INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS POSTDOC LECTURE SERIES

GABRIEL C. DRUMMOND-COLE

1. April 20: Rui Wang: On orbifold groupoids I

1.1. **Definitions.** Let me start with some motivation. Let M be equipped with an equivalence relation. When you take the quotient, you may want to remember more information, like how two elements are equivalent, by what path.

Definition 1.1. An *abstract groupoid* consists of a set G^0 of objects and a set G^1 of isomorphisms of G^0, where (G^0, G^1) is a category.

We can associate to this category structure maps

source	$s : G^1 \to G^0$
target	$t : G^1 \to G^0$
(associative) composition	$m : G^1_t \times_s G^1 \to G^1$
unit	$u : G^0 \to G^1$
inverse	$i : G^1 \to G^1$

So we call this $\mathcal{G} = ((G^0, G^1), (s, t, m, u, i))$. It is natural that this gives us an equivalence relation on G^0, where $x \sim y$ if there is an arrow connecting x and y. This is the *coarse space* $|\mathcal{G}|$ of this groupoid.

If there is a groupoid $\mathcal{G}$ whose coarse space is X then we say $\mathcal{G}$ is a *groupoid representation of* X. This representation is not unique.

Next

Definition 1.2. (Lie groupoid)

(1) For the Lie groupoid we need some topology. This is a groupoid where G^0 and G^1 are both smooth manifolds and all structure maps are smooth. In particular, s and t are submersions. Then $G^1_t \times sG^1$ is a smooth manifold.
(2) We call a Lie groupoid *proper* if the map $(s,t) : G^1 \to G^0 \times G^0$ is proper and étale if $\dim G^0 = \dim G^1$ (which implies that every structure map is a local diffeomorphism). We call a proper étale Lie groupoid an *orbifold groupoid*. So for $x \in G^0$ we can consider $Gx = \{\alpha \in G^1 | s(\alpha) = t(\alpha) = x\}$ and for an orbifold groupoid this is a finite set.

Example 1.1. (1) Take G^0 a point and $G^1 = G$ a group. Then the source and target map send everything to the point and the composition, inverse, and unit are in the group. This is a *point groupoid* $[\cdot^G]$. If G is a Lie group then this is a Lie groupoid; if G is finite then this is an orbifold groupoid. $|[\cdot^G]| = \cdot$.
(2) Suppose M is a manifold. Take $M^0 = M$ and $M^1 = M$. The source, target, and composition all fix x. This has $|M| = M$ and is also an orbifold groupoid.

(3) Let M be a manifold and $\{U_\alpha\}$ a locally finite open cover of M. Take $U^0 = \amalg_\alpha U_\alpha$ and $U^1 = \amalg_{\alpha,\beta} U_\alpha \times_M U_\beta$. The source of (x_α, x_β) is x_α and target x_β. Then $|U| = M$.

Later after I introduce an equivalence, I'll prove

Lemma 1.1. *The two representations of M, M and U, are Morita equivalent.*

(4) Let X be a manifold and K a Lie group acting on X. Then you can define a Lie groupoid $(XrltimesK)^0 = X$ and $(X \rtimes K)^1 = X \times K$. The source map takes $(x,k) \mapsto x$ and the target map takes $(x,k) \mapsto x \cdot k$. This is a Lie groupoid. Here $\rho : K \to Diff(X)$ makes this a semidirect product.

The coarse space is X/K, which in general has no good structure, this is no more than a topological space. If the K action is proper and free (this gives you a K-principal bundle), then X/K is a manifold. If the action is locally free, then X/K has an orbifold structure. This can be constructed from the slice theorem, which says that the K-orbit of $x \in X$ has a slice which takes an action of K and then you get a neighborhood with a K action. You can look at $K \times_{Kx} W_x$, and then you can put these together to get X_K, which is an orbifold groupoid Morita equivalent to $X \rtimes K$. I'll say more tomorrow about this.

(5) Let M^n be a manifold with g a Riemannian metric. Then take the $O(n)$-frame bundle, then $Fr(M) \rtimes O(n)$ is also Morita equivalent to M.

Theorem 1.1. *(Satake) An effective orbifold M is equivalent to U is equivalent to $Fr(M) \rtimes O(n)$.*

Can you do this for a general orbifold groupoid? I think no one has an answer to this up to now.

Now let's talk a little bit about equivalence.

1.2. **Equivalence.** If I have two objects $\mathcal{G} \to \mathcal{H}$, abstract groupoids, a functor Φ is a map $G^0 \to G^0$ and $G^1 \to G^1$ which commutes with every structure map. If $\mathcal{G}$ and $\mathcal{H}$ are Lie groupoids, then this is a *strict* homomorphism if Φ^0 and Φ^1 are both smooth maps.

Definition 1.3. $\mathcal{G}$ and $\mathcal{H}$ are *strongly equivalent* if there are functors $\Phi : \mathcal{G} \to \mathcal{H}$ and $\Psi : \mathcal{H} \to \mathcal{G}$, and then $\Phi \circ \Psi \sim \mathrm{id}$ and $\Psi \circ \Phi \sim \mathrm{id}$.

In categorical language, you can also talk about *weak equivalence*, if $\Phi : \mathcal{G} \to \mathcal{H}$ is

(1) essentially surjective (so $y \in H^0$ has an $x \in H^0$ such that $x \sim y$ in H), and
(2) full and faithful (arrows between x, x' in Gg are in bijection with arrows $\Phi(x), \Phi(x')$ in $\mathcal{H}$).

Proposition 1.1. *Strong and weak equivalences are the same.*

In the smooth category, $G^0_\Phi \times_s H^1 \xrightarrow{t \circ pr_2} H^0$ should be not just surjective but also a submersion. The full and faithful condition says that $G^1 \cong G^0 \times G^0_{\Phi,\Phi} \times_{s,t} H^1$ is not just a bijection but a diffeomorphism.

Then strong and weak are no longer equivalent.

Proposition 1.2. *In the smooth category, strong is always weak but weak may not be strong.*

Weak equivalence may not be symmetric in the Lie category which is why you need Morita equivalence.

Definition 1.4. We say that $\mathcal{G}$ is Morita equivalent to $\mathcal{H}$ if there exists $\mathcal{K}_1$ which is weakly equivalent to both $\mathcal{G}$ and $\mathcal{H}$.

