INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS SYMPLECTIC ALGEBRAIC TOPOLOGY OH YONG-GEUN

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1. September 10, 2013

Okay, I'm going to give a lecture in English. The grade will be based on pass or fail, but as long as you do a few homeworks, that will be fine.

I'm going to talk about some algebraic structures that appear in the topology of symplectic manifolds. That algebraic structure is quite complex, so it requires more complicated homological algebra than classical algebraic topology. I'm not going to talk very much about symplectic manifolds. I'll talk more about homological algebra of A_∞ structures.

Let me start by motivating where this kind of structure arises. The story begins with Stasheff or maybe some earlier work, but Stasheff is the one who introduced the precise concept.

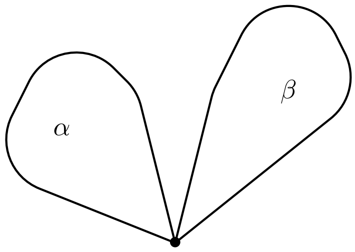
So this is Stasheff's A_n structures. In the first few lectures I'm going to explain what these A_n structures are about. Everything begins with a peculiar structure in the based loop space. Let's say X is a topological space and x_0 some base point. You're given some distinguished point. Let me denote, we have $\mathcal{L}X$, the free loop space, paths from the unit interval to X such that the initial and final point coincide:

$$\mathcal{L}(X) = \{\gamma : [0,1] \to X | \gamma(0) = \gamma(1)\}$$

Then I'll denote by $\Omega(X)$ the subset of the free loop space such that the loops start and end at the basepoint.

$$\Omega(X) = \{\gamma \in \mathcal{L}(X) | \gamma(0) = x_0\}$$

This is the "based loop space." It has a product. You can concatenate two loops.



The product $*: \Omega X \times \Omega X \to \Omega X$ works as follows:

$$\alpha * \beta = \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

This is not commutative. If you do this with the other order, you will not get the same loop.

If we have a third loop γ, we could compare $(\alpha*\beta)*\gamma$, that's one way, that gives

$$= \begin{cases} (\alpha*\beta)(2t) & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

and this unravels as

$$= \begin{cases} \alpha(4t) & 0 \le t \le \frac{1}{4} \\ \beta(4t-1) & \frac{1}{4} \le t \le \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

I divided the unit interval in half and then the left side in half again.

Now we consider $\alpha * (\beta * \gamma)$. We get

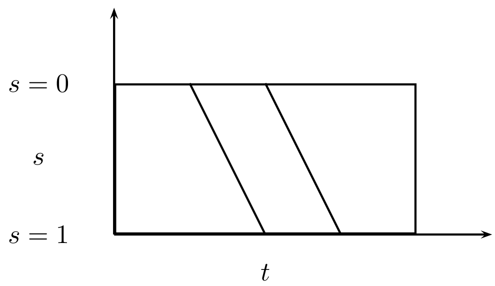
$$= \begin{cases} \alpha(2t) & 0 \le t \le \frac{1}{2} \\ \beta(4(t-\frac{1}{2})) & \frac{1}{2} \le t \le \frac{3}{4} \\ \gamma(4(t-\frac{3}{4})) & \frac{3}{4} \le t \le 1 \end{cases}$$

in general, $(\alpha*\beta)*\gamma \neq \alpha*(\beta*\gamma)$ as an element in ΩX but thy ar homotopic and the homotopy can be given by an explicit reparameterization. That is, there exists a homotopy

$$\Gamma : \underbrace{[0,1]}_{s} \times \underbrace{[0,1]}_{t} \to X$$

such that $\Gamma(0,\cdot) \equiv (\alpha*\beta)*\gamma$ and $\Gamma(1,\cdot) = \alpha*(\beta*\gamma)$.

So this is a path in the loop space, and the path is given by connecting the speeds



like this We have s=0 at the top.

Your first homework is to find the explicit formula for $\Gamma(s,\cdot)$. The middle interval of s is given by $\frac{1}{4}+\frac{s}{4} \le t \le \frac{1}{2}+\frac{s}{4}$, as a hint.

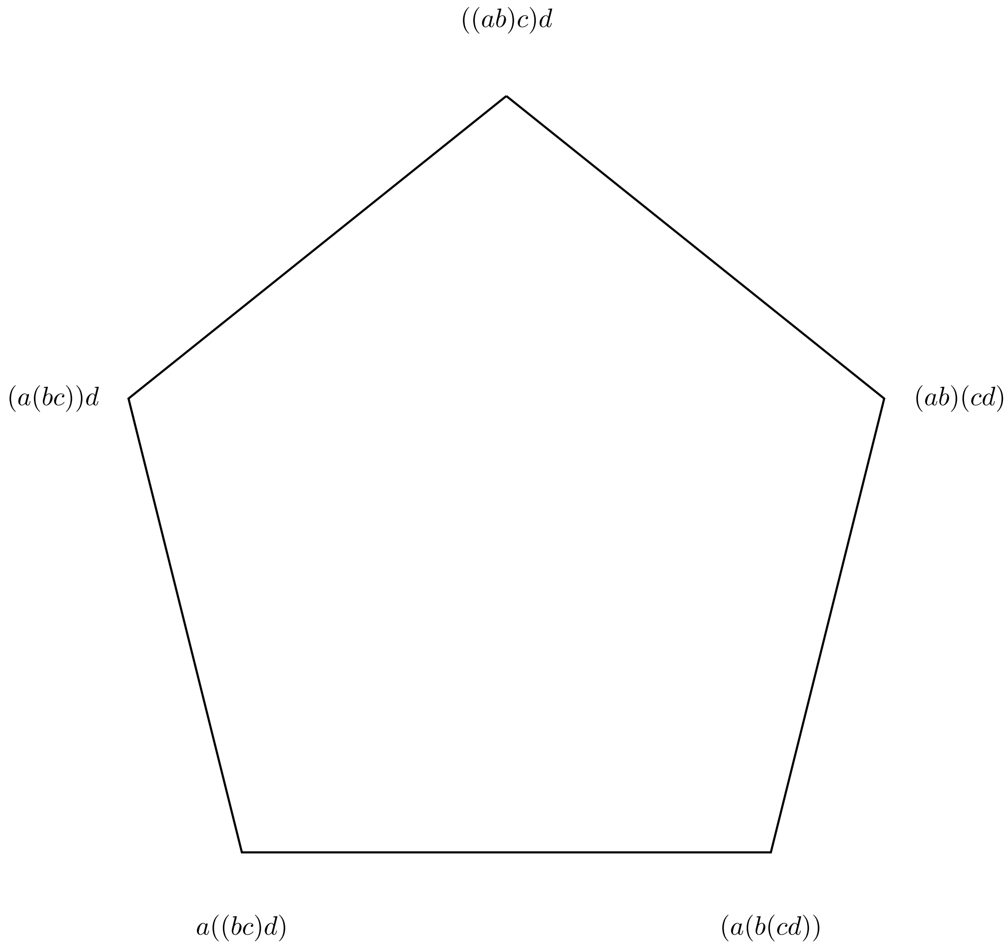
You can keep doing this with many loops and of course there are many different ways of taking the product.

