INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS MIRROR SYMMETRY WORKING GROUP

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1. February 10: Calin Lazariou

You can call this a Chern-like presentation of the deformation theory of pairs (X, E), where X is a smooth complex manifold of dimension n and E a holomorphic vector bundle over X. The joint deformation theory was studied in many ways by many. Here are some references, not the original ones but some intermediate repackagings:

(1) L. Huang, On joint moduli spaces, Matt. Ann. 302 (1995) 1, 61–79
(2) a derived version, D. Huybrechts and R. P. Thomas, Deformation and obstruction theory for complexes via Atiyah and Kodaira-Spencer classes, Math. Ann. 346 (2010) 3, 545–569
(3) E. Martinegro, higher brackets on moduli spaces of vector bundles, Ph.D. thesis, Sapienza, Rome.
(4) K. Chan, Y-H. Suen, A Chern-Weil approach to deformations of pairs and applications, preprint

The aim is to fit deformations of (X, E) into Manetti's (Deligne-Kontsevich-Manin's) dgla approach.

You have a functor $(X, E) \to Def_{X,E}$, you take a flat family over a polydisk and a parametric E in any polydisk. You can read about the details in Manetti's introductory lectures. This is all à la Kodaira-Spencer. A deformation just means a flat proper morphism, just the algebro-geometric analog of a fiber bundle.

So you want to factor this assignment into $\Delta_{X,E}$, some sort of dgla, and then you want a natural equivalence to $Def_{\Delta_{X,E}}$. This idea is old because you could do this for X. There's some choice because you only want this up to natural transformation, you want the natural transformation to itself be natural in (X, E). You can read this in Manetti's lectures. Because of the freedom of a natural isomorphism, I have a choice in terms of $\Delta_{X,E}$.

There's a huge conjecture that you can do this for more or less any geometric object, you can start with Poisson manifolds, foliations, anything, and do this. There are a lot of examples.

Some motivation comes from Kodaira-Spencer theory. For X itself, you have $\Omega^{0,*}(TX), \bar\partial_{TX}, [\ ,\ ]$ where the bracket is induced by the bracket of sections of TX and the product in $\Omega^{0,*}(X)$. So $\alpha \otimes t, \alpha' \otimes t'] = (\alpha \wedge \alpha') \otimes (t \otimes t')$. Then the deformation functor is Maurer Cartan elements.

You have a similar story for deformations of E itself, keeping X fixed, and the dgla is $\Omega^{0,*}(End(E)), \bar\partial_E, [,]$, where $End(E)$ is a bundle of Lie algebras. Here again $[\alpha \otimes t, \alpha' \otimes t'] = (\alpha \wedge \alpha') \otimes [t, t']$. This is sometimes called the linearized Kodaira-Spencer dgla

So in both cases you can find these factorizations, you can realize this. Now how do you realize this for (X, E)?

To quote a famous physicist, this is well-known to those who know it. The literature isn't so clear, so that's why I gave you the references.

There is a realization that is relatively well-known (to Huybrechts and Thomas and maybe a few others). This is the Atiyah dgla. Let me describe how this works.

First of all, let's describe the Atiyah extension of TX by $End(E)$. There's a short exact sequence

$$0 \to End(E) \to \mathcal{D}(E) \xrightarrow{\sigma} TX \to 0$$

where $\mathcal{D}(E)$ is a particularly nice holomorphic vector bundle that has nice properties.

Definition 1.1. *The extension $\mathcal{D}(E)$ is the sheaf of locally defined holomorphic linear differential operators on E having scalar symbol and order at most one.*

Having an element of $\mathcal{D}(E)$ means that for $U \subset X$, I have an operator D_U, which should be $g_U + d_U$, split into the order zero and order one parts. Then g_U is a local holomorphic section over U of $End(E)$, so a map $\mathcal{O}(U, E) \to \mathcal{O}(U, E)$. The d_U part has an expansion $d_U^k \frac{\partial}{\partial x^k}$ where d_U^k is a holomorphic section of $End(E)$.

If you give me such an operator, it has a symbol $\sigma(D_U) = d_U^k \partial_k$. In principle, this is valued in $TX \otimes End(E)$. We say that $\mathcal{D}_U$ has *scalar principal symbol* if $d_U^k = \hat{d}_U^k \otimes id_{E|_U}$ for a holomorphic function $\hat{d}_U^k$. So then $\sigma(D_U) \in \mathcal{O}(U, TX)$.