You can prove that this is an equivalence relation. The only thing you need to check is why it is transitive. If you have a Lie groupoid $\mathcal{G}$ which is Morita equivalent to $\mathcal{H}$ via $\mathcal{K}_1$ which is Morita equivalent to $\mathcal{W}$ via $\mathcal{K}_2$, then a tricky construction of the fiber product $\mathcal{K}_1 \times_H \mathcal{K}_2$ is a Morita equivalence from $\mathcal{G}$ to $\mathcal{W}$.

Assume $\mathcal{H}$ is a Lie groupoid and we have $\phi : G^0 \to H^0$ a smooth map. Assume that $G^0_\phi \times_s H^1 \to H^0$ is a submersion. I define the pullback of H^1, denoted $\phi^* H^1$, as $G^0 \times G^0_{\phi,\phi} \times_{s,t} H^1$, which is a smooth manifold by our submersion property. You pull back the groupoid structure of H over G^0. Let me call this $\phi^*\mathcal{H}$.

Lemma 1.2. *Iff $t \circ pr_2$ is surjective, the pullback $\phi^*\mathcal{H}$ is weakly equivalent to $\mathcal{H}$.*

In my previous example where $\mathcal{U}$ was given by charts, then $G^0 = \mathcal{U}^0$, which maps into M, the inclusion map from every component. You can easily check that this is a submersion and surjective. Then it turns out that $\mathcal{U} = j^* M$. With the same thing you can prove all the equivalences that I listed before.

I will say a last example. Say you have a principal G bundle P over M, and let H be a finite group with a homomorphism $\rho : G \to \operatorname{Aut} H$. Then I can construct a new groupoid $\mathcal{G} = P \times_G [\cdot^H]$, and here I mean that $\mathcal{G}^0$ is $P \times_G \cdot = M$, and $G^1 = P \times_G H$, and you get a natural group structure using this homomorphism.

It's not obvious showing that this is Morita equivalent to a global quotient; it is $(G \ltimes_\rho H) \ltimes P \cong_{\text{Morita}} \mathcal{G}$. That is an example I realized ery recently. Maybe I should stop here.

2. April 21: Rui Wang: On orbifold groupoids II

Today I'll focus on the effective case and talk about group actions on a groupoid. Let's start with some preparation. Suppose X is a connected smooth manifold and G is a finite group. We say that G acts on X through ρ if ρ is a homomorphism from G to the diffeomorphism group of X. Now I'll say some very simple properties of this kind of thing. We know this group is finite, which is closed, and this is smooth, which is an open condition; then connectedness gives us some kind of rigidity for the action.

So for $x \in X$, we have the isotropy group $G_x = \{g \in G | gx = x\}$. We also have the fixed points of g, $Z_g = \{x \in X | gx = x\}$. Let me give several lemmas. I'll assume the action is effective, that is, the kernel of ρ is trivial.

Lemma 2.1.
1. *for all $x \in X$ there exists an arbitrarily small U_x such that if $g \in G_x$ then $gU_x = U_x$ and if $g \notin G_x$ then $gU_x \cap U_x = \varnothing$.*
2. *For U open and nonempty in X, if g_1 and g_2 in G have the same restriction to U then $g_1 = g_2$.*
3. *For open nonempty $U \subset X$, if there is $f : U \to X$ with $f(x) \in Gx$ for any $x \in U$, then f is a group element acting, restricted to u*
4. *If we have two finite groups acting on manifolds, if $|f| : |X| := X/G \to |Y|$, we say this has a* smooth lifting *if there is a map $f : X \to Y$ such that the diagram commutes. Say it is an* embedded lifting *if f is an embedding.*
5. *If $|X| \to |Y|$ has an embedded lifting f, then*

(a) *There is an injective group homomorphism* $\phi_f : G \to H$ *such that for all* $g \in G$, $\phi_f(g)(f(x)) = f(gx)$

(b) *Any two embedded liftings* f_1 *and* f_2 *differ by a group element* h_{12} *in* H, *uniquely determined, such that* $f_2 = h_{12} \circ f_1$.

Now let me introduce the meaning of an (effective) orbifold.

Definition 2.1. So let's assume X is a topological space, and fix $n \geq 0$ (the dimension of X). Introduce an atlas for this topological space. If there is a smooth atlas we'll say this has an orbifold structure. So we want

(1) $\widetilde{U}_\alpha$, an open connected manifold of dimension n equipped with a finite group G_α acting (effectively). This gives an action groupoid $G_\alpha \rtimes \widetilde{U}_\alpha$. The coarse space is U_α, and I'll assume this is an open subset of X.

(2) $\bigcup_\alpha U_\alpha = X$.

(3) (compatibility) If $x \in X$ and $x \in U_\alpha \cap U_\beta$, then there is a chart $x \in U_\gamma$ which is a subset of $U_\alpha \cap U_\beta$ which has an embedded lift to both U_α and U_β. Then $\bar{\lambda}_{\gamma\alpha} : G_\gamma \to G_\beta$ is an injective homomorphism and likewise for β.

We say that U is an *orbifold representation* of X.

We say that U' is a refinement of U if every chart of U' has an embedding into U (with embedded liftings).

We say the two orbifold representations are *equivalent* if they have a common refinement.

Now I want to construct some special coordinates. Assume (X, U) is an orbifold representation and construct a new one $\mathcal{O}_X$, which I will explain.

(1) for each $x \in X$, choose $\widetilde{U}_x \subset \widetilde{U}_\alpha$, and let G_x be the isotopy group of G_α restricted to $\tilde{x}$, where $\tilde{x}$ is a lift of x. You can choos this arbitrarily small so that G_x acts on this $\widetilde{U}_x$, and every other element disjoints this set. So the coarse spaces are the same regardless of the choice of lift.

(2) for any two points $x, y \in X$, we say $x < y$ if $x \in U_y$. This is not a partial order. I claim that the embedding data gives an embedding λ_{xy} from a G_x-invariant subset $\widetilde{U}_x^y$ of $\widetilde{U}_x$ to $\widetilde{U}_y$.

I'll call this data (U_x, Λ_x) and together they are $\mathcal{O}_{U,X}$.

