-1}})$.

Theorem 1.1. *(Arnol'd (n = 2), Cohen) The cohomology with rational coefficients*

$$H^* Conf_k(\mathbb{R}^n) \cong (\bigwedge_{1 \leq i < j \leq n} \omega_{ij})/\omega_{ij} \wedge \omega_{jk} + \omega_{ij} \wedge \omega_{ik} + \omega_{ik} \wedge \omega_{jk}$$

As a corollary,

$$H_*(B_k(\mathbb{R}^n), \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & n \text{ odd} \\ \mathbb{Q} \oplus \mathbb{Q}[n-1] & n \text{ even} \end{cases}$$

The proof is, well, the main point is to compute the rank, for this we use the Serre spectral sequence for the inclusion of

$$\mathbb{R}^n \backslash \{x_1, \ldots, x_{k-1}\} \to Conf_k(\mathbb{R}^n) \to Conf_{k-1}(\mathbb{R}^n).$$

An analogous result is known over the integers.

Anyway, well, configuration spaces are "multi-local" on M, for example, $B_k(U)$ for U an open set in M homeomorphic to $\amalg\mathbb{R}^n$ is a basis for the toplogy of $B_k(M)$. So we can recover $B_k(M)$ as a colimit.

Moreover, $B_k(U \sqcup V)$ is just $\amalg B_i(U) \times B_j(V)$. In principle, the computation for $\mathbb{R}^n$ gives you everything, you just have to piece it together.

The configuration space of k points in $\mathbb{R}^n$ receives a map from the framed embedding space $Emb^{fr}(\amalg_k(\mathbb{R}^n), \mathbb{R}^n)$. Since this is a framed embedding, these are blobs in $\mathbb{R}^n$. The map is to remember where 0 went. The map is a homotopy equivalence.

This embedding space maps to a different space, a mapping space, the pointed maps of the n-sphere to itself of degree k. I view the same picture as instructions for how to map things in [missed some explanation]

More generally, I have a map $Conf_X(\mathbb{R}^n) \to \Omega^n\Sigma^n X$, where X is a pointed space and $Conf_X$ is

$$\amalg_{k\geq 0} Conf_k\mathbb{R}^n \times_{\Sigma_k} X^k/\sim$$

where $(p_1, \ldots, p_k; x_1, \ldots, x_k) \sim (p_1, \ldots, p_{k-1}; x_1, \ldots, x_{k-1}))$.

If $X = S^0$ then $Conf_X$ is the disjoint union of $B_k\mathbb{R}^n$.

Theorem 1.2. *(May) This map is a homotopy equivalence if X is connected.*

Let me give an example, like $X = S^0$ and $n = 1$. Then $\amalg B_k(\mathbb{R})$ is homotopy equivalent to $\mathbb{N}$, but ΩS^1 is homotopy equivalent to $\mathbb{Z}$. So more generally this is a group completion, this is Segal's group completion theorem.

I said this had something to do with globalizing local data to the manifold. What is the global version of this result? There's a theorem due to McDuff, there is a map $Conf_X(M) \to \Gamma_c(\widehat{TM} \wedge_M X)$, this is the fiberwise smash of the fiberwise one point compactification of TM with X, and this map is a homotopy equivalence if X is connected. We recover May's theorem locally, the tangent bundle of $\mathbb{R}^n$ is trivial, this becomes $\mathbb{R}^n \times \Sigma^n X$, compactly supported maps from $\mathbb{R}^n$ into this is maps from the sphere, so that's really May's theorem.

To prove this, one first has to show that both sides send handle attachments to quasifibrations. The second point, use May as a base case for an induction.

I started by saying we were interested in configuration spaces, and I've done a classic bait and switch.

The problem is that we care about the case $X = S^0$, and S^0 is not connected. This is not a fatal problem, what's the fix? Well $Conf_{S^n}(M)$ is filtered by cardinality and the associated graded pieces are Thom spaces of bundles on $B_k(M)$. Then we should be able to get the things we want from the Thom isomorphism.

Now I can state some computations.

Theorem 1.3. *(Bödigheimer–Cohen–Taylor) if n is odd and M is orientable then $H_*(B_k(M))$ (from now on I'll be rational) is $Sym^k(H_*(M))$, which is the simplest guess you might make. In particular, the homology depends only on the homology $H_*(M)$.*

Let me give you a couple of words about the proof. Handle attachments become quasifibrations, so the Serre spectral sequence applies and because we chose n odd, all the spectral sequences collapse, and the calculation of Cohen is a base case for induction.

The second computation is in the even dimensional case, due to Félix and Thomas, we need stronger hypotheses,

Theorem 1.4. *Let n be even and M be compact, orientable, and nilpotent as a topological space. Then for any $q > kn$,*

$$H^*B_k(M)[qk] \cong H\left(SymH_*(M)[q+n] \oplus H_*(M)[2q+2n+1], \Delta\right)[qk, (q+n)k].$$

This is awful but it only depends on the cohomology of the background manifold. This only depends on the ring $H^*(M)$.

What we want to do is apply rational homotopy theory to the target of McDuff's map, which these compactly supported sections $\Gamma_c(\widehat{TM} \wedge_M S^q)$. So Haefliger gives the minimal model for this space, which is most of the battle. As the sphere gets higher and higher dimensional, the groups start to spread out with velocity proportional to their cardinality and become separate and you can read them off by picking off some range of degrees.

We have this nasty consequence that we have to do infinitely many computations to know all of the answers.

It would be lovely if we could just set $q = 0$.

I have one more classical result to put on the board before we take a break, and that's cut from a slightly different cloth from the stuff we've been seeing so far, and that's homological stability, the subject of splitting points apart.

If I have k points in any manifold, I can have $k-1$ points by forgetting the last one, so I can make all the choices at once, apply homology and sum up over all ways of forgetting a point. This is no longer biased. Then I can add these, and the first map, it's not equivariant, the second map is equivariant, and gives a map between homology of unordered configuration spaces. The reason this is a good map is the following theorem of Tom Church, let me state it imprecisely

Theorem 1.5. *This map is an isomorphism in a range of degrees that tends to ∞ with k.*

Homological stability results are an industry, there is some contention about who proved what, I don't want to take any sides, but Church used this map, so that's why I mention him.

I want to talk about a theorem

Theorem 1.6. *(K.) Let M be smooth.*

There is an isomporphism of bigraded coalgebras

$$\bigoplus_{k \geq 0} H_*(B_k(M)) \cong H_*^{Lie} H_c^{-*}(M, Lie(\mathbb{Q}^w[n-1])$$

So let me start from the inside and work out. So $\mathbb{Q}^w$ is the orientation sheaf of my manifold in degree $n-1$. The *Lie* is the free Lie algebra functor, which is a complex of sheaves. Then I can take its compactly supported Lie algebra, and I can take its Lie algebra homology.

There are a couple of things I haven't explained. The first grading is degree. The left hand side is also graded by cardinality; the right hand side is graded by what I'll call *weight*, which is defined by saying that the generator of the Lie algebra has weight 1 and the bracket has weight 2 and so on and so forth.

I have not told you the coalgebra structure, I'll tell you that later.

Let's remember a couple of things about Lie algebras and their homology. So $H_*^{Lie}(\mathfrak{g})$ is just $Tor_*^{U\mathfrak{g}}(\mathbb{Q},\mathbb{Q})$, which looks hard to compute, but you can look at the Chevalley–Eilenberg complex $H(Sym(\mathfrak{g}[1]),[,])$ (up to a sign on the bracket). The free Lie algebra on one generator r is either $\mathbb{Q}r \oplus \mathbb{Q}[r,r]$ if r is odd, or $\mathbb{Q}[r]$ if r is even.

Let me unpack the isomorphism in low weights. In weight one, what does this say? On the left I have $H_*(M)$, the configurations of one point. This should be $H_c^{-*}(M,\mathbb{Q}^w)[n]$. So that's Poincaré duality.