Lemma 1.1.
(1) *Everything is well-defined. $\mathcal{D}(E)$ is a well-defined coherent, locally free sheaf so it is the sheaf of local holomorphic sections of a holomorphic vector bundle again $\mathcal{D}(E)$. This is trivial to prove, you have to prove the gluing condition. Coherence is quite trivial.*
(2) *The map σ is an epimorphism of sheaves whose kernel is $End(E)$. The kernel means you kill the degree one part d_U. To be a sheaf these should be compatible when you patch and so you get a global section of $End(E)$. Locally it's the sheaf of sections.*

So what's the dgla? There's a bracket on this.

Definition 1.2. *The* Atiyah dgla *is $\Omega^{0,*}(\mathcal{D}(E))$ which is $\Omega^{0,*}(X) \otimes \mathcal{D}(E)$ (here identifying $\mathcal{D}(E)$ with its sections). There's a natural $\bar{\partial}_E$ acting here. The bracket is given by, it's enough to define it, for $[\omega \otimes D, \omega' \otimes D']$ where D and D' are locally defined scalar symbol differential operators of order at most once, and ω, ω' are local sections, the bracket is*

$$\omega \wedge \omega' \otimes [D, D'] + \omega \wedge \mathcal{L}_{\sigma(D)} \omega' \otimes D - (-1)^{|\omega||\omega'|}(\omega' \wedge \mathcal{L}_{\sigma(D')} \omega \otimes D'$$

Martinegro's thesis says that

Theorem 1.1. *Atiyah's dgla models the joint deformation theory of (X, E) in the sense of Manetti.*

This is completely categorical. You never see smooth forms, you only see holomorphic or antiholomorphic forms. It's what an algebraic geometer wants to have. But it's not what a differential geometer wants, they want a Dolbeault realization to go back to smooth forms.

Let me make an observation, the *Atiyah class* of E is the extension class of the Atiyah extension in $Ext^1(TX, End(E))$. If you take a Dolbeault resolution, that's naturally isomorphic to $H^{1,1}(End(E)$.

Let's take an auxilliary metric, take the Chern connection, and its curvature. All of the construction will depend on this metric choice, but a change of metric induces an isomorphism. So this is defined up to canonical isomorphism. So let h be an arbitrary Hermitian metric on E and ∇ the Chern connection, the unique connection compatible with h so that $\nabla^{0,1} = \bar{\partial}_E$. We have $\nabla F_\nabla = 0$ for F_∇ the curvature of ∇. We have $F_\nabla \in H^{1,1}$.

It's simple to prove that this class coincides with the Atiyah class.

Lemma 1.2. *The class $[F_\nabla]$ coincides with the image of the Atiyah class in Dolbeaut cohomology.*

You now have this way to represent this sheafy thing in terms of a closed differential form. It mixes $0, 1$ and $1, 0$.

So now let's see what's the result.

Proposition 1.1. *There exists a well-defined isomorphism of smooth vector bundles phi_h depending on h from $\mathcal{D}(E)$ to $End_{smooth}(E) \oplus TX$, which I'll call $\mathcal{E}$. This is given by:*

$$D_U = g_U + d_U^k \otimes id_{E|_U} \mapsto g_U - d_U^k \otimes \partial_k \llcorner \bar{h}_U^{-1} \partial \bar{h}_U$$

where h_U is the matrix of h isn a local holomorphic frame of E above U.

[missed a bunch of discussion]

I end up with only some special sections that come from holomorphic sections, but if I replace $\mathcal{D}(E)$ with a smooth version I get everything.

Remark 1.1. *We can use ϕ_h to transport the holomorphic structure of $\mathcal{E}$ to a holomorphic structure $\bar{\partial}_{\mathcal{E}}^{(h)}$ on $\mathcal{E}$. Then ϕ_h becomes an isomorphism of holomorphic vector bundles.*

So also, you can use ϕ_h to transport the bracket, the Atiyah dgla to the bracket on $\Omega^{0,*}(\mathcal{E})$. Hence the image through ϕ_h of the Atiyah dgla is an h-dependent dgla of the form $\Omega^{0,*}(\mathcal{E}), \bar{\partial}_{\mathcal{E}}^{(h)}, [\ ,\ ]_h$.

