KNOTS GROUP LECTURE NOTES

GABRIEL C. DRUMMOND-COLE

1. Youngjin Bae: grid diagrams and knot Floer homology, October 15

Let me recall from last week, from a knot K in S^3 we can construct a grid diagram with grid index k. We associate a Heegard system associated to this grid diagram. This is H_Γ which is $(\mathbb{T}^2, \vec{\alpha}, \vec{\beta}, \vec{w}, \vec{z})$. We can associate to this $(\widehat{CF}(H_\Gamma), \partial)$ where the generators are $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ in $Sym^k \mathbb{T}^2$. If I take some points of my grid, those are the generators. There are $k!$ generators,

$$\partial \vec{x} = \sum_{\vec{y}} \sum_{\phi \in \pi_2(\vec{x}, \vec{y}), \mu(\phi)=1, n_{\vec{w}}(\phi)=0=n_{\vec{z}}(\phi)=0} \#(\mu(\phi)/\mathbb{R})\vec{y}$$

Recall that I defined some gradings on X.

(1) the relative M-grading, $rM : X \times X \to \mathbb{Z}$. This has $rM(\vec{x}, \vec{y}) = n_{\vec{x}}(D) + n_{\vec{y}}(D) - 2n_{\vec{w}}(D)$. I should let $\gamma_{\vec{x},\vec{y}}$ be an oriented nullhomologous curve in $\mathbb{T}^2$. Recall that my horizontal lines went from a point in $\vec{x}$ to a point in $\vec{y}$ and my vertical lines are the reverse. Then D is a two-chain whose boundary was $\gamma_{\vec{x},\vec{y}}$. Then $n_{\vec{x}}(D) = \sum_{x \in \vec{x}} n_x(D)$ where $n_x(D) = \frac{1}{4}$ if x is on the corner of D, $\frac{1}{2}$ if it's on the edge, $\frac{3}{4}$ if it is on an internal corner, and 1 if x is in the interior. Claim: rM is independent of the choice of $\gamma_{\vec{x},\vec{y}}$ and D. This is because a different D corresponds to adding and subtracting annuli which have $n_{\vec{x}}(A) = n_{\vec{y}}(A) = n_{\vec{w}}(A) = 1$. This all should be oriented and have signs.

Here $\pi_2(\vec{x}, \vec{y})$ is homotopy classes of maps $D^2 \to Sym^k \mathbb{T}^2$ which take a specified point to $\vec{x}$, another to $\vec{y}$, and the rest of the boundary into $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$.

[pictures]

I want to relate this grading to the symplectic grading. For $\vec{x}$, $\vec{y} \in X$ and $\phi \in \pi_2(\vec{x}, \vec{y})$, then $\mu(\phi) = \mu'(\phi) = n_{\vec{x}}(D(\phi)) + n_{\vec{y}}(D(\phi))$. This is the formal dimension of the moduli space of pseudo-holomorphic maps which represent ϕ. Then $\phi : \mathbb{D}^2 \to Sym^k \mathbb{T}^2$ are in one to one correspondence with k-fold branched covers of $\mathbb{D}^2$ mapping to $\mathbb{T}^2$.

To prove the lemma, the first step is that when $D(\phi)$ looks like just a rectangle with corners x and y, then $\mu'(\phi) = \frac{1}{2} + \frac{1}{2} = 1$. Then the complex to measure μ_ϕ, the moduli space of complex structures with four points on the boundary.

[Much discussion]

Step two, we already know that $\mu(\phi_1 * \phi_2) = \mu(\phi_1) + \mu(\phi_2)$ where $* : \pi_2(\vec{x}, \vec{y}) \times \pi_2(\vec{y}\vec{z}) \to \pi_2(\vec{y}, \vec{z})$ is concatenation. Then $\mu'(\phi_1 * \phi_2)$ can also be checked to satisfy the same equation, I'll leave it as an exercise.