This homotopy depends on α, β, and γ and the parameter s. We formalize the explicit definition of Γ, this homotopy, and define $m_3 : \Omega X \times \Omega X \times \Omega X \times [0,1] \to \Omega X$ by $m_3(\alpha,\beta,\gamma,s) = \Gamma(s,\cdot)$.

Similarly we can try to construct a homotopy of homotopies. Consider four loops α, β, γ, δ, and consider their quadruple product $\alpha * \beta * \gamma * \delta$, and there are many ways of doing this, to find a parameterized based loop, you need to use some parentheses, for example

(1) $((\alpha * \beta) * \gamma) * \delta$
(2) $(\alpha * (\beta * \gamma)) * \delta$
(3) $\alpha * ((\beta * \gamma) * \delta)$
(4) $\alpha * (\beta * (\gamma * \delta))$
(5) $(\alpha * \beta) * (\gamma * \delta)$

We can see that these each share one set of parentheses with the next one and we can draw the so-called pentagon. I'll put each of these quadruple products as a vertex of the pentagon.



On each edge we can put the explicit homotopy induced by Γ.

So this defines a loop in $\Omega(X)$. Each point on an edge defines an element in the loop space, and this loop is actually contractible in the loop space. Again, this can be done with an explicit contraction to the barycenter of the pentagon. I'll construct that later, but my claim is that we can construct an explicit piecewise linear construction.

This pentagon is an example of the Stasheff polytope K_4. So for example, K_2 is a point, there's only one way to multiply. K_3 is the unit interval, which is where we defined the homotopy. K_4 is the pentagon there, and generally, they keep going, and in general, K_n is an $(n-2)$-dimensional polytope which will be constructed inductively by some general properties.

Then, it's a fact that we have a continuous map $m_n : (\Omega X)^n \times K_n \to \Omega X$ that satisfies certain (explicit) properties sometimes called A_n properties. All of this will be explained later. Such a space is called an A_n-space.

Then Stasheff asked the following question. What conditions on a topological space Y makes it homotopy equivalent to ΩX for some topological space X. It has a remarkable answer. Stasheff found:

Theorem 1.1. *The answer to this question is exactly this A_n property. A topological space Y is homotopy equivalent to ΩX for some X if and only if Y is an A_∞-space (an A_n-space for all n).*

This structure appeared in symplectic geometry. Well.

You can now apply a cohomological functor. Recall the loop space product $*$. Although it is not associative on the space itself, but it induces a coproduct on cohomology, $H^*(\Omega X, \mathbb{R}) \to H^*(\Omega X \times \Omega X, \mathbb{R}) \cong H^*(\Omega X) \otimes H^*(\Omega X)$, so that loop product induced this homomorphism, and this is an example of a so-called coproduct. I'll explain what coproducts mean explicitly next time. That is coassociative, the dual notion of associative.

In other words, m_2 induced a natural coproduct $m_2 : H^*(\Omega X) \to H^*(\Omega X) \otimes H^*(\Omega X)$.

Maybe it would be better to work with homology, where we can get a natural product $m_2 : H_*(\Omega X) \otimes H_*(\Omega X)$. Let's think about the degree. Let's look at the induced homomorphism $(m_n)_\# : (H_*(\Omega X))^{\otimes n} \to H_*(\Omega X)$. Consider $p_1, \ldots, p_n$, singular chains in ΩX of dimension or degree $d_1, \ldots, d_n$. Denote $\alpha_i = [p_i]$. Then $(m_n)_\#(\alpha_1, \ldots, \alpha_n)$, to evaluate this you apply to representatives $m_n(p_1, \ldots, p_n)$, and you have an s which comes from K_n, and the dimension of K_n is $n-2$, and so this gives us a map from something of dimension, there is a resulting chain, that dimension is $d_1 + \cdots + d_n + n - 2$. Maybe it's easier to see this on chains. $(m_n)_\#(p_1, \ldots, p_n) = m_n(p_1, \ldots, p_n : K_n)$, so recall that m_n is a map from $(\Omega X)^n \times K_n \to \Omega X$. The image of this m_n will increase the dimension by $n-2$. This defines a chain of dimension $\sum d_j + (n-2)$. So this is sort of a chain map of degree $n-2$. This $(m_n)_\#$ is a map of degree $n-2$. So you regard, well, $C_*(\Omega X)$ as a graded vector space. So $C_*(\Omega X) = \bigoplus C(\Omega X)$. This map satisfies, the A_n relations induce certain relations on these. The second fact is that the $(m_n)_\#$ will satisfy certain quadratic relations called A_∞ relations.

From the A_∞ relations, we can derive many facts. It induces, for example, an associative product on $H_*(\Omega X)$. This is called an A_∞ algebra. The conclusion is that $(H_*(\Omega X))$ with m_n forms an A_∞ algebra.

Maybe I can write the quadratic relations. Next time maybe I'll explain all these unexplained terminologies. Maybe I'll finish today's lecture by saying one word, which is that such an A_∞ algebra also occurs in symplectic geometry for the Floer cohomology of Lagrangian submanifolds.