Then the main result is

Theorem 1.2. *Chan-Suen,*

$\bar{\partial}\mathcal{E}$ has this form on $End(E) \oplus TX$, restricted to $\mathcal{E}$ it's $\mathcal{E} \to \Omega^{0,1}(\mathcal{E})$, where it's

$$\begin{pmatrix} \bar{\partial}_{End(E)} & \beta \\ 0 & \bar{\partial}_{TX} \end{pmatrix}$$

[had to leave because of time]

2. February 24

I want to sketch the homotopy-theoretic interpretation of (tree-level) Chern-Simons perturbation theory. That's already a much larger subject, so I want to focus on pieces. There are two versions of Chern Simons, two topological versions, introduced and studied in great detail by Witten, and there is a connection to open strings. I want in a particular case to focus on this. In general you end up looking at graphs with trivalent vertices, and you build out of them various diagrams. There are certain rules, called "Feynman rules" that associate to Γ some objects called “amplitudes.” What are these amplitudes mathematically? They are multilinear functionals on some space of fields which is an infinite dimensional space. Of course you run into functional-analytic difficulties. You need a nice completion of this space, you want these to be continuous. I want to suppress all this, kind of standard in physics.

If you look at these graphs they are one-dimensional CW-complexes, so they have a genus, which is called the loop order of the perturbation expansion.

If you look at $g = 0$ then you look at tree-level perturbation theory. In that case you end up with trivalent trees. You know from general results that this is related to A_∞ algebras. So tree level topological Chern-Simons has to do with A_∞ algebras. So make this precise.

There’s also a non-perturbative side of things which doesn’t go through trees like this. Physicists don’t have a complete theory or definition in the non-perturbative case.

For me the version I’m interested in is either deformations of holomorphic vector bundles (B-model) or deformations of flat vector bundles on special Lagrangians of some Calabi-Yau (A-model).

We’re really interested in pairs (X, E) where X is a Calabi-Yau and E is in some derived category.

You won’t just get an A_∞ algebra, you’ll get a strictly cyclic minimal A_∞ algebra (up to functional-analytic subtleties) describing the tree-level Chern-Simons perturbation theory (tree-level effective potential), the D-brane superpotential. As you will see, ensuring strict cyclicity is subtle.

In the simplest case, we’ll need a Calabi-Yau manifold and we’ll need the Calabi-Yau metric which in a certain sense breaks topological invariance.

Let me explain mathematically the framework. It’s a very simple algebraic framework. You can take this as a definition up to functional analytic subtleties of “cubic string field theory.” This is a certain formulation of the string field theory of open strings.

Take a strictly unital differential graded algebra $(\mathcal{H}, Q)$ where $|Q| = +1$, so $\mathbb{Z}$-graded. Also, this is strictly cyclic with respect to a non-degenerate chain-map pairing $\langle\ ,\ \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$. This comes from a trace $\langle u, v\rangle = tr(uv)$ for some nondegenerate $tr : \mathcal{H} \to \mathcal{C}$. You can give this a degree, I’ll do the simplest case where the degree is -3, which corresponds to a Calabi-Yau 3-fold.

Take this as a definition, an abstract definition of string field theory á la Witten, in the case $|\mathrm{tr}| = -3$. This means $tr(uv) = 0$ unless $|u| + |v| = 3$.

So how to do perturbation theory? What’s the definition? I’ll behave as if $\mathcal{H}$ is finite dimensional for simplicity. It goes like this.

So what is gauge fixing? I’ll choose auxilliary data. I’ll take a Hermitian scalar product $\mathcal{H} \times \mathcal{H} \to \mathcal{C}$, antilinear in the first variable. Then let $Q^\dagger$ be the Hermitian conjugate of Q.

Define an operator $c : \mathcal{H} \to \mathcal{H}$ by $h(u, v) = \langle cu, v\rangle$. Then the degree of $|cu| = 3 - |u|$ and you find $\langle c^2 u, v\rangle = \langle u, c^2 v\rangle$.