Step three is that if I start with some $\vec{x}$ and $\vec{y}$ I can make a sequence of $\vec{x}_0, \ldots, \vec{x}_n$, so that the difference between $\vec{x}_i$ and $\vec{x}_{i+1}$ is one of these squares from step one and at the end $\vec{x}_n = \vec{y}$. We have $\psi := \phi_0 * \cdots * \phi_{m-1} \in \pi_2(\vec{x}, \vec{y})$ We have $D(\phi)$ and $D(\psi)$ only differ by adding and subtracting these same annuli. We have $\mu'(A) = 2$ by inspection and $\mu(A)$, well, A can be decomposed into two rectangles, and so that's 2 as well. This concludes the proof.

Up to now we have $rM(\vec{x}, \vec{y}) = n_{\vec{x}}(D\phi) + n_{\vec{y}}(D\phi) - 2n_{\vec{w}}(D\phi)$ which we can interpret as $\mu(\phi) - 2n_{\vec{w}}\phi$.

Proposition 1.1.

$$\langle \partial \vec{x}, \vec{y} \rangle = 1$$

if and only if we have a rectangle in the grid diagram in $\mathbb{T}^2$ with no w or z connecting our x and y, y in the upper left.

Proof. We have the left side if and only if we have a 2-chain connecting these in $\mathbb{T}^2$ containing no w or z with $\mu(\phi) = 1$. We should have $D(\phi)$ positively oriented. Then $n_{\vec{x}}(D(\phi)) \geq \frac{1}{2}$ and likewise for y, so they are both $\frac{1}{2}$. Then we have two choices, only one positively oriented.

The other direction is easy. □

Today let me stop here.

2. October 22

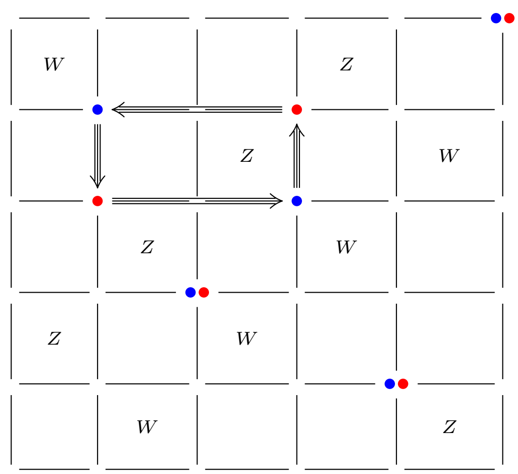
To explain Heegaard-Floer efficiently, I need many colors of chalk.

Let me recall, when I consider a knot diagram, we can turn it into a grid diagram and then associate some data $(\mathbb{T}^2, \vec{\alpha}, \vec{\beta}, \vec{w}, \vec{z})$, this is the grid diagram Γ_K and then this gives us $\widehat{CF}(\Lambda_K), \partial)$. The boundary is something like counting empty rectangles. The generators of this chain complex are $X = \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset Sym^g(\mathbb{T})$ where g is the grid index of K. I started some gradings.

Definition 2.1. *The (relative) Alexander grading is a map $rA : X \times X \to \mathbb{Z}$ where $(\vec{x}, \vec{y}) \mapsto n_{\vec{z}}(D_\phi) - n_{\vec{w}}(D_\phi)$ where, let me recall the definition of D_ϕ.*

First, $\phi \in \pi_2(\vec{x}, \vec{y})$. This gives us a map $\mathbb{D}^2 \to Sym^g\mathbb{T}^2$, which is in correspondence with a branched covering $\hat{\mathbb{D}}^2$ with a map $\hat{\phi}$ to $\mathbb{T}^2$. Then D_ϕ is the image of $\hat{\phi}$, which is a 2-chain in $\mathbb{T}^2$, and there is a well-defined intersection number with the $\vec{z}$ and $\vec{w}$. Changing ϕ corresponds to adding or subtracting annuli so since each annulus has one z and one w, this total index is well-defined.