In weight two, on the left I have $H_*(B_2(M))$, and I'm saying this is

$$H[Sym^2 H_*(M) \oplus H_*(M,\mathbb{Q}^w)[n-1], \bullet$$

where $\bullet$ is the intersection product. This is when n is even.

There's the obvious inclusion $Conf_2(M) \to M^2$, and the homotopy cofiber is $Th\ TM$. In spectra or anything spectral, I can put this before $Conf_2(M)$ as $\Sigma^{-1}Th\ TM$. Then if I quotient by Σ_2, I get a twisted version of $\Sigma^{n-1}(M)$, the desuspension of the Thom space, and that goes to $B_2(M)$ to $Sym^2 M$, and I can imagine splitting this rationally, and that's exactly the intersection product. It's something you might come up with thinking about basic topological things.

This is, I like to think of this computationally. Let's do the example $M = \mathbb{RP}^2$. Since $n = 2$, this Lie algebra $Lie(\mathbb{Q}^w[1]$ is $\mathbb{Q}^w[1] \oplus \mathbb{Q}[2]$. However, $H_c^{-*}(\mathbb{RP}^2, Lie(\mathbb{Q}^w[1]),$ this is an Abelian Lie algebra $\mathbb{Q}[-1] \oplus \mathbb{Q}[2]$. You can just work this out. Since it's Abelian, there's no differential in the Chevalley–Eilenberg complex, so the homology $H_*B_k(\mathbb{RP}^2)$ is the weight k piece of $\mathbb{Q}[x_0] \otimes \wedge[y_3]$ where the weight of x is one and the weight of y is two.

This says that $H_*(B_k(\mathbb{RP}^2))$ is isomorphic to $\mathbb{Q} \oplus \mathbb{Q}[3]$ as long as $k > 1$.

Another one you can do in the same way is to say that

$$\dim H_j B_k(\sharp_h \mathbb{RP}^2) = \begin{cases} \binom{h+i-2}{h-2} & i \in \{0,1,2,k+1\} \\ \binom{h+i-2}{h-2} + \binom{h+i-5}{n-2} & 3 \le i \le k \\ 0 & \text{otherwise} \end{cases}$$

So we have some corollaries. The homology of $H*(B_k(M))$ depends only on $H_*(M)$ if n is odd and on the cup product if n is even. We also recover the computations of Bödigheimer–Cohen–Taylor and Félix–Thomas. As another corollary, we can compute with twisted coefficients,

Corollary 1.1.

$$H_*(B_k(M),\mathbb{Q}^w)$$

depends only on $H_c^{-*}(M),\cup$.

What about stability?

Theorem 1.7. *Let M be connected and $n > 1$. Well $1 \cap (\quad) : H_*(B_{k+1}(M)) \to H_*(B_k(M))$ is an isomorphism for $* < k$ if M is an orientable surface and $* \le k$ otherwise.*

This is basically the result of Church combined with that of Randal-Williams, but this has the advantage (aside from a very slight improvement in the range) of being very easy to prove. Well $1 \cap (quad)$ is $\frac{d}{d[pt]}$ in the Chevalley–Eilenberg complex. Then the second thing to observe is that if $wt\alpha > |\alpha|$ then $[pt]|\alpha$. There are ten proofs of stability but this is by far the easiest one.

Let me revisit some of the concepts that we saw in the first half. So let's remember that $Conf_k(\mathbb{R}^n) \cong Emb^{fr}(\amalg_k \mathbb{R}^n, \mathbb{R}^n)$ Call that embedding space $E_n(k)$. That collection $E_n(k)$ is an operad, and the disjoint union $B_k(\mathbb{R}^n)$, let me just call that $B(\mathbb{R}^n)$, that's an E_n-algebra. Concretely this means I have maps that look like $E_n(k) \times B_{i_1}(\mathbb{R}^n) \times \cdots B_{i_k}(\mathbb{R}^n) \to B_{i_1+\cdots+i_k}(\mathbb{R}^n)$. What do these maps look like? [pictures of configurations in little disks]. In fact, this E_n algebra $B(\mathbb{R}^n)$ is equivalent to the free E_n-algebra generated by the space S^0.

Let's revisit globalization now under the name of factorization homology. Let A be an E_n algebra in chain complexes over $\mathbb{Q}$. The example to have in mind is singular chains on $B(\mathbb{R}^n)$. What's an E_n algebra? It takes the data of a framed embedding of k copies of $\mathbb{R}^n$ into ℓ copies of $\mathbb{R}^n$, and goes to a map $A^{\otimes k} \to A^{\otimes \ell}$. That's a symmetric monoidal functor from the category of natural numbers with these embeddings to chain complexes with tensor product.

At this point I should probably say that one needs to, we should choose a homotopy theoretic foundation, these categories are not just plain categories, they're enriched over simplicial sets or spaces. This is there in the background somewhere. How do I want to think of this algebra? It prescribes the value of my invariant on simple examples, that is, disjoint unions of $\mathbb{R}^n$. This category of disks moves into the category of manifolds, and there's a homotopy way to prolong that, this is $\int_{(\ \ )} A$ and is the *factorization homology* with coefficients in A. In the example of this E_n algebra, the factorization homology of M with coefficients in this particular E_n algebra is $C_*B(M)$.

This is totally categorical, no reason to believe this knows about topology, but

Theorem 1.8. *(Francis) The factorization homology can be calculated by induction along a handle decomposition of M.*

This is just more modern language for what has been happening since the fifties and sixties.

Let me revisit one last thing. I did that first calculation of Arnol'd and Cohen. Since $Conf_2(\mathbb{R}^n) \cong S^{n-1}$, this is the arity two piece, which controls the binary operations, an E_n algebra has two binary operations up to homotopy, one called m_0, which is a commutative multiplication up to homotopy, corresponding to the zero cell, and an operation, a shifted Lie bracket, m_{n-1}, of degree $n-1$. One way of making this precise is that

Theorem 1.9. *(Cohen) $H_*(E_n)$ is $Poiss_{n-1}$, the operad controlling shifted Poisson algebras.*

Another way to say this is

Theorem 1.10. *(Fresse) There is an essentially unique map of operads from $\Lambda^{1-n}L_\infty \to E_n$, where this is the shifted version of the operad controlling homotopy Lie algebras.*

As a corollary, I get an adjunction between Lie algebras on the one hand and E_n algebras, the left adjoint I'll call the n-enveloping algebra, and the thing we're interested in, by the way adjoints work, we have that $C_*(B(\mathbb{R}^n)) \cong U_n Lie(\mathbb{Q}[n-1])$. So we're going to apply factorization homology to both sides. I already asserted that the factorization on the left hand side is chains on configurations in M. A theorem of Francis and Gaitsgory says that the right hand side is $C_*^{Lie}(\Omega_c^{-*}(M, Lie(\mathbb{Q}[n-1]))$. There's a big glaring flaw which is that this is only for framed manifolds.

We can fix the definition of factorization homology quite easily. We can replace E_n algebras with $n - disk$, this is an E_n algebra with a compatible action of the orthogonal group. I erased framed and I have a perfectly good definition. Everything works except to get the equivalence of free with universal enveloping I used Fresse's map of operads. Since I'm running a little low on time and patience, I'll abbreviate a little bit and say that the way to fix this is

Theorem 1.11. *(K.) $C_*(B(\mathbb{R}^n))$ carries an $n - disk$ algebra structure which is equivalent to*

$$C_*^{Lie}(\Omega_c^{-*}(\mathbb{R}^n, Lie(\mathbb{Q}^{det}[n-1]))).$$

The main ingredient here is to recognize that the shift by n was not always a shift by n, it was a tensoring with $\tilde{C}_*(S_n)$. This is just $\mathbb{Q}$ in degree n if there is no $O(n)$ action. But I get a shift with a sign action if there is an $O(n)$ action. So $\tilde{C}_*(S^n) \cong \mathbb{Q}^{det}[n]$ as $O(n)$ modules.