All is fine when $c^2 = \mathrm{id}$. This cancels certain anomaly. You have to prove in your precise model that $c^2 = \mathrm{id}$. This can be proved in the cases that I’ll consider by using a Calabi-Yau metric. If you don’t have this condition some things will break.

How do we do perturbation theory under these assumptions, when $c^2 = \mathrm{id}$? You can define a propagator $H = [Q, Q^\dagger] = QQ^\dagger + Q^\dagger Q$, where $[,\ ,\ ]$ denotes the graded commutator. You can consider the kernel of Q, the kernel of $Q^\dagger \cong (\mathrm{im}Q)^\perp$. Define $K = \ker Q \cap \ker Q^\dagger = \ker H$. You have $\ker Q = K \oplus \mathrm{im}Q$, $KerQ^\dagger = K \oplus ImQ^\dagger$, and $\mathcal{H} = K \oplus \mathrm{im}Q \oplus \mathrm{im}Q^\dagger$. This provides a homological splitting of $\mathcal{H}, Q$. So define $\pi_Q = Q\frac{1}{H}Q^\dagger$ and $\pi_{Q^\dagger} = Q^\dagger \frac{1}{H} Q$ where $\frac{1}{H}$ is the inverse of H on $K^\perp$. Then if you define G as $\frac{1}{H}(\pi_Q + \pi_{Q^\dagger})$ is a Green operator, it gives $HG = 1 - P$, where $P = 1 - \pi_Q - \pi_{Q^\dagger}$ is the orthoprojector on K. Then you define the propagator as $Q^\dagger G = \frac{1}{H}Q^\dagger = \frac{1}{Q}\pi_Q$.

Now I'll associate a minimal model of my differential graded algebra using these things.

There's a recursive definition of the A_∞ products that you can write down, but applying the Kontsevich-Soibleman construction to $(\mathcal{H}, Q, \cdot)$ with trivalent trees gives a minimal A_∞ algebra which is homotopy equivalent to $(\mathcal{H}, Q)$. This is not surprising, but the proposition is that $c^2 = \mathrm{id}$ implies that the A_∞ algebra is strictly cyclic with respect to $\langle , \rangle$.

What do I mean by strictly cyclic? I mean that $\langle u_1, r_n(u_2, \ldots, u_{n+1})\rangle = (-1)^{n(|u_2|+1}\langle u_2, r_n(u_3, \ldots u_{n+1}, u_1)\rangle$.

Now we can define the effective potential.

$$W = \sum_{n\geq 2} \frac{1}{n+1}(-1)^{\frac{n(n+1)}{2}}\langle \phi, r_n(\phi^{\otimes n})\rangle$$

where $\phi \in \mathcal{H}^1$. The equation $\partial_\phi W$ is equivalent to the Maurer-Cartan equation

$$\sum_{n\geq 2}(-1)^{\frac{n(n+1)}{2}} r_n(\phi^{\otimes n}) = 0$$

This is very abstract, let me give some applications.

Let X be a smooth compact Calabi-Yau 3-fold. Let E be a holomorphic vector bundle over X. Take $\mathcal{H}$ as $\Omega^{0,*}(End(E))$, of course we should take an (L^2) completion.

The differential Q is $\bar{\partial}_E$. I pick a Hermitian metric. Let h be induced by any Hermitian metric on E. If I pick one, then $h(u,v) = \int_X vol^g(u,v)_E$.

We can define c by $\bar{*}_E U = \Omega \wedge cu$. I don't choose just any Kähler metric, I choose the Calabi-Yau metric. Then $c^2 = 1$ if g is the Calabi-Yau metric.

The trace of u is $\int_X \Omega \wedge tr_E u$.

3. March 10

4. Examples of higher order contributions in Chern Simons theory

This has been done in two ways, the B model and the A model. In my approach I'll work with a Calabi-Yau three-fold X with a graded (in the sense of Seidel) special Lagrangian L and E a graded flat vector bundle over L.

I want to wrap some A-type branes of different degree, lift the Maslow degree from $\mathbb{Z}/2$ to $\mathbb{Z}$. I put all these branes on top of each other. You want to do some Lagrangian intersection theory in the ordinary Fukaya category. I can do a little displacement and then I don't have to worry about multiple Lagrangian cycles. I can put a higher rank vector bundle over L. What I'm going to do is to put an arbitrary rank flat graded vector bundle, where flat means in the sense of a flat superconnection. As far as I know this hasn't been much studied. It's not obvious that you can do the displacement.