Example 2.1. *Here is a pair where if the red is $\vec{x}$ and the blue is $\vec{y}$ then $rA(\vec{x},\vec{y}) = 1$:*



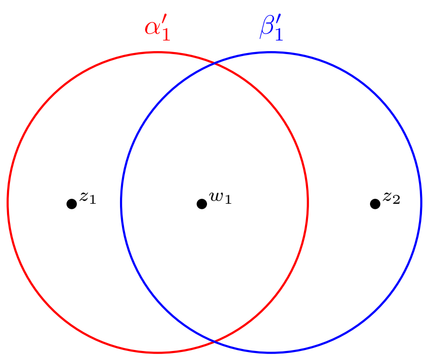
Remark 2.1. *There is an absolute Maslov grading such that $M(\vec{x}_0) = 1 - k$ for k the grid number and $\vec{x}_0$ the lower left corner of $\vec{w}$. Recall that the relative Maslov grading $rM(\vec{x},\vec{y}) = n_{\vec{x}}(D_\phi) + n_{\vec{y}}(D_\phi) - n_{\vec{w}}(D_\phi)$.*

The boundary preserves the Alexander grading and decreases the Maslov grading by one. If we count differentials which are empty regions with no w or z then the difference in relative Alexander grading is zero.

Remark 2.2. *There is an absolute grading satisfying $\Delta_K(T) = \sum_{a,m}(-1)^m rk\, \widehat{HFK}_m(K,a)T^a$ where on the left you have the Alexander polynomial. The m is the Maslov grading.*

Theorem 2.1. *Let $\Gamma_K = (\mathbb{T}, \vec{\alpha}, \vec{\beta}, \vec{w}, \vec{z})$. Then $H_*(\widehat{CF}(\Gamma_K), \partial) \cong \widehat{HFK}(K) \otimes V^{g-1}$ where V is a vector space with two generators over $\mathbb{Z}_2$, with one generator in $(0,0)$ grading and the other in $(-1,-1)$.*

Let me give a sketch of the proof. In step one, change our grid diagram into Γ'_K so that $(\mathbb{T}^2, \alpha'_1 \cup \vec{\alpha}', \beta'_1 \cup \vec{\beta}', \vec{w}, \vec{z})$ so that α'_1 and β'_1 satisfy a condition



and these do not meet any other α or β curves.
[discussion of an example, a heuristic of why this is possible.]

Claim 2.1. *We have*

$$\widehat{HF}(\Gamma'_K) \cong \widehat{HF}(\Gamma''_K) \otimes V$$

where $\Gamma''_K = (\mathbb{T}^2, \vec{\alpha}', \vec{\beta}', \vec{w}', \vec{z}')$*; here* $\vec{w}'$ *and* $\vec{z}'$ *start from index* 2.

[some argument about whether this is possible without increasing genus]

Let me restrict this to the case where I don't need to stabilize with genus.

I want to decompose $\widehat{CF}(\Gamma'_K)$, look at the generators. These consist of two parts. I want $X' = \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$. Then $X' \times \{x\}$ and $X' \times \{y\}$, where x and y are the points at the intersection of the Venn diagram above.

Then $\widehat{CF}_x$ is the sub chain complex generated by $X' \times \{x\}$ and $\widehat{CF}_y$ is the same thing with respect to $X' \times \{y\}$. To justify this notation we need to have that the boundary operator respects x and y, preserves which piece we are in. Every chain from x to y contains a marked point.

The boundary operator on $\widehat{CF}_x$ is $\partial'|_{X' \times \{x\}}$ and likewise for y. This only cares about the region outside the Venn diagram. Then we have $\partial'|_{X' \times \{x\}} = \partial'' \otimes \mathrm{id}_{\{x\}}$ and likewise for y. Then we have the tensor. That's everything except the grading and I do not want to do the grading. By induction, there, well, we may need to put additional curves and destabilization. So by induction we conclude our theorem. If we drop to one z and one w then we get $\widehat{HFK}(K)$. So that's the theorem.

Let's take a break.

Let me talk about a variation in the link case. For the (relative) Alexander grading for ℓ-component oriented links, here the relative Alexander grading $rA : X \times X \to \mathbb{Z}^\ell$, and $rA(\vec{x}, \vec{y}) = \sum_{j=1}^{k_1} n_{z_{i,j}}(D_phi) - n_{w_{i,j}}(D_\phi)$ for $i = \{1, \ldots, \ell\}$.

If I go vertically from w to z and horizontally from z to w like usual, with an oriented link, k_i is the number of z in the ith component.