Now I will relate this to what I talked about yesterday. From an orbifold groupoid, how do you get an orbifold representation, and vice versa? I'll assume $\mathcal{G}$ is an orbifold groupoid and I want to give it a chart. For every point $x \in G^0$, the isotopy group G_x is $g \in G^1$ such that $gx = x$, this is a finite set. I want to know how this acts on a neighborhood of x. Pick $g \in G_x$, which, the source of g and the target of g are x so you get a local diffeomorphism and a neighborhood of g in G^1, call it V_g, and a neighborhood of x, call it $\widetilde{U}_x$ (this is bad notation) and a map $V_g \to \widetilde{U}_x$, and s and t are local diffeomorphisms onto their images. I can take the intersection of all such neighborhoods over G_x. Now you can ask how G_x acts on this set. You use this local diffeomorphism to send your patch from $\widetilde{U}_x$ to V_g and then back via s^{-1} and then t. Then G_x acts on the open set. $\widetilde{U}_x$. This essentially depends on the proper étale property. For the embedding it is very similar. If x goes into another chart, then x is connected to gx by an arrow. Then a similar argument tells you that you have a family of arrows.

So from $\mathcal{G}$ I can get $U_{\mathcal{G}}$ and then $\mathcal{O}_{\mathcal{G}}$. Now let me quickly say how you can come back. When you have $\mathcal{O}_{\mathcal{G}}$, for $\mathcal{G}_0$ I take

$$\amalg_{x\in X}\widetilde{U}_x$$

and for G^1 I take

$$G^1 = G_x \times \left(\amalg_{w<x,w<y}\widetilde{U}^x_w \cap \widetilde{U}^p_w\right) G_y \Big/ \sim$$

where you need an equivalence relation because this union is too big. [discussion about equivalence].

You can prove that G^1 is a smooth manifold and the structure maps are well-defined. the source and target are not so hard. You must show that m is well-defined and associative, and this is tricky. It's kind of amazing that you get this from the effective condition. Equivalence and Morita equivalence are both connected by these constructions.

You can easily define K acting on X for X a manifold. But for X a groupoid that's harder. With charts, you want a group action on a manifold, that's already a problem. If you break S^1 into two charts, you'll have to switch charts to act by rotation. If you want K acting on $\mathcal{G}$, you can't expect $k : \mathcal{G} \to \mathcal{G}$ is a strict homomorphism. That's not possible even in the smooth category. You need to introduce subtle considerations.

[some discussion]

For the effective case, when you have K acting on the coarse space continuously, I want that for $(k,x) \in K \times X$ you can find a local lifting $V^x_k \subset K$ and $\widetilde{U}^k_x \subset \widetilde{U}_x$ and $F|_{V^x_k \times U^k_x}$ has a local lift into $\widetilde{U}_{kx}$, which I assume is compatible with the group structure, the lifting $\widetilde{e}^{(e,x)} = id_{\widetilde{U}_x}$ and $\widetilde{\ell k}^{(\ell k,x)} = \widetilde{\ell}^{(\ell,kx)} \circ \widetilde{k}^{(k,x)}$.

If I assume K is compact and a technical properness condition then I get a slice theorem

Theorem 2.1. *For any point $x \in X$ you can find a chart $K \times_{K_x} (G_x \rtimes \widetilde{W}_x)$ so that the coarse space is U_{kx}, and for any $x <_K y$, then we can construct λ^K_{xy}, and using the general procedure we can get the new groupoid $\mathcal{G}_K$ and show that it is Morita equivalent to $\mathcal{G}$ and that the action of K on $\mathcal{G}_K$ is strict.*

3. April 25: Rui Wang: On orbifold groupoids III

Some references are Adem–Leida–Rua, orbifolds and stringy topology, Moerdijk–Mrčun Lie groupoids, Chen–Hu, for the Abelian case, Hu–Wang for the non-Abelian case, and Chen–Ruan as another reference. By the way, this Chen is not this Chen, this Hu is not this Hu, and this Wang is not me.

Okay, so let $\mathcal{G}$ be a groupoid, then we can make a classifying space $B\mathcal{G}$ which is the nerve. The cells G_n is $(g_1,\dots,g_n)$ in the product of $G^1 \times \cdots G^1$ where the source of one is the target of the next. Then cross this with the standard simplex Δ^n. Take the disjoint union over all n and then quotient by the face relations from $d_i : G_n \to G_{n-1}$ where for d_0 you forget g_1, for d_n you forget g_n, and for d_i you multiply g_i and g_{i+1}. For G_1 you take G^1 and for G_0 you take G^0 with maps given by source and target.

Facts include, if ϕ is a strict homomorphism $\mathcal{G} \to \mathcal{H}$, then you get a map $B\mathcal{G} \to B\mathcal{H}$ and if ϕ is a weak equivalence then $B\mathcal{G} \to B\mathcal{H}$ is, so the homotopy groups coincide. If these are Morita equivalent, then the two homotopy groups are isomorphic.

Definition 3.1. Let $\mathcal{G}$ be a Lie groupoid. Then $\pi_n^{orb}(\mathcal{G})$ is equal to $\pi_n(B\mathcal{G})$.

If $\mathcal{G} = G \rtimes X$ then $B\mathcal{G}$ is homotopy equivalent to $EG \times_G X$ for a contractible principal G-bundle EG over BG So we get a long exact sequence of homotopy groups

$$\cdots\pi_n(X) \to \pi_n(EG \times_G X) \to \pi_n(BG) \to \cdots$$

so in particular we have $\pi_2(BG) \to \pi_1(X) \to \pi_1^{orb}(\mathcal{G}) \to \pi_1(BG) \to \pi_0(X)$

If G is finite, and X is connected, then $\pi_0(X) = 1$ and $\pi_1(BG) \cong G$ and $\pi_2(BG) = 1$, so we get a surjective map $\pi_1^{orb}(\mathcal{G}) \to G$ and this tells us that certain orbifold groupoids cannot be global quotients. For instance,

Example 3.1. The weighted projective space $\mathbb{WP}(a_0, \ldots, a_n)$ is not Morita equivalent to a manifold quotiented by a finite group unless $a_0 = \ldots = a_n = 1$.

We see this by seeing tha $\pi_1^{orb}(\mathbb{WP}(\vec{a})) = \pi_1(B(S^1 \rtimes S^{2n+1})) = \pi_1(ES^1 \times_{S^1} S^{2n+1}) = 0$.

Next let's talk about homologies. By using the classifying space, consider $H^*_{orb}(\mathcal{G}, R)$, defined as $H^*(B\mathcal{G}, R)$. If this has no torsion, this ring, then this is the same as $H^*(|\mathcal{G}|, R)$. With $\mathbb{Z}$ coefficients, for $B\mathcal{G}$, for a global quotient this is equivariant cohomology. In particular, if X is just a point, then this is $H^*_G(\cdot, \mathbb{Z}) = H^*(G, \mathbb{Z})$.