If $\mathcal{L}$ is a line bundle then $H^1(End(\mathcal{L})$ is in correspondence with infinitesimal deformations of L. So I can deform my line bundle and do ordinary intersection.

So I think that working in higher rank bundles is a richer theory. Because it is $\mathbb{Z}$-graded and not just $\mathbb{Z}/2$-graded it has coherent higher order corrections.

How do I construct such a theory? I take E a $\mathbb{Z}$-graded vector bundle, write it as $E = \bigoplus E_n$, and I consider a flat graded superconnection of total degree one ∇. Because of the off-diagonal conditions I'll have twisting where E_n maps to E_{n+1} and so on. This is a way to package the enhanced triangulated category of flat vector bundles over L.

You can think of such (E, ∇) pairs as providing a model for the ∞-Calabi-Yau 3-category of flat vector bundles over L.

So I'm going to let d be the differential twisted by ∇.

The objects are *classical* topological A-branes. This leads to a graded string field theory, a graded Chern-Simons theory on L. Three are two directions to go, the categorical and the one-object version (where you pick an object and look at its endomorphisms).

We only look at deformations of E with L fixed. That's similar to open string field theory. You can also do open-closed string field theory and consider deformations of the pair (L, E) inside X. If you do this, already you have a number of mathematical questions. What is the analogue of the Atiyah complex governing deformations of (L, E)? In the B-model I have joint deformations, this should be describable, what is the analog of the Atiyah differential graded Lie algebra?

What am I going to do? I take this bundle, I take $\mathcal{V} = \bigwedge^*(T^*L) \otimes End(E)$ where $End(E)$ has $Hom(E_m, E_n)$ in degree $n - m$. Then $\mathcal{V}$ has a grading combinining the degree in $End(E)$ and the rank in $\bigwedge T^*L$.

Then there's a product on sections of $\mathcal{V}$ and then $\mathcal{V}$ is $\mathbb{Z}$-graded. Then $\mathcal{H} \coloneqq C^\infty(L, \mathcal{V})$, well $(\mathcal{H}, \bullet, 1, d = d_\nabla)$ is a unital differential graded algebra.

In particular the associated differential graded Lie algebra governs deformations of E where L is fixed. That's why this is open string field theory. You'll get obstructions if you want to simultaneously deform L.

All of the framework I gave before is satisfied. We have a trace $\int_L str()$, where this really only acts on the $End(E)$ part. The super trace lands in $\bigwedge T^*L$.

Only with this I can define the toplogical Chern-Simons theory. You have an action functional $S : \mathcal{H}^1 \to \mathbb{C}$, so $\phi \in \mathcal{H}^1$ is called the string field and the functional is

$$S(\phi) = \int_L str[\frac{1}{2}\phi d\phi + \frac{1}{3}\phi^3].$$

This is graded and has a superconnection, so two twists. Note that this is $\mathbb{Z}$-graded.

Claim 4.1. *This is the correct theory in this case.*

I'm really interested in when X is compact. As far as I know, $\mathbb{Z}/2$-graded Chern-Simons theory is not useful.

[some discussion of the super-trace]

Okay, so it's a gauge theory, but the gauge transformations are of Kontsevich type, the gauge group is the group of units of $\mathcal{H}$ (of degree 0.) Then $\phi \mapsto \phi^g = g\phi g^{-1} + g d g^{-1}$. But the gauge group is very complicated. But $S(\phi)$ is invariant only if g belongs to the connected componentn of the identity.

Consider the 1-loop determinant, correcting the vacuum expectation. This is always ill-defined but it's well understood how to define it, so in zeta function regularization, where the volume of the kernel of the kinetic operator (the kernel of the Laplacian) is regularized by the resolvent method (this is always an eta function).

To compute this, you choose gauge fixing data, any metric on L and any Hermitian metric on E (because L is special Lagrangian).

The result is the following:

Theorem 4.1.

$$\tilde{Z}_{scl} = \frac{||\quad||^{H}_{\mathcal{L}}}{||\quad||_{RS}}$$

Where $||\quad||_{RS}$ is the graded [unintelligible] norm, defined by E, ∇. The theorem is that this is independent of the metrics and depends only on the topological class of E, ∇, only on the element in the triangualated category.