2. September 12, 2013

Missed

3. September 24, 2013

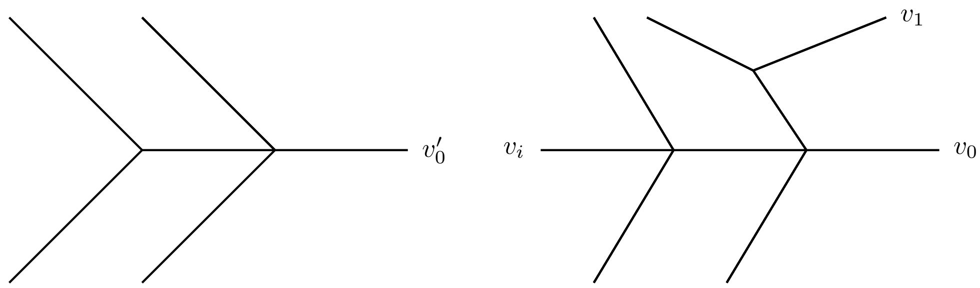
Maybe I should briefly remind you what we did last time. Recall that we talked about ribbon trees and also looked at rooted ribbon trees and actually the way how we defined ribbon trees is as embedded, you draw this tree but we sort of regard some extra edges as lying on the disk. We call this a ribbon tree because you construct this fictitious disk by thickening the tree. You are given the tree and you can thicken this by gluing several, so, the thickening of trees, what I mean by this, you look at the edges, instead of edges you replace them with bands, and then you are given bands like this [picture] and then what you can do is you identify the thin boundaries by taking the barycenter and then identify the half bands and do the same thing, and you will get some kind of space which is homeomorphic to the disk. By this thickening process, you give the flat metric, and since these are flat, you can identify them, and you will get some singularities, you won't have a flat metric at these barycenters. Then when you are given this metric, it canonically defines a conformal structure. The conformal structure uniquely extends over the isolated singular points. This is the way we realize this fictitious disk. That's why we call these things ribbon trees, these are the ribbons.

Anyhow, the rooted ribbon trees, you identify one distinguished vertex v_0 on the boundary, and all other exterior vertices are outgoing vertices and there is an automatic orientation on each edge toward v_0. We denoted the set of isotopy classes of stable (no vertices of valence two) ribbon trees (or rooted ribbon trees) by G_{n+1} (or $G_{n,1}$).

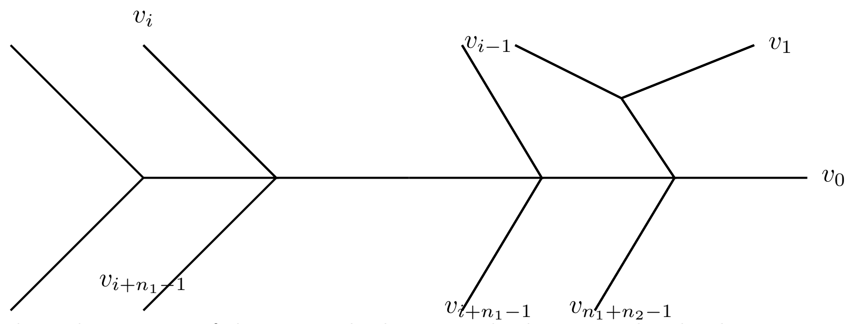
We gave a partial ordering, well, the ribbon tree is an embedding (T, i) of the tree T into the disk D^2 and $i^{-1}(\partial D^2)$ is the exterior vertices of T. In the rooted case we have $(T, v_0), i$. Denote by $[T, i]$ or $[(T, v_0), i]$ the isotopy class of (T, i) or $((T, v_0), i)$.

We denote for simplicity $\mathbf{t}$ as $[T, i]$ or $[(T, v_0), i]$. We say $\mathbf{t} < \mathbf{t}'$ if $\mathbf{t}'$ is obtained by collapsing a sequence of interior edges of $\mathbf{t}$. The maximal element, we call the unique maximal element a *corolla*. There are no interior edges. An n-corolla or $n+1$-corolla or $(n, 1)$-corolla has no interior edges.

Then we introduced the grafting operation, which I denote as, let $\mathbf{t}_1$ be $[(T, v_0), i]$ (or let me skip this i, assume it) and $\mathbf{t}_2 = [(T', v')]$. This operation $*_i : G_{(n_1,1)} \times G_{(n_2,1)} \to G_{n_1+n_2-1,1}$ occurs by the following picture (this is a little confusing)



and you glue these together with some renumbering:



We now go back to the axioms of the associahedron. We had two, so the third axiom is that K_n is a CW-complex. The set of cells in K_n is in one to one correspondence with $G_{(n,1)}$. Axiom four is as follows. Denote by $F(\mathbf{t})$ the face associated to $\mathbf{t} \in G_{(n,1)}$. Then $F(\mathbf{t})$ is an open cell of codimension (# of internal edges). The corolla is the generic cell, and if the tree has more interior edges, then the associated cell has higher codimension. Note that this codimension is the same as the number of parentheses.

[Darko Milinkovic: What are the faces of a CW complex? Maybe I'm missing axioms one and two.] K_n is an $n-2$ dimensional polyhedron. K_n has a cell decomposition where each cell is the cone on its boundary.

Axiom five is as follows, maybe this is a proposition, well, given the topology of K_n, we have the following: The closure $\overline{F(\mathbf{t'})}$ is $\bigcup_{\mathbf{t}\leq\mathbf{t'}} F(\mathbf{t})$ and the boundary is the same except with scrict inequality: $\partial\overline{F(\mathbf{t'})} = \bigcup_{\mathbf{t}<\mathbf{t'}} F(\mathbf{t})$.

There is one more axiom which I'll state later. But maybe I need to state axiom six to make this proposition true so I'll hold off on it.

Proposition 3.1. *K_n is homeomorphic to an $(n-2)$-cell.*

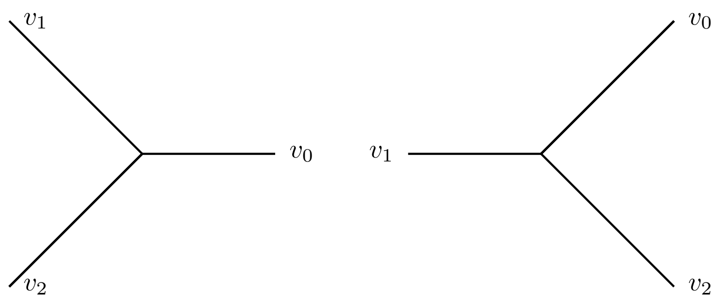
I'll give the proof of this proposition later, but I'll give some examples. K_2 is a point. There's only one isotopy class.