Before I go to Chen–Ruan cohomology, let me look at S^2 with k orbifold points. The orders of the k orbifold points are given by $(m_1, \ldots, m_k)$ which is $\{\lambda_1, \ldots \lambda_k | \lambda_i^{m_i} = 1 \text{ and } \lambda_1 \ldots \lambda_k = 1\}$. You can consider $\pi_1^{orb}(\mathcal{G}, x)$ as the deck transformations of a universial groupoid $\widetilde{\mathcal{G}}$ over $\mathcal{G}$.

For example, if Σ is a Riemann surface with orbifold points of order $\vec{m}$. Then $\widetilde{\Sigma}$ is a smooth Riemann surface if and only if either $g \geq 1$ or $g = 0$ and $k \geq 3$ or $g = 0$ and $k = 2$ and $m_1 = m_2$.

So let's go to Chen–Ruan cohomology. Assume $\mathcal{G}$ is an orbifold groupoid. Let $S^k_{\mathcal{G}}$ be k elements in G^1 where all elements have the same source and target. This has a projection to G^0, let me denote it by p. Then you can get the semidirect product of $\mathcal{G}$ with $S^k_{\mathcal{G}}$, where it acts via p. For any point in $S^k_{\mathcal{G}}$, then $\mathcal{G}$ maps it. So h acts on $g_1, \ldots, g_k$ as $hg_1h^{-1}, \ldots, hg_kh^{-1}$. By this way we get a new groupoid, $\mathcal{G}^k = \mathcal{G} \rtimes S^k_{\mathcal{G}}$, and the objects is $S^k_{\mathcal{G}}$ and the morphisms are the fiber product with $\mathcal{G}^1$ over $\mathcal{G}^0$. Then $|\mathcal{G}^k| = \{(x, (g_1, \ldots, g_k)_{G_x})\}$. If $\mathcal{G} = G \rtimes X$ then $S^1_{\mathcal{G}} = \bigsqcup \{g\} \times X^g$. Then $\mathcal{G}^K$ is the disjoint union over $\vec{g}$, conjugacy classes—

[couldn't follow.]

4. June 1: Juhyun Lee, On the classification of tight contact structures I

My abstract was too ambitious, I think.

In these talks I will only concentrate on the classification theory of surface bundles over the interval or the circle S^1, especially the trivial torus bundle $T^2 \times I$ with a boundary condition, or T^3_A, a torus bundle with monodromy $A \in SL_2\mathbb{Z}$, or a higher genus surface bundle.

There are multiple ways to classify tight contact structures on $T^2 \times I$. In the first lecture I'll talk about Giroux's method and in the second I'll talk about Honda's method for the thickened torus and the torus bundle with monodromy with trace greater than 2. In the third lecture I'll consider the higher geneus bundles with pseudoAnosov monodromy. About ten years ago Honda classified these (which satisfy an extremal condition). At the end I'll briefly explain the ideas of proving this without the extremal condition.

I'll give references for these kinds of topics. For Giroux's method, unfortunately, the good reference is Giroux's paper, which is written in French. The first one is *Structure de contact en dimension trois et bifurcations des feuilletages de surfaces.* For Honda's methods there are two series, *On the classification of tight contact structures*, both I and II. Also for the last type, *On the classification of tight contact structures of hyperbolic* 3*-manifolds.*

So for now I'll always assume that M or V is a 3-manifold, possibly with boundary, and I'll assume that everybody knows the definition of a contact structure. So (M,ξ) is a coorientable contact structure. An embedded disk $D \subset (M,\xi)$ is called overtwisted if T_pD and ξ_p coincide. The intersection of TD and ξ_p is called the characteristic foliation of the disk. If there is no such disk then (M,ξ) is called tight.

Almost thirty years ago, Eliashberg classified all overtwisted contact structures. They are in one to one correspondence with homotopy classes of [missed]. The known results for classifications for tight contact structures I'll introduce at the last part along with open problems.

From now on I'll start to introduce some properties of the characteristic foliation. It is the intersection of the contact structure and the tangent bundle of an embedded surface.

I want to talk about some properties of this characteristic foliation. For a C^∞ generic closed orientable surface $\Sigma \subset (M,\xi)$ we can say this characteristic foliation is of Morse–Smale type, meaning that it has the following properties:

(1) There are finitely many singularities and closed orbits, which are nondegenerate in the sense that the Poincaré-Witten map is not the identity.
(2) The α and ω limit set of each flow line is a singular point or closed orbit.
(3) There are no saddle saddle connections.

The α-limit is to flow in the positive direction, the α limit, to flow in backward time gives the ω limit. So saddle-saddle connection means there is no flow line from a saddle into another saddle.

I'll define a convex surface. An embedded surface Σ in (M,ξ) is called a *convex surface* if there is a contact vector field near Σ which is always transverse to Σ. The flow of this vector field preserves the contact structure. By a Hamiltonian function, we can extend to a globally defined vector field. If a surface is convex, there is an $\mathbb{R}$-invariant neighborhood in which the contact structure ξ is rewritten as $fdt + \beta$ where $f : \Sigma \to \mathbb{R}$ and β is a one-form on Σ and this does not depend on t. You should check this by yourself by putting $\alpha_t = f_t dt + \beta_t$. Among solutions that satisfy the contact condition you can find one that has the form I said.

I told you, I mentioned the Morse–Smale type vector fields. Near the surface Σ we can find a contact vector field that is not everywhere transverse, then we can smoothly isotope to [missed] and we find a Morse Smale type vector field, then we can construct $fdt + \beta$.

Let me list these properties again. We defined a convex surface, and this always has a *dividing set* which consists of Γ_Σ, the set of points in Σ in which the conctact vector field $V(x)$ is in ξ_x. This dividing set has the following properties:

- The characteristic foliation $\xi \cap T\Sigma$ is generated by $\ker \beta$ and
- $\Gamma = f^{-1}(0)$.

Then

(1) Γ is a disjoint union of closed 1-dimensional curves (Σ is closed) with $df > 0$, and
(2) $\Gamma \pitchfork \xi\Sigma$, and $\beta \wedge df > 0$
(3) Γ does not depend on the contact vector field up to isotopy,
(4) $\Sigma\backslash\Gamma = \Sigma_+ \sqcup \Sigma_-$ where $f > 0$ or $f < 0$.
(5) This is an ample set, never empty.