That's a generalization to the graded case. What does this give you for a Lens space? It's a new invariant. What is it?

5. March 17

I'll give more detail at Gabriel's request.

Let me give an example first, I was told that this gauge group is simple.

Already the group of automorphisms of a complex vector bundle of rank r over L a three-manifold, that is, sections from L to the automorphisms of E, this is a complicated group. It's infinite dimensional and depends on the topology of L. I was talking about gauge groups, which are generally considered complicated. This plays a crucial role in ordinary Chern-Simons theory.

You find some compact form of this infinite dimensional Lie group. It's hard to study stabilizers for flat conections on E.

A stabilizer of a flat connection depends in subtle ways on the choice of A. The problem of classifying these stabilizers is not solved. For rank one it's trivial. It will obviously depend on π_1 and you get pretty complicated things, because it's a global thing.

I have similar problems in graded Chern-Simons, if you think for a moment, there is a derived category of flat connections, $D^b(L)$, think of them as pairs (E, A), where E is a graded vector bundle and A is a flat superconnection of total degree 1. You can take this not just for dimension three.

This can be constructed directly as an enhanced triangulated category, analogous to $D^bCoh(X)$, the derived category of coherent sheaves. You start getting an insight from here that this will not be a trivial extension.

There's Grothiendieck–Riemann–Roch. You write it for morphisms of algebraic varieties $X \xrightarrow{\pi} Y$. There are pushforward, pullback, upper and lower shriek things associated to this.

Now the question is, what is the analogue of this in the flat world? There should be relations between $R\pi_*$, $L\pi^*$, $R\pi_!$, and $L\pi^!$. You get ordinary Riemann Roch for a point or something. The analogue is what Bismutt and Lott wanted to do. They have a fibration in the smooth category, where π is a fibration, locally a submersion with isolated points where it is not. You want this to be almost locally trivial. Then you can associate to this these derived categories $D^b_{flat}(X)$ and $D^b_{flat}(Y)$ and you have these π-associated functors. My claim, never proven, is that there should exist a version of relative Riemann–Roch in this setting.

My claim is that the work of Bismutt–Lott does a small part of this. This is a suggestion for why this might be interesting.

I want to talk again about graded Ray–Singer torsion. So what's ordinary Ray–Singer torsion? So L is an n-manifold, take it closed for simplicity. E is an ordinary vector bundle, flat, with a flat connection A. By using the spectral theory of the Laplacian $\Delta = d_A^\dagger d_A + d_A d_A^\dagger$ deppending on a choice of Hermitian metric on L. This operator can be completed to $\bar{\Delta}_A$ L^2 sections from L to $End(E)$. This has a nice spectrum. If you have a trivial bundle, then there's a beautiful relation between the geometry of the underlying manifold and the spectrum. Clearly somehow this spectrum reflects the topology of (L, E, A). So you can construct things that should be invariant under the choice of metric.

I chose a metric on E and a metric on L. So one such construction is $T(L, A)$, the Ray–Singer torsion, which they proved to be independent of the two metrics. More precisely the norm is independent.

In the case (this is general background) when you take E trivial and A trivial, this reduces to the ordinary Reidemeister torsion of L.

This can be formulated using the singular chain complex twisted by a non-Abelian local system of A.

Let me just give you an expression. What do I have in this paper?

(1) You can generalize the definition of $T(L, A)$ to the case when A is a graded superconnection of total degree 1 on E a graded vector bundle on L.

> **Observation 5.1.** *To define a Reidemeister versino of what I will do, you need to have a notion of holonomy of a graded superconnection. In 2007, Igusa explained how to do this using Chen's iterated integrals. Then there's a paper of Abbas that showed an interpretation in ∞-categories. Combining them you can get an interpretation in D^b_{flat} of what Igusa did.*

My definition of $T(L, A)$ is

$$\prod_{q=1-N}^{N+2} [det'^{reg} \Delta^{(q)}]^{\frac{(-1)^q q}{2}} = e^{-\frac{1}{2}\sum_{q=1-N}^{N+2} (-1)^q q \zeta'_{\Delta^{(q)}}(0)}$$

where N is the bound on the grading in my bundle.