The final step is to use the fact that $\Omega_c^{-*}(M, Lie(\mathbb{Q}^w[n-1]]))$ is formal so I can replace it with its cohomology.

2. September 7, 2015: Hwajong Yoo, Galois Symmetry I

My motivational problem is the well-known thing, Fermat's last theorem, which says that for any integer $n \geq 3$, the equation

$$x^n + y^n = z^n \qquad x, y, z \neq 0$$

has no rational solutions. This was finally proven by Wiles and Wyles–Taylor in 1994. I'll talk about how this proof goes in detail.

So first short-term goals,

1. I'll talk about constructing Galois representations from elliptic curves and modular forms.
2. Then I'll explain about Serre's modularity conjectures.
3. Then I'll talk about the $R \cong \mathbb{T}$ theorem, which is Wiles and Taylor–Wiles, and implies Fermat's las't theorem. This implication is Frey, Serre, Ribet.
4. It's enough for us to talk about $R \cong \mathbb{T}$ in the *semistable case*
5. If time permits, I want to talk about a generalization of such modularity conjectures, for example Skinner–Wiles and Calegari–Emerton and then recently Erickson–Wake.

I'll be talking about two dimensional irreducible representations and their theory.

So let me start with some background for Galois representations. Let K'/K be a normal and separable extension, not necessarily finite, of fields. Then the *Galois group* of K'/K, by definition, is $\{\sigma \in Aut(K') | \sigma|_K = \mathrm{id}_K\}$. This has the natural topology, the *profinite* topology, which I'll explain later. If you give a discrete topology in a finite set and then take the inverse limit, you get that topology. A finite index subgroup is an open subgroup. This makes this group compact and Hausdorff. This is a totally disconnected space. If the group is finite then everything is clear. The inverse limit of such a thing gives the profinite topology.

Fix a separable closure of K, then $G_K \coloneqq Gal(\bar{K}/K)$. A *Galois representation* is a continuous homomorphism $\rho : G_K \to GL_n(L)$ for L a topological field. The nature of the representation is totally dependent on L. What we'll use is most likely $L = \mathbb{C}$ or the p-adic numbers or finite fields. If L is $\mathbb{F}_{p^r}$, a finite field, or $\bar{\mathbb{F}}_p$, it has the discrete topology. For $L = \mathbb{C}$ we take the usual topology on $\mathbb{C}$. For $L = \mathbb{Q}_p$ or $\bar{\mathbb{Q}}_p$ there is the natural p-adic topology.

In each representation, there is a name. We say $\rho : G_{\mathbb{Q}} \to GL_n(L)$ is called an Artin representation if $L = \mathbb{C}$. It is called a mod p representation if $L = \bar{\mathbb{F}}_p$ or $\mathbb{F}_{p^r}$. It's called p-adic if $L = \bar{\mathbb{Q}}_p$ or $K/\mathbb{Q}_p$.

An example or exercise. If you know the profinite topology, so if you have open V in $Gal(K'/K)$, the index is finite. The exercise says $Im\ \rho$ is finite if ρ is Artin or mod ℓ. The Artin case is a little difficult but not hard. For the mod ℓ case this is obvious because of the discrete topology on L which means that id in the image is open and closed, because the field has a discrete topology. Then the kernel of ρ is open in $G_{\mathbb{Q}}$. So it is finite index but it's a normal subgroup, and the quotient is finite.

For $L = \mathbb{C}$, you have to use that $GL_n(\mathbb{C})$, the image of ρ, the image is a group, and the group is compact, so the image is compact. A compact subgroup of $GL_n(\mathbb{C})$ contains the identity. There are no small subgroups in $GLn(\mathbb{C})$ containing the identity. This implies that a small neighborhood contains nothing else.

If you restrict ρ to some pro-p group, then the image is finite if ρ is ℓ-adic and $\ell \neq p$. These are all kind of finiteness statements. So it looks like there is usually finite image. But it's not in the ℓ-adic case. So $Im\ \rho$ might be infinite. The example is the ℓ-adic cyclotomic character. Because many of you haven't seen such Galois representations, I'll explain this a bit carefully.

First of all, you have some primitive ℓth root of unity. The Galois group is naturally isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^\times$, which is $\mathbb{F}_\ell^\times$. SO by σ you can map to $\chi(\sigma)$ which is $\zeta_\ell \mapsto \zeta_\ell^{\chi(\sigma)}$. [explanation]

Then ζ_{ℓ^2} such that $(\zeta_{\ell^2})^\ell = \zeta_\ell$, is a primitive (ℓ^2)th root of unity. This is in fact isomorphic to $(\mathbb{Z}/\ell^2\mathbb{Z}) \cong \mathbb{F}_\ell^\times \times \mathbb{Z}/\ell\mathbb{Z}$.

Then

$$Gal(\mathbb{Q}(\zeta_{\ell^\infty})/\mathbb{Q}) = Gal(\cup\mathbb{Q}(\zeta_{\ell^n})/\mathbb{Q}) = \lim Gal(\mathbb{Q}(\zeta_{\ell^n})/\mathbb{Q}) \cong \lim \mathbb{F}_\ell^\times \times \mathbb{Z}/\ell^{n-1}\mathbb{Z}.$$

Here we have $\ell > 2$, I should say, 2 is not a prime.

So you get a 1-dimensional representation χ by $G_{\mathbb{Q}} \to Gal(\mathbb{Q}[\zeta_{\ell^\infty}]/\mathbb{Q}) \xrightarrow{\cong} \mathbb{Z}_\ell^\times \hookrightarrow \mathbb{Q}_\ell^\times$. By going from $\mathbb{Z}_\ell^\times$ to $\mathbb{F}_\ell^\times = GL_1(\mathbb{F}_L)$ I get this construction, the mod ℓ cyclotomic character.

Let $E/\mathbb{Q}$ be an elliptic curve over $\mathbb{Q}$. So for instance you can write this as $y^2 = x^3 + ax + b \cup \infty$, where ∞ will be the identity element,and the curve defined over $\mathbb{Q}$ means that a and b mean that they can be chosen in $\mathbb{Q}$. [picture].

So you can make a map $\ell : E \to E$ where $p \mapsto \ell p$, and then $E[\ell] \coloneqq \{p \in E(\bar{\mathbb{Q}}) | \ell p = 0\}$ which is isomorphic to $\mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$. SO the torus is isomorphic to a fundamental parallelogram.

The theory tells us that they are isomorphic as groups. If you want the structure here, you can find the torsion points. You can do the same thing for ℓ^2. The good thing is that this ℓ^2-torsion point, each ℓ-torsion point is stable by the Galois action. So you get an infinite sequence of finite modules which have Galois action, and each transition map is compatible, which is very important. Therefore you can glue them together.

You get the inverse limit of $E[\ell^n]$ for the ℓ-adic Tate module.

Naturally this one has a $G_{\mathbb{Q}}$-action, and it's isomorphic for ℓ to $\mathbb{Z}_\ell \times \mathbb{Z}_\ell$. If you tensor with $\mathbb{Q}_\ell$ you get $\rho : E, \ell : G_{\mathbb{Q}} \to Aut(V\ell E) \to GL_2(\mathbb{Q})$.

The properties I want to prove is that this is irreducible (use the Hasse bound that $|a_p| \leq 2\sqrt{p}$). I also want to state that this is *unramifield* outside of ℓN_E. A prime not dividing ℓN_E gives $Frob_p$, and each structure has some conjugacy class, a Frobenius element [unintelligible].

These three properties determine the representation $\rho_{E,\ell}$ uniquely, by the Brauer–Nesbitt theoremplus the Chebotarev density theorem.