The characteristic foliation determined the isotopy class of the contact structure, but we can say:

Theorem 4.1 (Giroux's flexibility theorem)**.** *Given (M,ξ), supposed the characteristic foliation $\xi\Sigma$ for a convex surface is given and there is another (singular) foliation of Morse–Smale type $\mathcal{F}$ on Σ which is divided by (transverse to) the same dividing set Γ_Σ. Then there is a contact structure ξ' near Σ so that $\xi'\Sigma = \mathcal{F}$ then there is an isotopy φ_s for $s \in [0,1]$ such that*

(1) *φ_0 is the identity and $\varphi_s|_{\Gamma_\Sigma}$ is also the identity.*
(2) *$\varphi_s(\Sigma) \pitchfork V$ for all s, and*
(3) *$\xi(\varphi_1(\Sigma))$ agrees with $\mathcal{F}$ under pushforward.*

We classify the thickened torus with some boundary condition up to isotopy relative to the boundary. The boundary conditions are specific fixed characteristic foliations. For the torus case, we assume the boundary is convex and the contact structure is transverse to the boundary. If you assume transversality, then there are two kinds of foliations. One is a linearized foliation. The other has two closed orbits that are spiralled to. We want both to have the same type.

Honda's method, the focus is on the dividing set and using bypasses. But Giroux's method is to flow along the foliation.

There are two types of structure. In a rotative structure, $\xi(T^2 \times \{t\})$ [missed].

Two structures have the same amplitude which is nonzero, then we can find a boundary isotopy among these. This is the classification theory of rotative structures.

The second kind is called elementary. For this case, there are some finite times in $[0,1]$ so that between t_{i-1} and t_i the structure is rotative. The fiber remains convex except at the endpoints. This is called normal form. We can show that for tight contact structures with fixed boundary, this is isotopic to normal form.

I'll stop here.

5. June 2: Juhyun Lee, On the classification of tight contact structures II

So I'm starting from giving a precise description of the the foliation coming from a dividing set. We have $\mathcal{F}$ a singular foliation of Morse-Smale type on Σ. Then there is a contact form ξ near Σ such that $\xi\Sigma = \mathcal{F}$. Two ξ and ξ' which satisfy $\xi\Sigma = \xi'\Sigma$ are topologically equivalent, then they are isotopic, so this contact structure is essentially unique. If Γ is isotopic to $\Gamma_{\Sigma,\xi}$ then we say that Γ divides $\mathcal{F}$.

Now I'll explain about bypass theory. For this we have to start from the standard form of a Legendrian knot embedded in a contact manifold (M,ξ). That means that if we choose an arbitrarily small neighborhood of the Legendrian knot, there is a normal form in the neighborhood of the knot, which looks like $(x, y, z)|x^2 + y^2 \leq \epsilon$ and $L = (0, 0, z)$ and $z \in [0, 1]$ with 0 and 1 identified. Then $(U, \xi|U)$ is contactomorphic to $(U, ker(sin(2n\pi z)dx + cos(2n\pi z)dy))$. [picture]

Transversally meeting convex surfaces on Legendrian boundary, then we actually see a local small neighbrhood of the Legendrian boundary. There's a collared neighborhood, one given by $y = 0$ and $0 \leq x \leq \epsilon$ and the other given by $x = 0$ and $0 \leq y \leq \epsilon$. The convex surface should have a dividing set with $2n$ arcs meeting the intersection.

A bypass is a technique to find another convex surface in a manifold. The blackboard is a given convex surface. There is a dividing set in it. If we can attach to it half of a convex disk with dividing set a single arc meeting the flat half, called the attachment arc. It should meet Γ_Σ at three points. [missed some, pictures].

I'll use the bypass technique to look at $T^2 \times [0, 1]$ with convex boundary. We start from s_0, the slope of T_0 and s_1 the slope of T_1. I'll assume that s_0 and s_1 are different. Because of the same reason yesterday, these s_0 and s_1, the angle is decreased. There are three types. We can have minimal or nonminal twisting. If we fix the slope of the boundary, then the slop eof any fiber is in the middle, always we move the clockwise direction. If not, we call this non-minimal twisting. Minimal twisting is divided into the nonrotative and rotative cases. Nonrotative means that s_0 and s_1 are equal. Because of time we'll concentrate on proving the the rotative case.

By acting by $SL_2(\mathbb{Z})$, we can put s_0 as 0 and s_1 as -1. This is basically from layering of very simple structures.

Proposition 5.1. *The number of isotopy classes of minimal tight contact structures between* 0 *and* 1 *is* 2*.*

We call these basic slices. On T^3 the tight contact structures were classified by different means. So there is a twisting structure $\ker(\sin 2\pi dx + \cos 2\pi dy)$ Then ξ_1 and ξ_2. The upper bound comes from, well, we assume that the slope of T_0 is 0 and T_1 is -1 [pictures]. So we can choose our foliations as vertical lines and cut along them. The line is Legendrian. We can take the annulus $\gamma \times I$, [pictures].

6. June 5: Juhyun Lee, On the classification of tight contact structures III

Definition 6.1. We call ξ a minimal twisting if every boundary-parallel convex torus T has a slope s of Γ_T contained within $[s_0, s_1]$.

There were two questions about this definition. In the last talk I explained about the two basic classes. One is the kernel of $\sin 2\pi z dx + \cos 2\pi z dy$, with $z \in [0, \frac{1}{4}]$. The boundary slopes are 0 and -1. In the middle, every torus, the fiber becomes convex, actually. This can be seen by observing that $\langle \frac{\partial}{\partial z}, \cos 2\pi z \frac{\partial}{\partial x} - \sin 2\pi z \frac{\partial}{\partial y} \rangle$. In between the slopes are in between. Any other convex torus is C^0-approximated by these [pictures].

In the nonminimally twisted case, there is a convex torus with a slope s not contained between s_0 and s_1.

I have to give a rough idea for classifying the other boundary. Last talk I showed you the isotopy number of tight contact structure, the basic slices with boundary slopes 0 and -1, there are two of these. The actual picture for this, we realize these by embedding the standard contact structure on T^3 and taking $[0, \frac{1}{4}]$ and $[\frac{1}{2}, \frac{3}{4}$ is another basic slice.

[some pictures and discussion, mainly related to the Farney tesselation.]