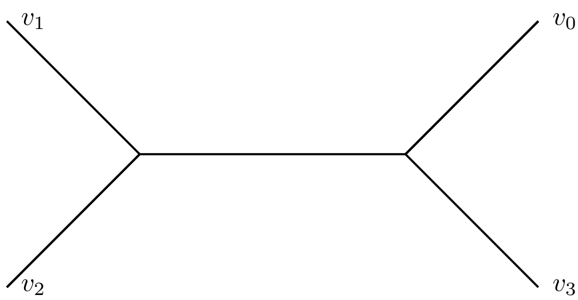
[Some discussion of K_1 and K_0] Let me give axiom six. There exist maps $\circ_i$: $K_r \times K_s \to \partial K_n \subset K_n$ where $n+1=r+s$. so that the image $\circ_i(K_r \times K_s)$ ofrms a facet of K_n.

The boundaries of K_n is the union over $2 \leq s \leq n-1$ and i from 1 to s of the closures of $\circ_i(K_(n+1-s) \times K_s)$. These are all possible ways of putting one parenthesis in a word of length n so that it's not the whole word.

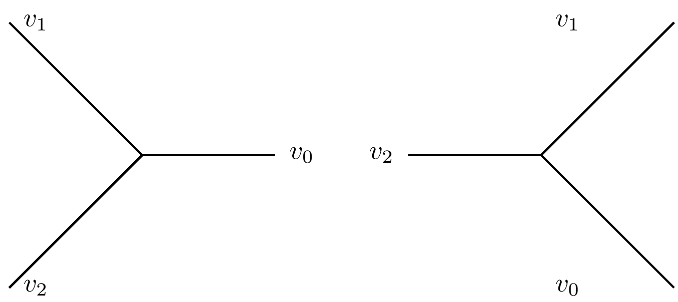
That's the axioms I'll use to describe these things.

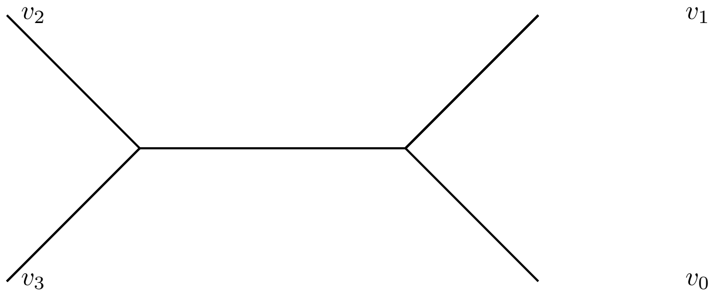
Let's go back and consider ∂K_3. How many boundaries? This is a zero dimensional set. This will be, $r+s=n+1=4$, and s must be 2. So there are two ways of putting one parentheses: $((ab)c)$ or $(a(bc))$. There should be two points. So let's see what $((ab)c)$ corresponds to in the graph. They should be $\circ_1(K_2 \times K_2)$ and $\circ_2(K_2 \times K_2)$.



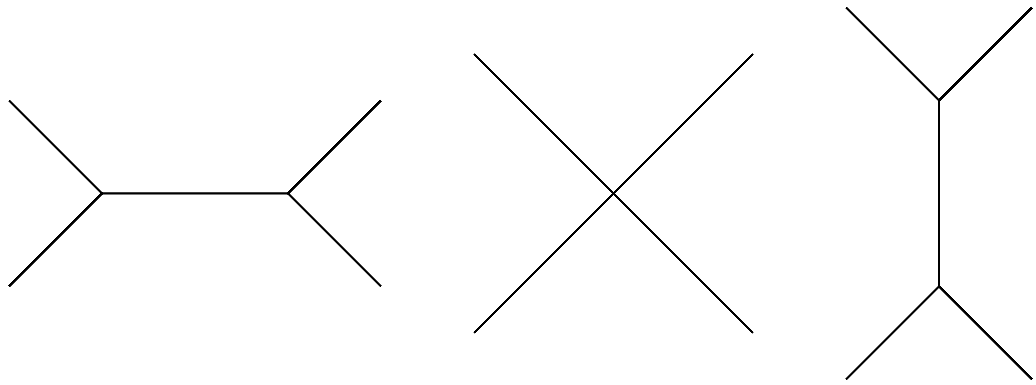


That's one. The other is:





You can think of this as one of these turning ot the other by passing through a corolla:



Now I've done something axiomatic and I want to turn instead to some geometric realizations of K_n. One could be metric ribbon trees. Let me set some notation. For a given $\mathbf{t} = (T, i) \in G_{n+1}$, we denote by $C^0(\mathbf{t})$ as the set of vertices, $C^1(\mathbf{t})$ as the set of edges, and $C^0_{int}(\mathbf{t})$ and $C^1_{int}(\mathbf{t})$ as the interior vertices and edges. Now for $\mathbf{t}$ in G_{n+1}, we associate the open cell

$$Gr(\mathbf{t}) = \{\ell : C'_{int}(\mathbf{t}) \to (0,\infty)\}$$

This is homeomorphic to $(0,\infty)^{\# C^1_{int}(\mathbf{t})} = (0,1)^{\# C^1_{int}(\mathbf{t})}$. Then we let $\overline{Gr(\mathbf{t})}$ to be the same thing including lengths zero and infinity:

$$\{\ell : C'_{int}(\mathbf{t}) \to [0,\infty]\}$$

This is homeomorphic to $[0,1]^{\# C^1_{int}(\mathbf{t})}$

We can make a grafting map for this. We provide a CW-structure on the union of the isotopy classes of $Gr(\mathbf{t})$. By the definition, $Gr_{(n,1)}$ is the resulting complex, which is homeomorphic to K_n.