The construction for modular forms, these are some function, complex valued function on the upper half plane satisfying some properties, there's an action from $SL_2(\mathbb{Z})$ and there's $\Gamma_1(N)$ which is unipotent matrices modulo N. This f is roughly a function on the quotient space $\Gamma_1(N)\backslash\mathcal{H}$. That's a modular form. There are Hecke operators acting on modular forms. Say f is a new form of weight $k \geq 2$ and level N and character ϵ. This means a normalized eigenform with respect to some Hecke operators. If you have this modular form then you can associate an ℓ-adic Galois representation, $K/\mathbb{Q}_\ell$ a finite extension, $\rho_{f,\ell} : G_{\mathbb{Q}} \to GL_2(K)$. This has properties

1. It's irreducible (Ribet). In the even case you can use the same bound, the Ramanujan–Peterssen bound.
2. It's unramified outside of ℓN_f
3. If p does not divide ℓN_f, then the characteristic polynomial of $\rho_{f\cdot\ell}(Frob_p)$ is $X^2 - a_p(f)X + \epsilon(p)p^{k-1}$.

So a fourier expansion of f is $\sum a_n(f)q^n$, that gives you a_p. The group $\Gamma_1(N)$ has the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Furthermore we know that a_n is the eigenvalue of the nth Hecke operator.

ϵ is some character $(\mathbb{Z}/N_f\mathbb{Z})^\times \to K^\times$.

So let's match these two pictures. The image is $GL_2(\mathbb{Q}_\ell)$ and on the other side to K. Then all Fourier coefficients must lie in $\mathbb{Z}$. So all Fourier coefficients, eigenvalues for Hecke operators are integral. The levels should be equal $N_f = N_E$. Then a_p must be equal between the two. The ϵ should be a trivial character and $k = 2$. Then the two representations look equal.

There is no evidence that equality holds, but if you compute this, list elliptic curves with respect to the conductor and construct the ℓ-adic representation, and you can ask if there's a modular form of weight two and the same level, the trivial character and coefficients lying in $\mathbb{Z}$, and then ask about a_p, that's the question by Taniyama–Shimura–Weil, that $E/\mathbb{Q}$ is modular. The meaning is that there exists ℓ and f such that $\rho_{E,\ell} = \rho_{f,\ell}$ over $\mathbb{Q}_\ell$.

There are several different equivalent conditions. You could also say that there's a condition, there's an f such that $\rho_{E,\ell} = \rho_{f,\ell}$ for all ℓ.

You can also associate an L function to elliptic curves and to modular functions, you could say $L(E,s) = L(f,s)$, that's equivalent. This is an infinite sum that only converges for the real part of s greater than $\frac{3}{2}$ for the $L(E,s)$ case. The same is true for $L(f,s)$, but there's a natural symmetry, an inversion action and you can flip this region and extend the function to the whole plane by analytic continuation. If you associate an L-function to a variety, you want to be able to, this is the Hasse–Weil conjecture, to analytically continue the $L(E,s)$ function.

Theorem 2.1. *If $E/\mathbb{Q}$ is* semistable, *E is modular. So for us we can say semistablity is that N_E is squarefree.*

Theorem 2.2. *(Frey, Serre, Ribet) This theorem plus some earlier work for $n \leq 11$ implies Fermat's last theorem.*

Finally, the modularity conjecture was fully proved by

Theorem 2.3. *(Breuil, Conrad, Diamond, Taylor, 2001) $E/\mathbb{Q}$ is modular in general.*

It's a good time to stop. Any questions?

3. October 12, 2015: Hwajong Yoo, Galois Symmetry V

Let me talk about representations coming from modular forms. For the first part, I'll talk about how to construct Galois representations from modular forms. People always just assume you have this, to f you have ρ_f. Today I want to focus on the ideas of how to associate this Galois representation.

For simplicity, let's assume that f is a newform of weight 2 for $\Gamma_0(N)$. This is a Hecke eigenform not from previous levels, and the Fourier expansion $\sum a_n q^n$ has $a_1 = 1$ and $T_p f = a_p f$.

The idea is that we want to associate some Galois representation $G_{\mathbb{Q}} \to GL_2(K_\lambda)$ to some ℓ-adic field.

I'll show you how to do this in this situation. In weight 2 we can use geometry. If the weight is bigger than 2 we need étale cohomology. In weight 1 we need a congruence argument. The representation always exists for the classical Hecke eigenform (for integral weight).

This is the construction of Eichler–Shimura. First, consider, let K_f be $\mathbb{Q}(\ldots, a_n, \ldots)/\mathbb{Q}$. This is a number field; the degree is finite. This is well-known theory of modular forms. Then, last time I defined the Hecke ring $\mathbb{T} = \mathbb{Z}[\ldots, T_n, \ldots) \subset \mathrm{End}(J_0(N))$ where $J_0(N) = Pic^0(X_0(N))$ where $X_0(N) = \Gamma_0(N)\backslash\mathcal{H}^*$.

So there is a natural map $\lambda_f : \mathbb{T} \otimes \mathbb{Q} \to K_f$ where T_n maps to a_n. We can then imagine the kernel of λ_f, an idea of $\mathbb{T} \otimes \mathbb{Q}$. This, we want to consider the intersection with $\mathbb{T}$. The ideal is generated (formally) by things like $T_n - a_n$, well, if a_n is in $\mathbb{T} \otimes \mathbb{Q}$.

Now $I_f \cdot J_0(N) \subset J_0(N)$. You can write this as the union of γx where $\gamma \in I_f$ and $x \in J_0(N)$. We can write an Abelian variety $A_f := J_0(N)/I_f J_0(N)$. The wonderful theorem of Shimura is that this is an Abelian variety (easy) of dimension $[K_f : \mathbb{Q}] = d$ (nontrivial.)

Note that $\mathbb{T}$ acts on A_f and this factors through I_f. So as a module over $\mathbb{T}/I_f$ this is roughly of order 2. But $(T/I_f) \otimes \mathbb{Q}$ is isomorphic to K_f which maps to $\mathrm{End}(A_f) \otimes \mathbb{Q}$.

The idea is that this Abelian variety, if you have one like this, then you can imagine any ℓ-torsion of A_f is isomorphic to $(\mathbb{Z}/\ell Z)^{2d}$

This is also a K_f-module. [example]. Take the inverse limit $A_f[\ell^n]$ for all n and you get the so-called ℓ-adic Tate module. It is roughly $\mathbb{Z}^(\ell)^{2d}$.

Then tensoring with $\mathbb{Q}_\ell$ over $\mathbb{Z}_\ell$ you get $(\mathbb{Q}_\ell)^{2d}$.

This is the wrong dimension, we want something 2-dimensional.

[some discussion]

Theorem 3.1.
1. *This ρ is unramified if $p \nmid N\ell$. The characterstic polynomial of $\rho(Frob_p)$ is $x^2 - a_p X + p$.*
2. *the determinant $\det p = \epsilon_\ell$ is the ℓ-adic cyclotomic character. Here $\epsilon_\ell(Frob_p)$ is p if $\ell \neq p$. This is where the Frobenius element is taking the pth power.*

(3) *ρ is absolutely irreducible (Ribet)*
(4) *$N(\rho)$, the conductor of ρ, $\prod_{p\neq\ell} p^{m_p(\rho)}$, is roughly $2 - \dim \rho^{I_p}$. This is exactly the prime to ℓ part of N.*
(5) *If $p \neq \ell$ but $p||N$, then $\rho|_{G_p}$ is isomorphic as a G_p representation to $\begin{pmatrix} \chi\epsilon_\ell & * \\ 0 & \chi \end{pmatrix}$ where χ is an unramified quadratic character and $\chi(Frob(p)) = Q(p)$.*
(6) *If $\ell \nmid 2n$, then take $p = \ell$. There are two cases, the "ordinary" and "supersingular" corresponding to $\#E[\ell](\bar{\mathbb{F}}_\ell)$ being ℓ or 1. So $\rho|_{G_\ell}$ is ordinary if and only if a_ℓ is n ℓ-adic unit.*

[Too hard for me to follow]

4. October 19: Hee-Joong Chung Chern–Simons theory and its relation to 3-dimensional $N = 2$ supersymmetric conformal field theory

Thank you for the introduction. So as you heard I will talk about the Chern–Simons theory and its relation to $N = 2$ supersymmetric conformal field theory.