So known results include Σ_g-bundles over S^1 with some monodromy ($g > 1$). Then it's also known that in the toroidal case there are infinitely many tight structures. The atoroidal case was known to be finite and there were conditions on the Thurston–Bennequin inequality that guaranteed a certain count. I wanted to remove these. I did it in the case where there is a convex fiber with $\#\Gamma_\Sigma$ is 1 or 2.

A longstanding problem is the existence problem. If the Betti number is greater than 0, the tight contact structure can be obtained by perturbing something. But a rational homology sphere which is atoroidal and not Seifert fibered, we don't know.

7. September 9, 2015: Dohyeong Kim, Moduli of certain $K3$ surfaces via GIT I

I will not talk about moduli of $K3$ until Friday. Today I'll talk about my motivation. In the second talk I'll talk about moduli spaces of certain $K3$ surfaces with Picard number 18. In the third talk I'll talk about those with Picard number (possibly) 16. I'm not sure. This is work in progress so some of these I only have a sketch for.

This is roughly the plan for the first, second, and third lecture.

Let me explain the diophantine equations that led me to the geometry of $K3$ surfaces. This is elementary but often causes some confusion, so let me be precise. I'll work with a ring R, which is a $\mathbb{Z}$-algebra which is finite (dimension over $\mathbb{Q}$ after tensoring with $\mathbb{Q}$ is finite) and flat. Typically I think of $\mathbb{Z}[\frac{1}{p_1}, \ldots, \frac{1}{p_r}]$, where p_i are primes or $R \subset F$ is a finite field extension.

So projective and affine equations, given $f_1, \ldots, f_n$ in $R[x_1, \ldots, x_m]$, with f_i a homogeneous degree $d_i > 1$ polynomial, then $f_i = 0$ defines a variety $V(\underline{f})$ in $\mathbb{P}^{m-1}$ and I want to consider its complement, which is affine if $n = 1$. with equation "$f_i t_i = 1$" heuristically. So for instance if $n = 1$ and $f_1 = x_m$ then the complement is $\mathbb{A}^{m-1}$, with coordinates $\frac{x_i}{x_m}$.

When $n = 1$ and f_1 is generic of degree ≥ 2, then $\mathbb{P}^{m-1} - V(\underline{f})$ is affine. You can prove this, say, by Veronese embedding. This becomes a hyperplane section in a bigger projective variety. But it doesn't have "nice" affine coordinates.

So $Y = \mathbb{P}^{m-1} - V(\underline{f})$, I want to point out that even though this is affine it's profitable to keep the projective coordinate system. So $Y = \{(a_1, \ldots, a_m) \in R^m | \text{for some} i, f_i(a_1, \ldots, a_m) \in R^\times\}/\sim$ where $(a_1, \ldots, a_m) \sim (\lambda a_1, \ldots, \lambda a_m)$ for $\lambda \in R^\times$.

This is a point of confusion. This is NOT that $f_i(a_1, \ldots, a_m) \neq 0$ as an element of R. People think of this automatically but this is not correct.

I'll explain why this is correct by giving an example. Look at $\mathbb{P}^1$ with x_1 and x_2, let $f(x_1, x_2) = x_1(x_1 - x_2)x_2$. Here $Y = \mathbb{P}^1 - \{0, 1, \infty\}$. Then $Y(\mathbb{Z}) = \varnothing$. You might think that removing 3 points from infinitely many gives you infinitely many. But $Y(\mathbb{F}_2) = \varnothing$ where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ because $\mathbb{P}^1(\mathbb{F}_2) = \{0, 1, \infty\}$.

So a second explanation. Schemes over $\mathbb{Z}$ is a fibred (not in a technical sense) space over $Spec\mathbb{Z}$. So if you have an equation $Y/\mathbb{Z}$, a scheme, then one has this picture. $\operatorname{Spec}\mathbb{Z}$ is a line with primes 2, 3, 5, and so on. Over each of them you have Y_2, Y_3, Y_5, and so on. Here $Y_p = Y \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$. So $\mathbb{Z}$-points are sections. Then a point σ is a section. So if you have infinitely many sections and you removed three. But for some reason, if you only had three sections. It doesn't have to be three.

Suppose $Y_2 \to Spec\mathbb{F}_2$ has r sections and all of them come from global sections $\sigma_1, \dots, \sigma_r$. Then $Y - \cup \operatorname{Im} \sigma_i$ has no $\mathbb{Z}$-points.

If X is proper then $X(R)$ and $X(\text{Frac } R)$ coincide.

In the case $Y = \mathbb{P}^1 - V(f)$ and $V(f)$ has at least three distinct $\mathbb{C}$-valued points, then $Y(R)$ is finite (Siegel theorem). For our example, in $Y(\mathbb{Z}[\frac{1}{2}])$, this conisists of $2, -1, \frac{1}{2}$.

In terms of our previous description of $Y(R)$, we have $\{f(a,b) \in R^\times\}$ is finite. Or when $R = \mathbb{Z}[p_1^{-1}, \dots, p_r^{-1}]$, then $R^\times = \{\pm 1\} \times \mathbb{Z}^r$, which is $\{\pm \prod p_i^{e_i} | e_i \in \mathbb{Z}\}$, then

$$\{(a,b) \in R | f(a,b) = \pm p_1^{e_1} \cdots p_r^{e_r}\} / \sim$$

is finite. This is called a "polynomial exponential equation."

What happens in higher dimensions? In terms of the last description, in concrete terms, we want to solve (solve may mean many different things)

$$\{(x_1, x_2, x_3) \in R^3 | f(x_1, x_2, x_3) = \pm p_1^{e_1} \cdots p_r^{e_r}\} / \sim$$

for the same $R = \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_r}]$. But there are some problems.

We need to find analogues for the statement that $f(x_1, x_2) = 0$ has finitely many $\mathbb{C}$-valued points and for "$Y(R)$ is finite." This is answered, people wanted to generalize Siegel's theorem in the 80s and got different formulations, so answered by Lang, Vojta, Bombieri–Lang. The answer is roughly as follows.

$Y(R)$ can be infinite (this is imprecise now) only for copies of $\mathbb{G}_m$, a multiplicative group in Y. If $\mathbb{G}_m$ injects into F then $\mathbb{G}_m(R) = \infty$ implies $Y(R) = \infty$. This is the only reason that finiteness can fail. One should be slightly careful, working with affine Y. In general you should replace this with an arbitrary algebraic group.

More precisely,

Conjecture 7.1. Let $Y_0 \subset Y$ be

$$\left(Y_0 = \bigcup_{\sigma : \mathbb{G}_m \hookrightarrow Y} \operatorname{Im}(\sigma)\right)^-$$

(the closure), then $Y(R) - Y(R_0)$ is finite.