Okay, for the second one, the moduli space $\overline{\mathcal{M}}^{main}_{n+1}(D^2)$. Consider $n+1$ distinct points on $S^1 = \partial D^2$. Denote this by

$$Conf_{n+1}(S^1) = \{(z_0,\ldots,z_n) \in (S^1)^{n+1} | z_0,\ldots,z_n \text{ are distinct and arranged counterclockwise}\}.$$

Then $PSL(2,\mathbb{R})$ acts on S^1. Recall that $SL(2,\mathbb{R})$ is two by two matrices of determinant one and $SL(2,\mathbb{R})$ is $PSL(2,\mathbb{R})$ modulo the relation that a matrix is equal to its negation.

Identify $D^2\backslash\{1\}$ with the upper half-plane, $z \in \mathbb{C}$ such that $Im\ z \geq 0$. These are equivalent via a conformal diffeomorphism.

Then $PSL_2(\mathbb{R})$ acts on the upper half-plane by $z \mapsto \frac{az+b}{cz+d}$ if the element of $PSL(2,\mathbb{R})$ is the class of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. This action acts the same by an element or its negation and preserves the upper half plane and the real line union $\{\infty\}$. For a given $(z_0,\ldots,z_n)$, we may assume $z_0 = \infty$. Then we arrange $z_1 < z_2 < \cdots$.

So you can identify $z_1,\ldots,z_n$ around the circle. We can realize, or identify, these points, in $Conf_{n+1}(S^1)$ as a subset, if you put z_0 at 1 you still have two dimensions. You can identify the point as a subset of n points in $(0,1)$. I'll make this more precise next time.

Definition 3.1. *We define* $\mathcal{M}_{n+1}(D^2) = Conf_{n+1}(S^1)/PSL(2,\mathbb{R})$ *where* $PSL(2,\mathbb{R})$ *acts on* $(z_0,\ldots,z_n)$ *by* $(Az_0,\ldots,Az_n)$.

We identify two points if all coordinates are moved by the same fractional linear transformation. This quotient space has a natural differential structure and the dimension is $(n+1) - 3 = n-2$. I'll explain the cell structure of this which tells you this is another geometric realization of the Stasheff polytope after you compactify, next time I'll talk about compactification.

4. September 26, 2013

Maybe just for those who are interested in looking at more details of what I'm talking about, here's a reference, the original paper of Stasheff. This is very explicit and does not use any machinery for graphs or moduli space. It's hard to motivate where these relations come from. This is the most basic reference: Homotopy Associativity of H-spaces, I and II, in the transactions of the AMS, 1963, volume 108. My presentation is more modern.

Axiom six is a little more involved, which I will explain using the realization of the Stasheff polytope. This is very much like the relationship between degeneracy and face maps for the simplex, so this is an analog for Stasheff polytopes.

There exist a family of maps called face operators. I already introduced these with another notation, but I want to make this more formal. This is $\partial_k(r,s)$, but I used $\circ_k$ before. I want to make every data clearly now. This is a map from $K_r \times K_s \to K_n$, and $r+s=n+1$. So s is the only free parameter. n is given so r is determined. You should think of K_n as being like a simplex. For this case, there is a map, the face map, in all dimensions. This is the analog of the inclusion of low dimensional faces into the n-simplex. Here s runs from 2 to $n-1$ and k ranges from 1 to s. This should properly be $K_s \times K_r$. Maybe the parentheses way is better:

$$a_1 \cdots a_{k-1}(a_k \cdots a_{k+r-1})a_{k+r} \cdots a_n$$

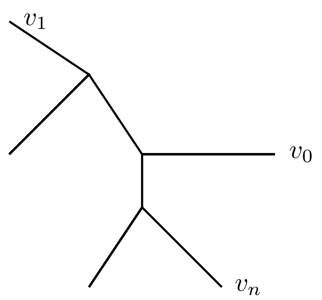
I insert K_r into the kth spot of K_s. This is the spot where you attach the first tree to the second tree.

There are also degeneracy maps $s_j : K_n \to K_{n-1}$. These range from j from 1 to n.

You have standard relations between the face maps and the degeneracies in the simplex. I'll write down the relations here but they're much more complicated.

i $s_j s_k = s_k s_{j+1}$ for $k \le j$ (I think this is the same as for the simplex).
iia $s_j \partial_k(r,s) = \partial_{k-1}(r-1,s) \circ (s_j \times id)$ for $j < k$ and $r > 2$. Similarly:
iia' $s_j \partial_k(r,s) = \partial_k(r,s-1) \circ (1 \times s_{j-k+1}$ for $s > 2$ and $k \le j \le k+s$.
iib $s_j \partial_j(n-1,2) = s_{j+1}\partial_j(n-1,2) = \pi_j$ for $1 < j < n$;
iib' $s_1 \partial_2(2,n-1) = \pi_2 = s_n \partial_1(2,n-1)$
iii $s_j \partial_k(r,s) = \partial_k(r-1,s)(s_{j-s+1} \times id)$ for $k+s \le j$

The geometric realization makes this very clear. The ribbon graph model makes this clear.



The degeneracy map is mapping from $Gr(n,1)$ to $Gr(n-1,1)$, just by collapsing the kth exterior edge followed by stabilization (deleting bivalent edges). The face map $\partial_k(r,s) : \overline{Gr}_{(r,1)} \times \overline{Gr}_{(s,1)} \to \overline{Gr}_{(n,1)}$, which is nothing but the grafting map, and the image lies in the boundary. Once you know those topologies, the relations I wrote down are an immediate consequence of the compactification.

I forgot to mention one remark. By definition, the face operator gives a boundary map and you can see the boundary is

$$\partial K_n = \bigcup_{s=2}^{n-1} \bigcup_{k=1}^{s} K_r \star_k K_s$$

where $K_r *_k K_s := \partial_k(r,s) K_r \times K_s$.

The number of faces is $\frac{n}{n-1}2-1$. You can count this. The similar realization of the face operator, well, that's even easier, remove the jth marked point. That's a little bit, maybe I'll say more about that model later.

Let's look at some special case. Well, maybe in a little bit. I'm ready to define an A_n-space. We say that (Y, m, e) satisfies the A_n relations [m is a product $Y \times Y \to Y$ and e a basepoint of Y] if there is a family of maps $M_i : K_i \times Y^i \to Y$ for $2 \leq i \leq n$ such that (I'll interchange Y^i and K_i factors when it's convenient for me)

(1) $M_2(e, y; *) = M_2(y, e; *) = y$ for $y \in Y, * \in K_2$. That is, $m = M_2$ makes (Y, m, e) an H-space.