I thought it would be better to give an overview of Chern–Simons theory. Today I'll talk about several aspects of Chern–Simons theory with compact gauge group $G = SU(2)$. I will mainly review the paper of Witten about the Jones polynomial and Chern–Simons theory.

There are many other references, which include several papers by Witten, there is a review paper by Kohno, "Conformal field theory and topology" and then Axelrod, Pietra, Witten.

This is a broad subject and I decided to give an overview rather than going into details.

This will be the first part. Today I will talk mostly about what is Chern–Simons theory. I'll talk about the Chern–Simons action and Wilson loops. Then I'll talk about perturbation theory and framings of three-manifolds. Then I will move onto quantization. For the last part I'll review how skein relations and the Jones polynomial arise from Chern–Simons theory. This is the contents of today.

Let's talk about the Chern–Simons action. Briefly speaking, what Witten did, before Witten the story of knot theory was done in a two-dimensional context. But he did Chern–Simons theory in the presence of a Wilson loop operator, identifying this theory with a conformal field theory. I won't talk about the conformal field theory today and will just quote results about them. For the second talk I'll talk more about Chern–Simons theory and talk about some relevant issues. In the third talk maybe I'll talk about my own work, about the relation between Chern–Simons theory and [unintelligible].

So the Chern–Simons action is given by

$$S_{CS} = \frac{k}{4\pi} \int \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

where A is the connection 1-form of a G-bundle on M_3.

Here k is called the Chern–Simons level. For G equal to $SU(N)$, it should be an integer.

We have a gauge group and we say "large gauge transformations" are those not in the component of the identity. So $S_{CS} \to S_{CS} - 4\pi y k w(g)$.

Here y is the Dynkin index, which is $\frac{1}{2}$ for the fundamental representation of $SU(n)$ and

$$w(g) = \frac{1}{48\pi^2 y}\int_M Tr(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)$$

the winding number.

The path integral is

$$Z = \int DAexp(i\int_{CS}).$$

Physics should be invariant under large gauge transformations, and that implies that k should be an integer.

If something is independent of the choice of metric we usually call it topological. So we have the Wilson loop. So $W_R(\gamma)$, for γ a loop in M, is defined as $Tr_R P\exp\left(\oint_\gamma A_i dx^i\right)$.

In nature we live in Minkowski space. In this theory we usually consider the Euclidean signature. Maybe you're not familiar with the notation P. This is the path-ordered product. It's something like

$$TrP\left(1+\sum_{n=1}^{\infty}\frac{1}{n!}(\oint_\gamma A_i(x)dx^i)^n\right)$$

which is

$$Tr(1+\oint_\gamma A_i(x)dx^i + \frac{1}{2}\oint_\gamma(\int_0^x A_j(x')dx'^j)A_i(x)dx^i\cdots)$$

This is the Wilson loop operator. The Wilson loop operator is independent of the choice of metric.

There is the so-called expectation value. The expectation of the Wilson loop observable can be written

$$\langle\prod_j W_{R_j}(\gamma_j)\rangle = \int DA\prod_j W_{R_j}(\gamma_j)exp(i\int_{CS})$$

where in general

$$\langle\mathcal{O}\rangle = \int D\Phi\mathcal{O}e^{iS}$$

So we can sometimes write $\frac{k}{4\pi}$ as $\frac{1}{\hbar}$.

Write the action in terms of the Lagrangian, $S = \int d^3x\mathcal{L}$ and then you can write the equations of motion in terms of the Lagrangian. So $Y = dA + A\wedge A$, the field strength, is 0. So the classical motion of Chern–Simons theory is via flat connections, and those are described by the holonomy of the theory.

That's $\rho : \pi_1(M)\to G$ modulo gauge transformation $\sim$ conjugation.

Mostly we are interested in cases where the number of homomorphisms is finite. From these, the "vacua" of the theory, we expand the field. Roughly, the partition function is

$$Z = \sum_\alpha Z(A^{(\alpha)})$$

and in this case, when we evaluate the partition function, we expand out the gauge field for our vacua,

$$A_i = A_i^{(\alpha)} + B_i$$

getting something like this

$$Z^{(a)} \sim e^{i\pi\eta(0)/2} = e^{i(k+c_2(G)/2)S_{CS}(A^\alpha)} \times T_\alpha.$$

I should have said, we always make a gauge choice. For this choice one needs to choose the metric.

Here T_α is the Ray–Stinger torsion, which

Here $c_2(G)$ is the quadratic Casimir operator of the adjoint representation. One introduces the gravitational Chern–Simons term

$$I(g)_{grav} = \frac{1}{4\pi}\int_M tr(w \wedge dmw \wedge w \wedge w)$$

There is some dependence of the framing of the manifolds.

Chern–Simons theory is supposed to be topological. We added in this geometric contribution, which we want to cancel.

Instead of having a topological theory, we have one that depends on the framing. Under chainging the framing the partition function changes something like

$$Z \to Z\exp(2\pi i S\frac{c}{4})$$

where $c = \frac{kd}{k+\tilde{w}}$ where $d4$ is the dimension of G.

Using the expectation of the Wilson line, [missed some], you can define a linking number

$$\phi(\gamma_a, \gamma_b) = \frac{1}{4\pi}\int_{\gamma_a} dx^i \int_{\gamma_b} dy^j \epsilon_{ijk}\frac{(x-y)^k}{(|x-y|^3)}.$$

The self-linking number is only well-defined if you choose a framing of the knot.

Under the change of framing $\langle W\rangle$ becomes, well, if h_R is the [unintelligible](for example $G = U(N)$, h_R is $\frac{c_R}{2(k+N)}$ and c_R are quadratic Casimir for R).

[too hard to understand]

5. November 23: Jae-Suk Park: Lectures on Homotopy Theory of Quantum Fields I

I'm going to give a series of lectures, maybe five lectures. Hopefully it will be very elementary.

I always try to understand quantum field theory mathematically. The biggest obstacle there is that the physicists' main tool, the path integral, is not defined.

There are many reasons it's not defined. It's an integral, so it should be attached to a measure. But the measure is not really defined. I want to get rid of the measure from the business, get rid of the integral, entirely.

Today I'll give some toy model that will be a background or something like that.

The cartoon of the idea is the following. Suppose I have a complicated looking jug and there is water inside, I want to figure out the volume. We could take a beaker with obvious units, some marks, and put the water into the beaker. Then the volume of this thing would be, say, 5.2, you measure. I want to realize this kind of procedure. So the integral is a linear map. So assume A is some algebra of integrands. This is unital, associative, and while I don't need it in general, let me assume that it's commutative, over a ground field $\mathbf{k}$, which I'll assume to be characteristic 0.

So x is an element here, and an integral associates a number $\iota(x) \in \mathbf{k}$. This is a linear map, $\iota : A \to \mathbf{k}$ is $\mathbf{k}$-linear, and I'll normalize it so that $\iota(1_A) = 1$.

Now we consider a bunch of functions $\{x_1, \dots, x_k\}$, such that $x_i \in A$. Then let me introduce dual variables $\{t_1, \dots, t_k\}$. Then I'll introduce

$$Z(t) := \iota(e^{\sum t_i x_i},$$

this is a generating function, so this will be

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j_1, \dots, j_n} t_{j_1} \cdots t_{j_n} \iota(x_{j_1} \cdots x_{j_n})$$

where $1 \le j_1, \dots, j_n \le k$.

For example, if I have x and y, then I want to know $\iota(x^n y^m)$. This is just a convenient device.

Let me try to introduce terminology. I'll call an element of this algebra a random variable, then this is a set of random variables, and we want to figure out all of these expressions $\iota(x_{j_1} \cdots x_{j_n})$.