I spent already 57 minutes but that's not so bad, I wanted to make sure that everyone understands clearly.

So $f = x_1 x_2 x_3 (x_1 + x_2 - x_3)$. You have the projective plane $\mathbb{P}^2$ and the coordinate lines x_1 and x_2 and a line at ∞ defined by $x_3 = 0$. If you let $x_3 = 1$ then you have the line $x_1 + x_2 = 1$ and these other lines [picture] are copies of $\mathbb{P}^1$ minus two points ($\mathbb{G}_m$ lying inside). For $R = \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_r}]$, you want $(a,b,c) \in R$ such that $abc(a+b-c) \in \mathbb{R}^\times$, in or]ther words $a, b, c \in R^\times$ such that $a + b - c \in R^\times$. So the set of triples of units such that $a + b \neq 0$, $a - c \neq 0$, and $b - c \neq 0$ and $a + b - c \in R^\times$ is a unit.

This is another illustration that $Y(R) - Y_0(R)$ is not the same as $(Y - Y_0)(R)$. I'll stop.

8. September 11, 2015: Dohyeong Kim, Moduli of certain $K3$ surfaces via GIT II

Remember that we were talking about $R = \mathbb{Z}[\frac{1}{p_1}, \dots, \frac{1}{p_r}]$, and Y/R is an affine variety

$$Y_0 = \left(\bigcup_{\sigma : \mathbb{G}_m \hookrightarrow Y} \operatorname{Im}(\sigma)\right)^-$$

The conjecture is

Conjecture 8.1.

$$\#(Y(R) - Y_0(R)) < \infty$$

This is a conjecture of a lot of people, Bombieri–Lang, Vojta, Lang.

I spent a long time talking about how $Y(R) - Y_0(R)$ is not the same as $(Y - Y_0)(R)$.

Now let's talk about moduli of $K3$ surfaces. Let $\mathbf{k}$ be a field. A proper projective surface $X/\mathbf{k}$ is called a $K3$ surface if

(1) $\Omega^2_{X/\mathbf{k}} = \mathcal{O}_X$, that is, it has a nowhere vanishing 2-form.
(2) $H^1(X, \mathcal{O}_X) = 0$.

This definition works for any field.

The moduli of complex analytic $K3$ surfaces is dimension 20. The moduli of algebraic $K3$ surfaces is a countable union of 19-dimensional components. When I say the moduli space of $K3$-surfaces by GIT, I mean a description of one of these components in terms of geometric invariant theory.

A *polarized* $K3$ surface over $\mathbf{k}$ is a pair $(X, \mathcal{L})$ where $X/\mathbf{k}$ is $K3$ and $\mathcal{L}$ is an ample line bundle.

If $\mathcal{L}$ is an ample line bundle on X, then the self-intersection number $\mathcal{L}^2$ is $2d$ for $d \in \mathbb{Z} > 0$. So $\mathcal{M}_{2d}$ is the moduli of $(X, \mathcal{L})$ with $\mathcal{L}^2 = 2d$. For each d this is a 19 dimensional, probably, algebraic stack.

I'll give very classic examples to describe $\mathcal{M}_{2d}$ for small d in terms of geometric invariant theory.

So suppose you have $\mathbb{P}^2$ and a curve of degree 6, smooth, let's say D is defined by the vanishing of f. Then this defines a surface in weighted projective space $\{w^2 = f \subset \mathbb{P}^3(1,1,1,3)\}$. A nice thing is that the canonical class K_X is trivial and thus, switching to, well, we see $(K_X - D) \cdot D = \chi(D)$ (Noether's formula) so $D^2 = -\chi(D)$ if D is a smooth curve.

Let's use this to compute a natural polarization here. Call this surface X_f. Then $\pi : X_f \to \mathbb{P}^2$, and π^{-1} of a hyperplane section is a curve of genus two. A hyperplane has 6 intersection points, so the double cover has genus two. Then $D \coloneqq \pi^{-1}(H)$ and $D^2 = -\chi(D) = 2g - 2 = 2$.

Then say (X, D_f) is in $\mathcal{M}_2$. This pair defines a $K3$ surface with degree 2 polarization. The space of degree 6 polynomials in $\{x, y, z\}$ has dimension $\left(\left(\begin{array}{c}3\\6\end{array}\right)\right)$, you can choos three times with repetition, this is $\binom{3+6-1}{6} = \binom{8}{6} = 28$. The dimension of GL_3 is 9, so this has dimension $28 - 9 = 19$. I haven't checked whether this is an isomorphism or something finite, but you have this map $P_6(x,y,z)//GL_3$.

Smooth quartics in $\mathbb{P}^3$ are examples of $K3$ surfaces, with $X_f \subset \mathbb{P}^3$, if you take a hyperplane section then the genus is 3 because this is a quartic curve. Then X_f, D is in $\mathcal{M}_4$. If you count the dimension it's also 19-dimensional. There are more examples.

This is a slight restatement of the original theorem.

Theorem 8.1. *(Y. André) Let $Y = \mathcal{M}_{2d}$. Then $\mathcal{M}_{2d}(R)$ is finite.*

Any map $\mathcal{G}_m \to \mathcal{M}_{2d}$ is constant.

Let $X/\mathbb{Q}$ be a $K3$ surface. We say that X has a good reduction at a prime p if X has a model $\mathcal{X}$ of $\mathbb{Z}_p = \lim_{\overleftarrow{n}} \mathbb{Z}/p^n\mathbb{Z}$, the ring of p-adic integers, sech that $\mathcal{X} \otimes_{\mathbb{Z}_p} \mathbb{F}_p/\mathbb{F}_p$ is a smooth $\mathbb{F}_p$-variety.

So $\mathcal{X}/\mathbb{Z}_p$ is a model for $\mathcal{X}/\mathbb{Q}$ if $\mathcal{X} \times_{\operatorname{Spec}\mathbb{Z}_p} \mathbb{Q}_p \cong X \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\mathbb{Q}_p$. A nice category of models needs you to fix an isomorphism but it doesn't matter at this point.

Theorem 8.2. *(Y. André, reformulation) The set of (X, D) over $\mathbb{Q}$ such that X has a good reduction at all p outside of $\{p_1, \dots, p_r\}$, up to isomorphism, for some fixed degree d, is finite.*

So this is isomorphism classes of degree $2d$ $K3$ surfaces. If you have good reductions at every prime, then this is finite.