Proposition 2.1. *We have the same, well, if* $\Gamma_{\vec{L}}(\Sigma, \vec{\alpha}, \vec{\beta}, \vec{w}, \vec{z})$ *is a* $(2k)$*-poitned Heegaard diagram for* $\vec{L}$ *then*

$$\widehat{HF}(\Gamma_{\vec{L}}) \cong \widehat{HFK}(\vec{L}) \otimes (\bigotimes^{\ell} V_i^{\otimes(k_i - 1)}).$$

I want to end this talk by mentioning the Maslov index and branched covering issue. Last week, we talked about, if we consider $\phi : \mathbb{D}^2 \to Sym^k \mathbb{T}^2$, then this corresponds to the following data, $\widehat{\mathbb{D}}^2 \xrightarrow{\hat{\phi}} \mathbb{T}^2$. If we consider just a two-fold branched cover of $\mathbb{D}^2$, it has many choices, many configurations. The one branched point, it looks like a disk, with three it has one genus, and with five branching points two genus, and so on.

There was some comment, maybe given by Calin, we need to care about the equivalence relation between these coverings. So when I choose $\phi \in \pi_2(\vec{x}, \vec{y})$, it's a homotopy class of this map, we need to care about the equivalence class, we can say something about the relation between these things. Actually it was given by Byung Hee's comment. If I consider the chain given by the genus one surface with three special points, if I consider the involution given by rotating with respect to the plane. In this homotopy situation, we can homotope two of the points close, and the corresponding branched cover becomes a pinched torus. Then also by a homotopy, we can pull it apart, so this is the same as the branched cover with only one point. We cannot distinguish the higher genus branched cover from the cover with just one point. That is the argument.

So by this argument, if there is higher genus we can pinch them and all the higher genus branched cover can be reduced into the disk case. That's all. Then I'm done.

3. November 19: Seonhwa Kim, knot Floer homology and knot genus

Why do I want to study knot homology (Khovanov homology, knot Floer homology)?

- Knot homology is obtained by categorification of quantum invariants such as the Jones polynomial and Alexander polynomial. These polynomial type invariants, my main playground is the Volume conjecture and this is a relation between quantum invariants and hyperbolic geometry.

My question in some sense is whether the relation between knot, well, the volume conjecture says that asymptotics of quantum invariants give classical geometric invariants. My question is what about the asymptotics of link homology. Until now, there is no conjecture between knot homology and certain asymptotics.

$$\lim_{N\to\infty} \frac{1}{N} \log |J_N(k, q = e^{\frac{2\pi i}{n}})| = vol(S^3 \backslash K).$$

Here we can think that N is the dimension of a representation. In Khovanov homology, there is one for each N. Anyway, this is just a dream but this is the motivation. The first target is knot Floer homology. I'm not familiar with knot Floer homology, so I want to know about the practical properties of this thing. So for instance knot Floer homology detects knot genus, so detects fiberedness. My curiosity is about how we can detect knot genus by knot Floer homology. The background of my curiosity is:

Remark. *The knot genus problem in a general* 3*-manifold is* NP*-hard.*

If some guy says some problem, given a three manifold and a knot, what is the genus of the knot, that problem is algorithmically hard. That is the reason I wonder how knot Floer homology detects knot genus.

A conjecture is that an unknot recognition algorithm is P, but we don't know. We have the conjecture that the Jones polynomial detects the unknot, but this is just a conjecture.

Theorem 3.1. *Computing the Jones polynomial is* $\#P$*.*

I want to study how to detect genus using knot Floer homology.

Knot Floer homology is a categorification of the Alexander polynomial and the Alexander polynomial gives a lower bound on knot genus.

Today's goal is to explain this last statement. First I should understand why the Alexander polynomial gives a lower bound.

Theorem 3.2. *(Seifert) Every knot in* S^3 *is a boundary of an orientable surface in* S^3*. We call such a surface a* Seifert surface*.*

Definition 3.1. *The knot genus* $g(K)$ *is the minimal genus of all Seifert surfaces.*

There are various names of this knot genus. It's also called the orientable genus. Another notion of genus is the "canonical" or "projection" or "Seifert" genus, which is related to the proof of the theorem. Seifert provided an algorithm to get a Seifert surface from a diagram. This other genus is the minimal genus provided by the algorithm over all diagrams.

for all n, we have a genus 1 knot with $g_S(K) = m$.