It is convenient to consider a different quantity that I'll call $e^{F(\underline{t})}$ where $F(\underline{t})$ is a formal logarithm of $Z(t)$. So I'll write this as

$$e^{\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{j_1, \dots j_n} t_{j_1} \cdots t_{j_n} \kappa_{j_1, \dots j_n}}$$

where $k_{j_1, \dots j_n} \in \mathbf{k}$.

Now I want to sketch a procedure to find out κ. If we know one we know the other. It's more convenient for whatever reason to think in terms of κ.

Then I want to consider an equivalence relation. The set of random variables $\{x_1, \dots, x_k\}$ is equivalent to $\{y_1, \dots y_k\}$ if $e^{\sum t_i x_i} - e^{\sum t_i y_i}$ is in the kernel of ι. I want to stretch this definition more.

I've fixed k, let me consider the following thing, let me consider the set $\{X_I, X_{I_1 I_2}, X_{I_1 I_2 I_3}, \dots\}$ where I goes from $1, \dots, k$. So here $X_{I_1, \dots, I_n} \in A$.

What we considered originally is a special example. Let me do it a different way.

Let V be a vector space of dimension k. I can choose a basis $\{e_i\}$ and then choose a map $V \to A$, which is completely specified by sending e_i to X_i. This is just a linear map. Now we think of this as a map from a finite dimensional vector space to A. I can also consider symmetric powers, $SV \to A$, where SV is the reduced symmetric power $V \oplus S^2 V + \cdots$

If I specify these maps $\varphi_1, \varphi_2, \dots$ where $\varphi_n : S^n V \to A$. Now you think that $\{X_I, X_{I_1 I_2}, \dots\}$ are the images of this thing in A.

You can assume we have a similar set $\{Y_I, Y_{I_1 I_2}, \dots\}$, we'll say it's equivalent to our given one if

$$e^{\sum_{n=1}^{\infty} \frac{1}{n!} \sum t_{j_1} \dots t_{j_n} x_{j_1, \dots j_n}} - e^{\sum_{n=1}^{\infty} \frac{1}{n!} \sum t_{j_1} \dots t_{j_n} y_{j_1, \dots j_n}}$$

is in the kernel of ι.

The claim is that the set $\{x_1, \dots x_k\}$, this is $\{X_I, 0, 0, \dots\}$, this is equivalent to $\{K_I 1_A, K_{I_1, I_2} 1_A, \dots\}$.

This is a tautology, but it's a very important claim. You've just proved that these two sets of integrals are equal. This tells you

$$e^{\sum t_i x_i} - e^{\sum_{n=1}^{\infty} \frac{1}{n!} \sum t_{j_1} \cdots t_{j_n} \kappa_{j_1, \dots j_n} 1_A}.$$

There's nothing to integrate on the right, this is

$$e^{\sum_{n=1}^{\infty} \frac{1}{n!} \sum t_{j_1} \cdots t_{j_n} \kappa_{j_1, \dots j_n}}.$$

This is a theorem.

Later, I defined a map $SV \xrightarrow{\varphi} A$. If $\varphi_n : S^n V \to A$, V was a finite dimensional vector space, then $\varphi_n(e_{j_1}, \ldots, e_{j_n})$ in A determines all of the map. So if I have another $\varphi' : SV \to A$, I am giving an equivalence relation $\varphi \sim \varphi'$. I'll make this less tautological by writing these as L_∞ morphisms and homotopies. Then this integration we're doing is invariant of the homotopy type of the L_∞ morphism.

Now today, let me call A an algebra of random variables, an element of A is a random variable, and this is a set of random variables. Then $\iota(X_{j_1} \cdots X_{j_n})$ is called a *joint moment*, $Z(t)$ is called the *moment generating function*, and in non-commutative probability theory, the moment generating function gives you the distribution of your random variables.

Then I'm claiming that the distributions are an invariant of L_∞ homotopy type.

The second issue is symmetry of the expectation. We're always doing the situation $A \to \mathbf{k}$ and $\iota(1_A) = 1$, but what is a symmetry? Let me consider a Lie algebra version first. A symmetry of the expectation is a Lie algebra $\mathfrak{g}$ together with a representation $\rho : \mathfrak{g} \to \mathrm{End}(A)$ such that $\iota \circ \rho = 0$. I mean $\iota(\rho(g)(x))$ for $g \in \mathfrak{g}$ and $x \in A$ is zero.

Then I have

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d, "\pi"] & \mathbf{k} \\ A/\mathfrak{g}.A = A_{\mathfrak{g}} \arrow[ur, dotted, "\iota_{\mathfrak{g}}"'] & \end{tikzcd}$$

An easy theorem is that there exists a symmetry such that $A_{\mathfrak{g}}$ is isomorphic to $\mathbf{k}$ as a vector space.

You may think it's hard to find a symmetry in a general situation but that is not true. Let me define $\rho \in \mathrm{End}(A)$ by the formula $\rho(x) = x - \iota(x)1_A$. This is obviously a linear map. What is $\iota(\rho(x))$? It's $\iota(x) - \iota(x) = 0$. This is the image of a 1-dimensional Lie algebra.

The coinvariants are clearly 1-dimensional.

Now you see that this would not be useful in practice, but it gives us an important lesson, which is that you can characterize ι completely in terms of its symmetry. So we want to characterize ι in terms of its symmetries. To write down this particular representation explicitly you'd need to know all the integrals. But in practice, this symmetry is not useful. So let me consider another thing. Let me consider a subalgebra called $\mathrm{Diff}_{\mathbf{k}}(A)$, a subalgebra of $\mathrm{End}_{\mathbf{k}}(A)$.

These are *linear differential operators* on A. If I have $\rho \in \mathrm{End}_{\mathbf{k}}(A)$, then I want to define a sequence $\ell_1^\rho, \ell_2^\rho, \ldots$, where $\ell_n^\rho : S^n A \to A$ is defined as

$$\ell_n^\rho(x_1, \ldots, x_n) = [[\cdots[[\rho, L_{x_1}], L_{x_2}]\cdots], L_{x_n}](1_A)$$

where L_x is left multiplication by x, $L_x(a) = x.a$.

Definition 5.1. *The map ρ is an order n differential operator on A if $\ell_{n+1}^\rho = 0$ and $\ell_n^\rho = 0$. For example, well, I won't do it. So order n differential operators satisfy this relation.*

Definition 5.2. *An* infinitessimal symmetry *of $\iota : A \to \mathbf{k}$ is a map $\mathfrak{g} \to \mathrm{Diff}_{\mathbf{k}}(A)$ such that $\iota \circ \rho = 0$.*

I want to do one very easy example. Let $A = \mathbb{R}[s]$. I'll define

$$\iota(x) = \int_{-\infty}^{\infty} x e^{-\frac{1}{2}s^2} ds \Big/ \int_{-\infty}^{\infty} e^{-\frac{1}{2}s^2} ds$$

where x is a polynomial in s.

I want to consider a representation of the one-dimensional Lie algebra where $\rho = -\frac{d}{ds} + L_s$. This is an example because $\iota \circ \rho = 0$, because this is basically doing a total derivative. So $\iota\rho(o) \propto \int_{-\infty}^{\infty} \frac{d}{ds}(oe^{-\frac{s^2}{2}})ds$ which is 0 for $o \in A$.

So if I do a computation, then $\rho(1) = s$ and $\rho(s^n) = -ns^{n-1} + s^{n+1}$, so $o \sim o'$ if $o' - o$ is in the image of ρ.

So $\iota(s) = 0$, and then $s^2 \sim 1$. If you work this out inductively, you find that $s^{2k+1} \sim 0$ because it will be equivalent to something times s, which is 0, and s^{2k} is equivalent to $(2k-1)!!1_A$. So basically this implies that

$$\iota(s^n) = \begin{cases} 0 & n \text{ odd} \\ (n-1)!! & n \text{ even} \end{cases}$$

Okay, so no computation involved.