Pick a metric g on L and a Hermitian metric g_E on E.

Then the degree of the Laplacian $d_A^\dagger d_A + d_A d_A^\dagger$ is zero, and thus it preserves $\Omega^q(L, End(E))$. So $\Delta^{(q)}$ is the restriction of the Laplacian to Ω^q.

In general, let A be any strictly positive (the spectrum is positive) self-adjoint operator on a seperable Hilbert space, densely defined, not bounded. Assume for simplicity that the spectrum of A is discrete.

Then $\zeta_A(z)$ is defined to be, for $Re\ z \gg 1$,

$$\zeta_A(z) = \sum_{\lambda \in \sigma(A)} \frac{n_\lambda}{\lambda^z}$$

where n_λ is the multiplicity of λ. It's easy to prove that ζ_A has an analytic continuation to a meromorphic function defined on $\mathbb{C}$ with poles on the negative real axis. So $\zeta'(0)$ makes sense.

Perturbation theory asks to find the spectrum $\sigma(A = A_0 + K)$ in terms of the spectrum of A_0. So K is a differential operator of order 1. When K is compact, this is easy.

The regularized determinant is

$$det^{reg} A = e^{-\zeta'_A(0)}.$$

If A is semipositive, then we define $det'^{reg} A$ to be det^{reg} restricted to the complement of $(ker A)$. We have good closedness properties.

So then the result which is just based on Ray–Singer

Proposition 5.1. *There is a Ray–Singer norm* $|| \quad ||_{L,A}$*, which is essentially* $T(L, A)$ *improved by a factor which depends on the choice of an auxilliary metric on* $H^{(q)}_{d_A}(L, End(E))$ *is independent of* g *and* g_E*.*

So for example,

$$Z_{scl} = c \frac{|| \quad ||_{HS}}{|| \quad ||_{L,A}}$$

where the left hand side is the semiclassical partition function for graded Chern–Simons theory. The full partition function is $Z_{scl}(1 + o(\lambda))$. Witten did this in the ungraded case and asked what these new, higher Chern–Simons invariants are, they have some relationship to intersection theory in the flat local system determined by A.

You can play the same game here, and even these things, not so much is known about them.

All this generalizes to the graded case.

All of this was proved for arbitrary finite rank vector bundles. So the decomposition of E is finite and each constituent bundle is finite.

If you think of this sequence from $\mathbb{Z} \to \mathbb{N}$ we have finite support. You can certainly generalize this by improving the space of functions. Compactness will fail and you will have to be careful, the spectrum may not be countable, you need to define a zeta function as an integral with respect to the spectral measure. In general it's not a sum of point measures. In general you have for a positive operator that the measure has support in the positive reals.

$$\zeta_A(z) = \int d\mu_A \frac{1}{\lambda^z}.$$

You can still prove that this converges for $Re\ z \gg 1$ and this has an analytic continuation as before.

This takes care of everything. In general you have operator theory telling you that this is a continuous piece, an essential piece, plus a discrete piece.

6. March 24

Let's look at a simple example. Let L be a 3-manifold, compact, connected, smooth, everything smooth. Take E to be the sum of two guys $E_0 \oplus E_1$ and I'll build a superconnection from these. E_0 is a flat vector bundle of rank r_0 over L with specified flat connection d_0. And E_1 is the same, rank r_1 and flat connection d_1.

Let's choose a morphism $\varphi_0 : E_0 \to E_1$, a smooth section of the bundle of morphisms. I want to view these as concentrated in grading 0 and 1 respectively. The degree of φ_0 is then 1. I want to construct a superconnection using these two connections and φ. Let $d = d_\varphi = d_0 + d_1 + [\varphi,]$, where this is the graded commutator. So this is $\begin{pmatrix} d_0 & 0 \\ \varphi_0 & d_1 \end{pmatrix}$, well, let me call this d and it's a connection on E. It's a derivation of total degree 1.

It's easy to see that $d^2 = 0$ is equivalent to $d^{(0)}\varphi_0 + \varphi_0\varphi_0 = 0$ where $d^{(0)}$ is the diagonal action of d_0 and d_1. I guess I really mean $d_1 \circ \varphi_0 - \varphi \circ d_0$, the differential induced by the flat connections—

[some discussion; I stopped taking notes at this time.]