(2) For $\rho \in K_r, \sigma \in K_s, r + s = i + 1$, we have

$$M_i(y_1, \dots, y_i; \partial_k(r, s)(\rho, \sigma)) = M_r(y_1, \dots, y_{k-1}, M_s(y_k, \dots, y_k{+}s{-}1; \sigma), y_{k+s}, \dots, y_i; \rho)$$

(3) $M_i(y_1, \dots, y_{j-1}, e, y_{j+1}, \dots, y_i; \tau) = M_{i-1}(y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_i; s_j(\tau))$

We call this collection $(Y, \{M_i\})$ an A_n-space. For homework (from Stasheff), check that this definition is consistent with the relations between $\partial_k(r, s)$ and s_j in Axiom six.

Let me explain what this relation actually means. The first A_2-space is nothing but an H-space. Why is this so? An H-space has M_2 which is a map $K_2 \times Y^2 \to Y$ but K_2 is a point, so if we set $m = M_2(*; \quad)$, this is a map $Y \times Y \to Y$, it defines a multiplication. The relation one basically says that $m(e, y) = m(y, e) = y$ so e plays the role of the identity for this multiplication map.

For general n, consider $i = 3$. For this, we know K_3 is homotopic to the unit interval, and M_3 is a map $K_3 \times Y^3 \to Y$. So this is a map $[0, 1] \times Y^3 \to Y$. The relations say that $M_3(y_1, y_2, y_3; \partial_k(2, 2)(\rho, \sigma))$ is just a point, this lies in the boundary of K_3 this is either $+$ or $-$, and so the second relation says that for $k = 1$, this is $M_2(M_2((y_1, y_2; \sigma), y_3; \rho)$ and for $k = 2$ it's $M_2(y_1, M_2(y_2, y_3, ; \sigma); \rho)$. So to simplify the notation here, we denote, well, we can omit ρ and σ since K_2 is a point. So $M_2(x, y; *)$ as xy. Then this becomes, the above formula becomes

$$= \begin{cases} (y_1 y_2) y_3 \\ y_1 (y_2 y_3) \end{cases}$$

This is the boundary so what this formula says is that this M_3 which is a map from $[0, 1] \times Y^3 \to Y$, obviously this is a homotopy. It's a homotopy between $M_3(\{0\} \times \bullet)$ and $M_3(\{1\} \times \bullet)$. These two maps are, respectively, $(y_1 y_2) y_3$ and $y_1 (y_2 y_3)$. So this M_3 provides a homotopy between these two different products. In other words, M_3 provides an associating homotopy between these two different triple multiplications.

Let's do one more, for $i = 4$. We know K_4 is a pentagon and there are five ways to put a product in four letters in parentheses. So we could have an edge that is a homotopy from $a(b(cd))$ to $a((bc)d)$ and the next goes to $(a(bc))d$. This is treating bc as one letter. The following one goes from that to $((ab)c)d)$, then $(ab)(cd)$, and then back to $a(b(cd))$. So there is an explicit map, restricted to the boundary we get this loop. So we have, concatenating this homotopy, we have a map $S^1 \times Y^4 \to Y$. That's provided by the M_3. Then the map M_4 extends this map which is already defined by M_3 along the boundaries and M_4 provides an explicit contraction of that map, extending it to $K_4 \times Y^4$. Of course, K_4 is homeomorphic to D^2. This is a homotopy of a homotopy, and the higher things can be regarded as homotopies of homotopies of homotopies.

There's a trivial way of constructing these A_n spaces. An H-space does not have to be associative. Any associative H-space admits an a trivial A_n structure for all

n by setting $M_i(y_1, \ldots, y_i; \sigma) = y_1 \cdots y_i$. You don't need any parentheses because the product is associative. The product comes from $m = M_2$. This satisfies, you can check, the A_n relations. This map does not depend on the K_n element.

Here is the motivating example, of based loop spaces. Let X be a topological space with x_0 a basepoint. Then $Y = \Omega X = \{\gamma : [0, 1] \to X | \gamma(0) = \gamma(1) = x_0\}$. Then ΩX is a (homotopy) H-space. Set the constant loop e and then consider the standard concatenation as the product given by

$$\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

Consider $i_L, i_R : Y \to Y \times Y$ defined by $i_L(Y) = (e, \gamma)$ and $i_R(\gamma) = (\gamma, e)$. We want to check that $m \circ i_L \sim id \sim m \circ i_R$ as a map $Y \to Y$. So $m \circ i_L(\gamma)(t)$ is $m(e, \gamma) = e * \gamma$ which is

$$\begin{cases} * & 0 \le t \le \frac{1}{2} \\ \gamma(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

You remain constant for time $\frac{1}{2}$ and then follow γ twice as fast. Similarly, $m \circ i_R(\gamma)(t)$ follows γ twice as fast and then remains constant for time from $\frac{1}{2}$ to 1:

$$\begin{cases} \gamma(2t) & 0 \le t \le \frac{1}{2} \\ * & \frac{1}{2} \le t \le 1 \end{cases}$$

We have an obvious homotopy that I don't want to write from $m \circ i_L$ to the identity given by changing the $\frac{1}{2}$ to an s:

$$H(s, \gamma)(t) = \begin{cases} * & 0 \le t \le \frac{1-s}{2} \\ \gamma(\frac{2}{1+s}t - \frac{1-s}{1+s}) & \frac{1-s}{2} \le t \le 1 \end{cases}$$

For the next couple of lectures I'll prove that (Y, m, e) is an A_∞ space. I'll relate this based loop space with another type of path space, $Z = \Theta(X)$ which is pairs $(r, \alpha) | r \ge 0, \alpha : [0, r] \to X$ with $\alpha(0) = x_0 = \alpha(r)$. This is an associative H-space. The multiplication map $\mu : Z \times Z \to Z$ is given by the following. The domain side adds and then we use concatenation. So

$$\mu((r, \alpha), (s, \beta)) = (r + s), ??$$

and I'll continue next time.