Our answer is the following, $Z(t) = \iota(e^{ts) = e^{\frac{1}{2}t^2}}$.

Let me just give some remark. Later I'll do some homological algebra to deal with this symmetry in a fancier way, but the first part is like Δ, the BV operator, and the L_s is the BRST operator. You do Batalin–Vilkoviski quantization, physicists are assuming that they are working with a translation-invariant measure twisted by the exponential of the action functional. But in an infinite dimensional case, you don't have a translation invariant measure. You may say this is trivial, but you can solve Kontsevich's matrix models this way.

There is another way to solve this, where you consider a family of symmetries twisted by the partition function, and this gives you a differential equation to solve.

If you think in terms of homotopy you have a linear term in t, you have a family of maps ρ, where the first one is $\varphi_1, 0, \dots$, with $\varphi_1(e) = s$. The second one, φ' is $0, \varphi'_2, 0, \dots$ where $\varphi'_2(e, e) = 1$. This implies that these two are homotopic.

That's what I want to say for today. Any questions?

6. December 21: Jae-Suk Park: Lectures on Homotopy Theory of Quantum Fields IV

The main goal for today is to introduce correlation algebras and affinely flat structure.

I will explain the context. So basically, our basic setting was the following. Somehow, a category, a certain category with objects $\mathcal{A}_{bC}$ are tuples $(\mathcal{A}, 1_{\mathcal{A}}, \cdot, K)$ where $(\mathcal{A}, 1_{\mathcal{A}}, \cdot)$ is a $\mathbb{Z}$-graded supercommutative associative unital algebra and $(\mathcal{A}, 1_{\mathcal{A}}, K)$ is a pointed cochain complex. I gave a motivation for why we are considering this guy.

If I have two such objects $\mathcal{A}_{bC}$ and $\mathcal{A}'_{bC}$ then morphisms between them are pointed cochain maps f, so $fK = K'f$ and $f(1_{\mathcal{A}}) = 1_{\mathcal{A}'}$. I told you about a functor from this category to the category of homotopy Lie algebras, which takes $\mathcal{A}_{bC}$ to $\mathcal{A}_L$ and f to $\underline{\phi}^f$. This is a functor $\mathcal{D}es$ to sL_∞-algebras. I told you this functor can be extended as a homotopy functor, where homotopies in the domain are cochain homotopies and in the codomain are L_∞ homotopies.

In the first category, the ground field is an initial object, $\mathbf{k}_{bC}$ is $\mathbf{k}$ as a $\mathbf{k}$-algebra, with no differential. Then if I have $\mathcal{A}_{bC}$, I consider morphisms to the initial object in the category. This guy, or more precisely something defined at the level of homotopy categories

$$\mathcal{A}_{bc} \xrightarrow{[c]} \mathbf{k}_{bc}$$

is some model of a path integral. The degree of c is 0, so it's a zero map on $\mathcal{A}^i$ for $i \neq 0$, and $c(1_A) = 1$, and $c \circ K = 0$. This is defined up to pointed homotopy, $c \sim \tilde{c}$ if $\tilde{c} = r \circ K + c$ where $r : A \to \mathbf{k}$ has degree -1.

Then we define a *homological random variable as an element $X \in \mathcal{A}$ such that $KX = 0$.* We say that two homological random variables are equivalent if they are in the same homology class. We call $c(x)$ the *expectation* of x, and this depends only on the homotopy class of x and c. To define correlations of random variables, we need to be able to compute something like $c(x^n)$. We have some problems. The first is that Kx^n is nonzero in general, since K is not necessarily a derivation of the product. Then $c(x^n)$ may depend on the representative of c. Another thing is that if $x \sim \tilde{x}$, this does not imply that $x^n \sim \tilde{x}^n$, even if $Kx^n = 0$. We have this problem around here. So there's a problem in defining correlations. This problem first occurs in BV quantization. In the Batalin–Vilkoviski quantization scheme, K may be regarded as the BV-BRST operator, which has a classical part and a quantum part, $-\hbar\Delta + Q$, where Q is a derivation but Δ is a second order operator. An element annihilated by K is called a quantum observable. The path integral should assign an expectation value to each quantum observable, depending only on the K-homotopy class.

But the product of two quantum observables will not necessarily be a quantum observable. If I let $\hbar$ go to zero, then on the BRST cohomology, you have no problem with the homology ring. But we don't have this at the BV-BRST level.

We'd like to overcome this difficulty. How to change, deal with this situation? The idea is very simple. Basically we consider e^{tX}, a formal expression

$$1 + tA + \frac{t^2}{2}X^2 + \cdots \in \mathcal{A}[[t]].$$

We see that to define the moments of the random variable, we need the equation that $Ke^{tX} = 0$, then $KX^n = 0$ for all n. Then the condition $Kx = 0$ is not enough.

That kind of deals with the first problem. What about the second problem? For illustration, consider the case that $Kx^n = 0$ for all n, and consider $\tilde{x} = x + k\lambda$. Then $e^{t\tilde{x}}$, we want to compare this to e^{tx}.

$$e^{t\tilde{x}} = e^{tx + tK\lambda}.$$

Let's even assume that $K(e^{t\tilde{x}}) = 0$, which isn't necessarily true. Then still $c(e^{t\tilde{X}}) - c(e^{tX})$ may not be zero.

We want to define our random variables, or observables, as a map from $(V, \underline{0})$, an sL_∞-algebra with zero sL_∞ structure (this is just a vector space) together with an L_∞ morphism $\underline{\varphi}$ to $\mathcal{A}_L$ (which itself has the descendent morphism ϕ^c to $\mathbf{k}$). We want $\underline{\varphi}$ to be something like this in a fixed homotopy type.

Then we can compose this to get a map $\underline{\kappa}_V = \underline{\phi}^c \underline{\varphi}$ to $\mathbf{k}$.

Then, everything is well-defined in the homotopy category, meaning that κ^V depends only on the two homotopy classes of $\underline{\varphi}$ and c.

Let me also define $\underline{\mu}^V$, also a map from $\underline{S}^n(V) \to \mathbf{k}$, so that $\mu_n^V(\alpha_1, \ldots, \alpha_n)$ is defined by $\sum_{\pi \in P(n)} \kappa^V(\alpha_{B_1}) \cdots \kappa^V(\alpha_{B_{|\pi}})$. We talked about the notation for partitions before.

The product is living in $\mathbf{k}$. So μ_1^V is κ_1^V and $\mu_2^V(\alpha_1, \alpha_2) = \kappa_2^V(\alpha_1, \alpha_2) + \kappa_1^V(\alpha_1)\kappa_2^V(\alpha_2)$.

We consider an L_∞ morphism$(V, \underline{0}) \xrightarrow{\varphi} \mathcal{A}_L \xrightarrow{\phi^c} \mathbf{k}$.

So let $\{e_i\}$ be a basis of V, assumed finite dimensional to make things easy.

So consider $\Gamma^{\underline{\varphi}}$ defined as

$$\sum_{n=1}^{\infty} \frac{1}{n! \sum_{i_1,\ldots i_n} t_{i_1} \cdots t_{i_n} \varphi_n(e_{i_1} \cdots e_{i_n})}$$

which is

$$\sum_i t_i \varphi_1(e_i) + \frac{1}{2} \sum_{i,j} t_i t_j \varphi_2(e_i, e_j).$$

A lemma you can compute is that $Ke^{\Gamma^{\underline{\varphi}}} = 0$.

Then $e^{\Gamma^{\underline{\varphi}}} - e^{\Gamma^{\underline{\tilde{\varphi}}}} = K\Sigma$ if φ and $\tilde{\varphi}$ are related by L_∞ homotopy.