If you consider a diagram, fix an orientation. Take a smoothing for each crossing using the orientation. Then we can get a diagram consisting of simple closed curves. Attach a twist at each crossing.

This always gives an orientable surface. Oszvath-Szabo use Seifert genus to mean knot genus but that is not standard terminology. When I first read the paper, this was confusing.

Remark. *For alternating links, $g_S(K) = g(K)$.*

We want to know about Seifert surfaces. So let's talk about the universal Abelian covering of the knot complement. If you want to define the Alexander polynomial, the most classical way is by using such a covering. You take $\tilde{X}$, this cover, which is Abelianization of π_1, which is $\mathbb{Z}$ in the case of a knot. The π_1 comes from the Wirtinger presentation of the knot diagram. The generators come from the arcs and the relations from the crossings. In the case of the trefoil, we get three generators and then conjugation relations. [pictures for Wirtinger presentation]

So let's make this Abelian cover for the trefoil. We can make Seifert surface for the trefoil and we can construct the cover using the surface. The meridian action moves us to another sheet in $\tilde{X}$. We split using this surface.

So in the Abelian covering, we get a kind of $S^1 \times \mathbb{R}$ where the real line is the meridian and the S^1 the longitude.

We think of $\tilde{X}$ as having an action of $\mathbb{Z}[\mathbb{Z}]$, the group ring.

The torsion polynomial, the homology of $\tilde{X}$ with coefficients in the local coefficient system which is this group ring, is the Alexander polynomial.

We want to calculate an actual example. [Pictures]

In our example, we have

$$H_1(\tilde{X}, \mathbb{Z}) = \langle t^i\alpha, t^i\beta | t^{i-1}(\beta - \alpha) = -t^i\alpha; -t^{i-1}\beta = t^i(\alpha - \beta)\rangle.$$

As a $\mathbb{Z}[t, t^{-1}]$-module, it is

$$H_1(\tilde{X}, \mathbb{Z}[t, t^{-1}]) = \langle \alpha, \beta | \beta - \alpha = -t\alpha, -\beta = t(\alpha - \beta)\rangle.$$

If we subsitute for β we get $(t^2 - t + 1)\alpha = 0$

The next subject will be knot Floer homology detecting knot genus.

4. Seonhwa Kim, December 3

First I think I need to change the title of this seminar to the learning seminar on *basic* knot *theory*.

Anyway, last time I said that the Alexander polynomial gives a lower bound for knot genus,

$$\underbrace{\deg}_{\max - \min} \Delta_K(t) \leq 2g(K).$$

The reason is that the degree of Δ_K is less than or equal to the number of generators of H_1 of a Seifert surface, which is two times the genus. If you take a minimal genus surface, this gives the inequality.

Let's remember the goal, which is to see how knot Floer homology determines knot genus.

I need to make a correction. Last time I said that the knot genus problem in a general 3-manifold is NP-hard. This is also contained in NP. So the problem is NP-complete.

So as you expect, I think knot Floer homology calculations are NP-complete. I think, this is my curiosity, is knot Floer homology NP-complete or not? I surveyed some results about the Alexander polynomial.

So $\widehat{HFK}$ determines the *Thurston norm*. This is the main part of showing that $\widehat{HFK}$ determines knot genus.

(1) In $H_2(M, \mathbb{Z})$ pick an element a representing embedded surface S. Set $\chi_-(S) = \max\{0, -\chi(S)\}$. Define $x(a)$ as the minimum of $\chi_-(S)$ for all $[s] = a$.

    If there are nontrivial sphere, torus, disk, or annulus representing any a (we could also work in $H_2(M, \partial M, \mathbb{Z})$. Then this is a semi-norm.

There are certain results about this norm using the Alexander polynomial.