So can we reduce the conjecture to this version of the theorem? We need to prove some set is finite. We know this finiteness theorem. Naively you can ask whether one finiteness can be reduced to the other. Yes, this is possible if $Y = \mathcal{M}_{2d}$.

I don't know where this is written, that $\mathbb{G}_m \to \mathcal{M}_{2d}$ is finite. The first has $\mathbb{C}$ as universal cover. If you can prove that $\mathcal{M}_{2d}$ has a period domain with hyperbolic metric, well, some certain hyperbolicity, then there is only a constant map from $\mathbb{C}$. You can check this using purely analytic differential geometry methods. Then we get the conjecture.

Let me go back to the example $Y = \mathbb{P}^2 - \{xyz(x+y+z) = 0\}$. [picture]

So $Y(R)$ can be described as (a, b, c) in R with $abc(a+b+c) \in \mathbb{R}^\times$ or as (a, b, c) in $R^\times$ with $a+b+c$ in $R^\times$.

Suppose we have the following construction, a hypothetical construction which I'll give next time, for any t in $(Y - Y)(\mathbb{Q}) = Y(\mathbb{Q}) - Y_0(\mathbb{Q})$ (these agree because $\mathbb{Q}$ is a field) we have a $K3$ surface (with some data) X_t such that $X_t/\mathbb{Q}$ has a good reduction at a prime p whenever $t \pmod p$ does not belong to $\{xyz(x+y+z) = 0\}$ and a fixed $X/\mathbb{Q}$ appears only finitely many times as X_t. Then the conjecture holds for Y.

That is, if a finite map $f : Y - Y_0 \to \mathcal{M}_*$ such that $f(t) = X_t$ has good reduction whenever $t \mod p$ is in $Y(\mathbb{F}_p)$.

9. September 15, 2015: Dohyeong Kim, Moduli of certain $K3$ surfaces via GIT III

I'll begin my last lecture. Today I'll give a construction of a certain $K3$ surface and compute its Picard group. In fact, I'll compute the Picard group with its intersection pairing. This is today's goal. I'll try to finish this. So my Y was $\mathbb{P}^2 - \{xyz(x+y+z) = 0\}$, the complement of this hypersurface, but I view $Y \ni y$ as a 5-tuple $(\ell_1, \dots, \ell_4, t)$ of four lines and a point. Taking the projective dual you get $(\ell_1^*, \dots, \ell_4^*, t^*)$, which is four points and a line.

I'll tell you how to construct a $K3$ surface. This can be described in maybe three ways. I think the most efficient one is the following. Given a line. Maybe this is not the best way to draw. [picture]

There are two conics, B_1 and B_2, passing through the four points and tangent to the line. The conic is smooth if $y \notin \{(x+y)(y+z)(z+x) = 0\}$.

Let me draw this picture again. We had three lines in the projective plane, and there were three copies of $\mathbb{G}_m$ defined by $(x+y)(y+z)(z+x)$. If t is elsewhere, then these two conics are smooth.

So first blow up $\mathbb{P}^2$ at four points, then define a 4-to one cover X_0 branched along $B_1 \cup B_2$. After blowing up, the intersections become disjoint. Then you have exceptional divisors E_i and you have $\tilde{E}_i$, inverse images in X_0, these are four to one covers of the projective line branched at two points, with ramification index 4; they're irreducible and disjoint. So you take X, branched along these $\tilde{E}_i$, this is an 8-fold cover of the blowup with specified branching.

The claim is that X is a $K3$ surface. Why? It has a natural map to $\mathbb{P}^1$. Then this passage to X_0 is a base change, you cover with $\mathbb{P}^1$ by a four to one map. Then the double cover is taken with respect to four distinct horizontal sections. You have a conic with four points, so if you take a branched cover with respect to these four points, you get a genus one curve. The singular fibers, if you count them, come from the degenerate conics. Then you get, eventually, taking a 4-fold cover, 12 special fibers each of which has two points. So you have 24 points, all of which are nodes, in an elliptically fibered surface, so it's $K3$.

A very general remark on computation of the Picard group. For $K3$ surfaces, a Picard group has no [unintelligible]component, and it can be identified with the singular homology group. You have a surface X and a finite map to smooth Y. This is how to produce nontrivial divisors of F. Suppose you have a divisor D in Y. Take the inverse image $\pi^{-1}(D)$. Try to generate $Pic(X)$ by choosing D with reducible $\pi^1(D)$. If it's reducible, meets the branch divisor in a special way, then this preimage may be useful.

[pictures]

Now let me claim that the subgroup of $Pic(X)$ generated by the components of $\pi^{-1}E, \pi^{-1}(\mathbb{S}_{ij})$, and $\pi^{-1}(E_j)$ has rank 18.

So let's write [pictures]

So these components generate a subgroup of rank 18 in $Pic(X)$. So we have $Y - \{(x+y)(y+z)(x+z) = 0\}$, so I write $\mathcal{X}$ over Y, for t we have $X(\vec{\ell}, t)$. And sitting in $\mathcal{X}$ is $\mathcal{D}_j$, we have a lot of divisors, $4 + 24 + 16 = 44$ divisors, but their intersection matrix does not depend on t. The image of the divisors in the Picard group of $\mathcal{X}$ is constant. This gives you lattice polarized $K3$ surfaces.

You can compose this classifying map with the locus of six points on t^*. Then given two conics you have only four bitangents. So $Y \to M$ factors through $\mathcal{M}_{0,4+2}$, the moduli space of $4+2$ points on a genus zero curve. You can show that $Y \to \mathcal{M}_{0,4+2}$ is finite so the other map $Y \to M$ is finite.

So $K3$ lattices are $U^{\oplus 2} \oplus (-E_8) \oplus (-E_8)$. But you need a sublattice generated by vectors of norm -2.

This is actually the end of lecture two. So far the GIT quotient was taken for $(\ell_1, \ldots, \ell_4, t)$ in $(\mathbb{P}^2)^4 \times \hat{\mathbb{P}}^2$ by PGL_3 acting on $\mathbb{P}^2$ (and its dual).

Now consider (C_1, C_2, t) where C_i are smooth conics. The moduli space of four lines has dimension 0. So it's the same as fixing four lines and varying t.

But two conics have moduli. Now there are a lot of copies of G_n. There are three families of copies of G_n like this. If you have a point outside of this, then you can do a similar construction, which is the following: [picture]