Then we consider $Z_V = c(e^{\Gamma^{\underline{\varphi}}})$, and this guy, the whole thing belongs to the kernel of K, so this only depends on the homotopy type of c. Now the differences will be K of something. Then it's not hard to show that this is

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1,\ldots i_n} t_{i_1} \cdots t_{i_n} \mu_n^\nu(e_{i_1}, \ldots e_{i_m})$$

It's long and complicated but straightforward to prove the lemmas. But the relation between κ and μ tells you this exponent is

$$\sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1,\ldots,i_n} t_{i_1} \cdots t_{i_n} K_n^V(e_{i_1}, \ldots, e_{i_n}).$$

Now consider maps. You can always make a map $\underline{\psi}$ from $(V, \underline{0})$ to A_L where $\psi_m :== \kappa_n^V \cdot 1_A$.

Then it could be that $\underline{\varphi} \sim \psi$. In general this is not true, but that's the answer of our problem, which is homotopic to our question.

So if the dimension of V is 1, say $\{e\}$ and $\underline{\varphi}$ has only φ_1, so $x = \varphi_1(e) \in \mathcal{A}$. Then say we have another $\tilde{\varphi}$ which is of the same form, with $\tilde{\varphi}_1 = \varphi_1 + K\lambda_1$, then it is not necessarily true that $\varphi \sim \tilde{\varphi}$ even though φ_1 and $\tilde{\varphi}_1$ are homotopic.

Let me return to our favorite example, the Gaussian, $\mathcal{A} = \mathbb{R}[s, \eta]$, with the degree of s equal to 0 and the degree of η equal to -1, and since η is anticommuting, we have $\eta^2 = 0$, and $\eta s = s\eta$. Then this is $\mathbb{R}[s]\eta \oplus \mathbb{R}[s]$, which is $\mathcal{A}^{-1} \oplus \mathcal{A}^0$. This is a graded unital commutative associative algebra.

Define K as $-\sigma^2 \frac{\partial^2}{\partial s \partial \eta} + s\frac{\partial}{\partial \eta}$ where $\sigma^2 \in \mathbb{R}^+$. Call $s\frac{\partial}{\partial \eta}$ as Q and $\frac{\partial^2}{\partial s \partial \eta}$ by Δ and then maybe $\sigma^2 = \hbar$.

Then we can calculate that $K^2 = 0$ and $\ell_2^K(s, s) = 0$, $\ell_2^K(\eta, \eta) = 0$, and $\ell_2^K(\eta, s) = -\ell_2^K(s, \eta) = -\sigma^2$. Extend this as a derivation of the product in both variables.

Then it turns out $\mathcal{A}_L = (\mathcal{A}, 1, K, \ell_2^K)$, and that's where it stops.

What is c now? From $\mathcal{A}$ to $\mathbb{R}$ it's the Gaussian integral, define

$$c(p(s)) = \frac{\int_{-\infty}^{\infty} p(s) e^{-\frac{s^2}{2\sigma^2}ds}}{\int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma^2}ds}}$$

or something very close to that. You can compute that the unit goes to the unit. You can easily see that $c \circ K$ is 0. You only need to check what happens to $\mathcal{A}^0$. An arbitrary element Λ in A^{-1} can be written as $\lambda(s)\eta$ then $K\Lambda$ is $-\sigma^2\frac{d\lambda}{ds}+s\lambda$, and if you put this here, it's a total derivative, so it vanishes in the numerator.

Now you consider a 1-dimensional vector space V with basis e, and consider a map $\varphi_1 : V \to A$, this is an ungraded vector space and a degree 0 map, so we need to specify $\varphi_1(e) = X$. Then $\underline{\varphi} = \varphi_1, 0, 0, \ldots$ is an sL_∞ morphism. Why? Because $\mathcal{A}$ is concentrated in nonpositive degrees, and so any such map is an sL_∞ morphism.

So $e^{\Gamma \underline{\varphi}}$ is just e^{ts}. So our question is what is $c(e^{ts})$? It's just proportional to

$$\int e^{\frac{s^2}{-2\sigma^2}+ts}ds$$

but let me do this integral in a strange way. Well, φ_1 is cochain homotopic to zero, which tells you that $c(s) = 0$. We know, then $s = K\eta$. We deform the action functional by a BRST-exact term. Then this ts term, this gives a non-zero contribution, this is $e^{\frac{t^2}{2}\sigma}$. This is why I emphasize that even if you have an L_∞ morphism which is a chain map chain homotopic to zero, that doesn't make it L_∞ homotopic to zero.

So for instance, $\varphi_1, 0, 0, \ldots$ is L_∞ homotopic to $(0, \tilde{\varphi}_2, 0, \ldots)$, where $\tilde{\varphi}(e, e) = \sigma^2 1_{\mathcal{A}}$. This shows that $c(e^{ts}) = e^{\frac{t^2}{2}\sigma^2}$.

So when can we compute this without computing anything? If the dimension of the cohomology is 1, you can always do that. What is that class? It's the class of 1. The unit will never be exact. Then if I have a 1-dimensional cohomology, I know everything is homotopic to some coefficient times 1. I told you, you can go to different K with different features, to deal with these situations.

Let me talk about complete space of random variables. If I start with $\mathcal{A}_{bC}$ and go to $\mathcal{A}_L$ via $\mathcal{D}es$, then I always get a smooth formal L_∞ algebra, always quasi-isomorphic to a zero L_∞ structure. Then the ultimate space, let me call it $(S, 0) \to \mathcal{A}_L$, this guy is a quasi-isomorphism $\underline{\varphi}^S$. Then we can do the same game, map to $\mathbf{k}$ via $\underline{\phi}^c$, and then we can define $\underline{\kappa}^S$ and $\underline{\mu}^S$, and those things go from $S(S) \to \mathbf{k}$. We know $S \cong H(\mathcal{A}_L, K)$. Remember that $\mu^S(h_1, \ldots, h_n)$ is $\sum_{\pi\in P(n)} \pm\kappa^S(h_{B_1})\cdots\kappa^S(h_{B_{|\pi|}})$, which tells you that $(H, 0) \to \mathcal{A}_L$ is a quasi-isomorphism, but we know that there will be many homotopy types of such quasi-isomorphism. This space of homotopy types of sL_∞ quasi-isomorphisms, this moduli space is the same as the solution space of Maurer–Cartan equations, $K\Gamma + \frac{1}{2}\ell_2^K(\Gamma, \Gamma) + \cdots$ where $\Gamma = \sum_i t_i\gamma_i + \cdots$. This may not make sense if H is not finite dimensional, but if it is then these maps correspond to Maurer–Cartan elements and homotopy corresponds to gauge equivalence.

Now it is possible to define $\underline{M}^S$ which starts $\underline{M}_1^S, \ldots$, where M_1^S is the identity, and $M_n^S : S^n(S) \to S$. We want that $\mu_1^S \circ M_n^S = \mu_n^S$. Remember the original formula had originally, $\mu_n(x_1, \ldots x_n)$ was $\iota(x_1 \cdots x_n)$. What I'm doing, I'm forced to have a product structure on H, and that's the structure M.

What properties does this have? It's symmetric, and $M_{n+1}^S(h_1, \ldots, h_n, 1) = M_n^S(h_1, \ldots, h_n)$. Note that $M_n^S(h_1, h_2, h_3)$ is not the iterated product $M_2^S(h_1, M_2^S(h_2, h_3))$.

Define $h_1 \cdot h_2 = H_2^S(h_1, h_2)$, then this is commutative. If M_3 is given by this iterated product, then the product is associative as well as graded commutative. This kind of thing I call correlation algebra. Maybe this is strange, but I'm running out of time. What can I say?

Okay. I'll return to this more systematically. What I'll say is that a complete space of random variables, there's this moduli space of these, and the basic idea is that M is some formal supermanifold and using this gadget, you can give this an affinely flat structure, some torsion free flat connection on the tangent space of that manifold. Then this is the same as a torsion-free flat connection, and then there are formal flat coordinates which encode all the information about correlations.

Now we start with some algebra, but end up with some flat structure here. Something is wrong, because the cohomology has a different structure. But in the special case, the cohomology is just an algebra. So we should restart, assuming that the space forms something much more general.