McMullen (98) defined the Alexander norm. Today's main topic is about these things. To do this, we need some definition. He defined the Alexander polynomial for a finitely generated group and ϕ a homomorphism $G \to F$, a surjective homomorphism to a free Abelian group. This is $\mathbb{Z}^b$. In this setting he defines an Alexander module, any subgroup H, $m(H) \subset \mathbb{Z}[G]$ is defined as $\langle (h-1) : h \in H \rangle$, the augmentation ideal.

The Alexander module $A_\phi(G)$ is defined as $m(G)/m(\ker \phi)m(G)$. This is a G-module and also an F-module.

Let me give another definition. Let (X, p) be a pointed CW-complex with $\pi_1(X, \pi) = G$ and $tildeX \to X$ the Galois covering corresponding to ϕ, with deck transformations F. Then $A\phi(G) := H_1(\tilde{X}, \tilde{p}, \mathbb{Z})$, which has a natural action of F coming from deck transformations.

This F acts on the cell structure so H_1 gets an F-module. I didn't understand why these two definitions coincide. So let's see.

Choos a pase point $*$ in the lift of p. Then $(g-1) \in m(G)$ is obtained by lifting the loop $g \in \pi_1(X, p)$ to a path in $\tilde{X}$ running from $*$ to $g*$. I had some trouble but maybe it was just a typo in the book, an upper instead of a lower index.

Then there are similar but different notions of Alexander modules, Alexander ideals, this is defined by taking first the free resolution $\mathbb{Z}[F]^r \xrightarrow{M} \mathbb{Z}[F]^n \to A \to 0$. Then M is an $n \times r$ matrix and if we take $E_i(A)$, an ideal in $\mathbb{Z}[F]$, no be generated by (determinants of) $(n-i) \times (n-i)$-minors of the matrix M. Call this the ith Alexander ideal. It turns out to be independent of the resolution. The proof was done by Fox.

I think there is a higher language proof but I don't know it.

The Alexander polynomial is the greatest common divisor of the elements of the Alexander ideal $I_\phi(G) := E_1(A_\phi(G))$, generates the smallest principal ideal containing $I_\phi(G)$. [Some discussion about what the *smallest* principal ideal means.]

Let me talk about twisted gounp cohomology. Let $M_A \to M$ be a covering space with Abelian Galois group A. Then A acts on $H^1(M_A, \mathbb{C})$. We can try to decompose this action into irreducible pieces.

Consider "crossed homomorphisms" f from G to a G-module B by $f(gg') = f(g) + gf(g')$. So f form an additive group $Z'(G, B)$ of 1-cocycles on G with values in B.

So the coboundaries $B^1(G, B)$ are those f given by $f(g) = ab - b$ for some $b \in B$. The first cohomology group of G is $H^1(G, B) = Z^1(G, B)/B^1(G, B)$.

So $Hom_G(A_\phi(G), B) \cong Z^1(G, B)$ for any F-module B considered as a G-module via $\phi : G \to F$, $h : A_\phi(G) \to B$ where $f(g) = h(g-1)$.

We can calculate

$$f(gg') = h(gg'-1) = h(g-1) + g(g'-1)) = h(g-1) + gh(g'-1) = f(g) + gf(g')$$

so f is a cocycle.

The Abelianization $ab(G)$, we can say $\mathbb{C}[ab(G)] = \mathbb{Z}[ab(G)] \otimes \mathbb{C}$, the coordinate ring of the character variety $\widehat{ab(G)} = Hom(ab(G), \mathbb{C}^*) \cong (\mathbb{C}^*)^{b_1 G}$.

Any character $\rho : ab(G) \to \mathbb{C}^*$ determies a multiplicative action of G on $\mathbb{C}$.

I should skip.

The Alexander norm on $H^1(M, \mathbb{Z})$ has $||\phi||_A = \sup \phi(g_i - g_j)$ where g_i ranges over the group elements in H_1 for which Δ_M has a nonzero coefficient. Here M is a connected compact orientable manifold with torus boundary.

We finally have the inequality, using Poincaré duality

$$||\phi||_A \leq ||\phi||_T + 0$$

if $b_1 M \geq 2$ or

$$||\phi||_A \leq ||\phi||_T + 1 + b_3(M)$$

if $b_1(M) = 1$ and $H^1(M, \mathbb{Z}) = \mathbb{Z}\phi$.