5. October 1

Let me remind you, let (Y, m, e) be an H-space. We say this satisfies the A_n relation if there exists a family of maps $M_i : K_i \times Y^i \to Y$ for $2 \le i \le n$ satisfying the axioms (1)–(3) which I do not want to repeat. We looked at some examples. We considered X a based topological space and denote $Y = \Omega X$ and $Z = \Theta(X)$. The lemma was

Lemma 5.1. *Z is an associative H-space.*

Then we have a natural map. Remember that $Z = \{(r, \alpha) | r \ge 0, \alpha : [0, r] \to X \text{ with } \alpha(0) = x_0 = \alpha_r\}$. The product is concatenation without reparameterization. We have a natural map $f : Y \to Z$ which is inclusion. We have a map $g : Z \to Y$ which is rescaling: $g(r, \alpha)(t) = \alpha(rt)$. These maps have the following properties. If you compose $g \circ f$ you get the identity. On the other hand, $f \circ g(r, \alpha) = f(\alpha(r \cdot)) = (1, \alpha(r \cdot))$

You can easily see that this is homotopic to the identity. What is the homotopy? Indeed, the homotopy is $h : [0,1] \times Z \to Z$ and is given by $h(s,(r,\alpha))(t) = ((1-s)r + s, \alpha((1-s)+sr)t)$.

Now $h(0,\cdot) = id_Z$ and $h(1,\cdot) = f \circ g$. Noting this, I'm going to derive the following theorems in general.

Theorem 5.1. $(\Omega X, m, e)$ *(e is the constant path and m the concatenation product) is an* A_∞ *space.*

This is an immediate corollary of the following theorem:

Theorem 5.2. *Suppose* Y *is a topological space and let* Z *be an associative* H*-space with multiplication* $\mu : Z \times Z \to Z$*. Suppose you are given maps* $f : Y \to Z$ *and* $g : Z \to Y$ *such that* $f \circ g$ *is homotopic to* id_Z*. Define a multiplication* $m : Y \times Y$ *by* $m(y_1, y_2) = g(\mu(f(y_1), f(y_2)))$*. Then* (Y, m, e) *becomes an* A_∞*-space, where* $e = g(e_Z)$ *and* e_Z *is the unit in* Z*.*

We are given a multiplication on Z and want to define one on Y, so we do it like this:

$$\begin{tikzcd} Z \times Z \arrow[r, "\mu"] & Z \arrow[d, "g"] \\ Y \times Y \arrow[u, "f\times f"] \arrow[r, "m"'] & Y \end{tikzcd} .$$

Proof. Denote by μ_k the k multiplication map since μ is associative: $\mu_k : Z \times \cdots \times Z \to Z$. Furthermore, for simplicity, we denote $z_1 \cdots z_k = \mu(z_1, \ldots, z_k)$. This will make it a little more intuitive. We'll define using μ_k, f, g, and the homotopy h, we define an $M_i : K_i \times Y^i \to Y$ inductively. I'll describe this precisely up to $i = 4$.

The M_2 case, we know K_2 is a point, we define $M_2 = m = g\mu_2(f \times f)$. In other words, $M_2(y_1, y_2) = g(f(y_1)f(y_2))$.

Second, let's see M_3. It's supposed to be $K_3 \times Y^3 \to Y$. We know K_3 is the unit interval $[0,1]$. What we'll do is define $M_3|_{\partial K_3 \times Y^3}$ and then extend to the interior. The inductive process you define first on the boundary and then extend it to the interior. You'll recall $\partial K_3 = K_2 *_1 K_2 \cup K_2 *_2 K_2$ where $K_2 *_i K_2$ is the image of $\partial_i(2,2)(K_2 \times K_2)$. This lies in the boundary of K_3. This is how the face map was defined.

We already defined M_2. There are two boundary points. So we define $M_3(x,y,z;0) = M_2(M_2(x,y),z)$. Similarly $M_3(x,y,z;1) = M_2(x, M_2(y,z))$. Now we choose and fix a homotopy h which I assume to exist from $f \circ g$ to Id_Z. This will let us interpolate these two maps. So h is a map from $f \circ g$ to id_Z, those are $h(0)$ and $h(1)$. So then we have a map $h(t)$ for time t. So now we interpolate the two maps. Recall how M_2 was defined. The first map is

$$g(\underbrace{f(g}_{f\circ g}(f(x)f(y)))f(z))$$

and the second is

$$g(f(x)\,\underbrace{f(g}_{f\circ g}(f(y)f(z)))).$$

Then we replace these with h. So consider first

$$s \mapsto g(h_s(f \cdot f) \cdot f).$$

At $s = 0$ this becomes $g(f \circ g(f \cdot f) \cdot f)$ which is the first boundary map. At $s = 1$ this becomes $g(f \cdot f \cdot f)$, which you should regard as the central point. Similarly, the homotopy

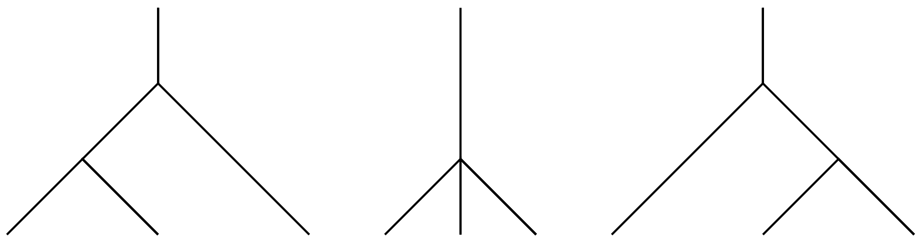
$$g(f \cdot h_s(f \cdot f))$$

connects $g(f \cdot f \circ g(f \cdot f))$ to $g(f \cdot f \cdot f)$. So by concatenating these homotopies you get a homotopy from one to the other.

Finally, well, denote these two homotopies as h_1 and h_2. So obviously, we define a concatenation homotopy h from $g(f \circ g(f \cdot f) \cdot f)$ to $g(f \cdot f \circ g(f \cdot f))$ by setting

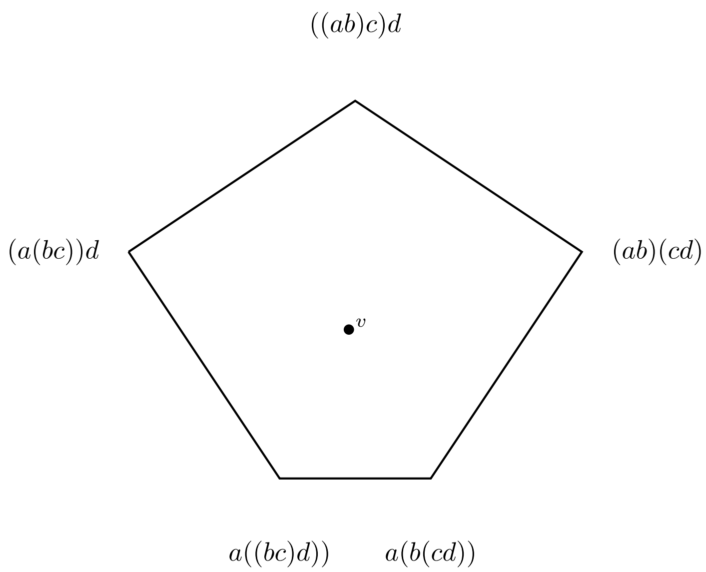
$$h(t) = \begin{cases} h_1(2t) & 0 \le t \le \frac{1}{2} \\ h_2(2 - 2t) & \frac{1}{2} \le t \le 1 \end{cases}$$

Pictorially, K_3 is $G_{r_{3,1}}$. You should think of this as looking like

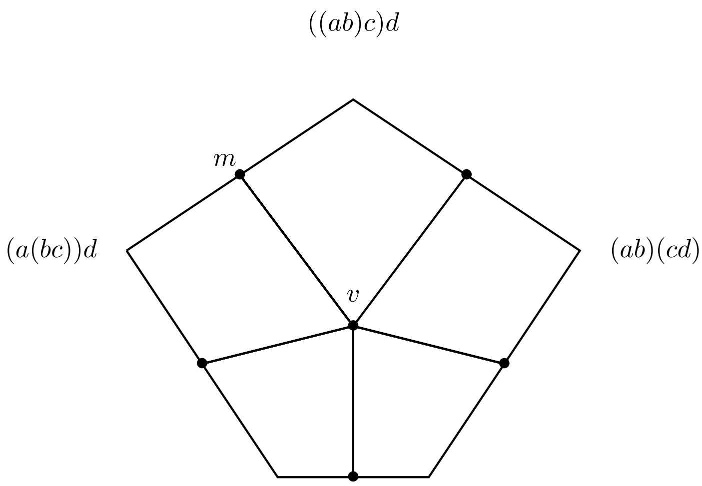


where you think of the two endpoints as being $K_2 *_1 K_2$ on the left and $K_2 *_2 K_2$ on the right, with the center point.

We can then look at one more, $M_4 : K_4 \times Y^4 \to Y$. Recall that K_4 is the pentagon



Here we first define $M_4|_{\partial K_4 \times Y^4}$ using the given homotopy h. We denote by m the midpoint of an edge and v the barycenter of the K_4.



We connect m to v and define $M_4(\cdot, \cdot, \cdot, \cdot, m)$, which is, say, at the midpoint between $((ab)c)d$ and $(a(bc))d$ equal to $g(f \circ g(f \cdot f \cdot f) \cdot f)$. We have a similar product explicitly given for each midpoint. We get rid of the $f \circ g$ and put $g(f \cdot f \cdot f \cdot f)$ at the center point v, which corresponds to the corolla graph. Now using the homotopy h from $f \circ g$ to id_Z, we define M_4 on the segment connecting each midpoint m to v. So for instance, $M_4|_{[m,v] \times Y^4}$ sends s to $g(h_s(f \cdot f \cdot f) \cdot f)$. In this way we have defined M_4 restricted to the line segments of this subdivision [ed: the pair subdivision]. Now for any other λv for $\lambda \in \partial K_4$, we interpolate linearly $M_4(\cdots ; \lambda)$ and $M_4(\cdots, v)$.

I'm using an axiom of a pentagon. K_i is the cone on its boundary. On each line segment I use the homotopy h to interpolate the two maps. The explicit formula is a little bit complicated but that's the idea. For higher M_i we just repeat this process inductively.

The way this was constructed, the A_n relations are automatic. These relations say what happened to the restriction of M_i to the boundary of K_i. But by construction this M_i satisfies the relations for any n. The A_n relation is nothing but the relation between, well, $M_i(y_1, \ldots, y_i; \partial_k(r,s)(\rho, \sigma)) = M_r(y_1, \ldots, y_{k-1}, M_s(y_k, \ldots, y_k + s - 1; \sigma), y_{k+s}, \ldots, y_i; \rho)$ for $r + s = i + 1$. But r and s are already defined. That's how our inductive construction was defined. □

Corollary 5.1. *Then applying the theorem to $Y = \Omega X, Z = \Theta X$, we have proved that ΩX is an A_∞ space.*

Not every space carries an A_∞ structure, only those homotopic to a based loop space do. Now applying the cohomology functor to this space we'll get an A_∞ algebra. Let's apply the cohomology functor to the A_n space $(Y, \{M_i\}_{i=1}^n)$ So $M_i : K \times Y^i \to Y$. Say the coefficients for cohomology is $\mathbb{R}$. Then $m_i = (M_i)^* : H^*(Y) \to H^*(K_i \times Y^i)$. Now K_i is contractible so this is canonically isomorphic to $H^{*-(i-2)}(Y)$ [some discussion, there's a comment that an explanation will be forthcoming] by integration over the fiber. This integration over the fiber can be defined. If Y is integrated over the fiber, well, we define a shifted complex, the cohomology ring is a graded Abelian group so denote $A^* = H^*(Y)[1]$ by defining $A^k = H^{k+1}(Y)$. This is the standard notation. Then define m'_k, well, I have a map $A^* \to H^{*+1}Y$, and well, I want to rewrite m_i as $m'_i : H^*(Y)[1] \to (H^*(Y)[1])^{\otimes i}$. The original m_i has degree $i-2$ but the degree shifting gives all m_i degree 1 for all i. In this business, the keeping track of signs is a complicated matter, very confusing, but if you shift the degree in this way, keeping track of the degree will be much easier because the sign rule will satisfy the Koszul sign. This degree shifting is very important.