INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS
STABLE HOMOTOPY TYPES IN FLOER THEORY
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Thanks for the invitation to give this lecture series here. The plan for the four lectures in this lecture series is something like the following. I'll start today with an introduction about how this fits into a bigger picture. Then I'll discuss Conley indices and Thom spaces, and the relation between these, which you could call stable Morse theory. This is the plan for day one. In day two, I'll introduce spectra, which follow naturally, and then say a little bit, I'll define homology of spectra and then I'll say a little bit about why the homology of spectra becomes Morse homology, I'll say why it's better to have the spectrum than its homology. I want to say something about usefulness of the spectrum versus its homology.

That's days one and two. On day three, I hope to talk about finite dimensional approximations of action of a Hamiltonian system in $\mathbb{R}^{2n}$, then the same in T^*N. Then I'll describe symplectic homology as a spectrum, then Viterbo functoriality. You can take these as definitions if you haven't heard of them. This might take all my time but for now I'll write day four as parameterized homotopy theory and some version I will call fiberwise symplectic homology which then has some sort of applications. I'll say a little more about the applications at other times but this is just an outline of the plan.

2. Introduction

First just a little short introduction to Hamiltonian Floer homology, and it will be very short. This is usually defined as Morse homology of some action $A : \mathcal{L}M \to \mathbb{R}$. For $\mathcal{L}M$ the loop space of an exact symplectic manifold M. There's a lot of things involved in this. You need to perturb, you need to do a lot of things. For those of you who don't remember this, you define a chain complex $CF_*(A)$ as generated by critical points of A and then you have a differential, which counts, well

$$\partial x = \sum_{z \in M_0(x,y)} \epsilon(z).y$$

where x and y are critical points for A, z is a gradient trajectories of dimension 0, and ϵ is a sign. Even though this sign is important and I'll have something interesting to say about Viterbo functoriality, I don't want to say very much about it.

What I want to do is to define this in a very different way, using topology, defining "spaces" or spectra refining this. Here I mean that their homology gives you back their Floer homology. We'll only do this in the cotangent bundle. I should say about this fact that this refines the notion of Floer homology in cotangent bundles. One can say it's motivated by Floer homology, but these were actually there before Floer homology, these are the motivation for Floer homology. But before no one has put it into spectra. Using knowledge of spectra to get information out of it is the new part.

Okay. One result from this is a theorem by me and Abouzaid which says that

Theorem 2.1. *For $n > 3$ odd and 8 not dividing $n+1$, there exists Lagrangian immersions $S^n \to T^*S^n$ which are smoothly isotopic to embeddings (i.e., the zero section) but not Lagrangian isotopic to Lagrangian embeddings.*

This was not known before and comes directly out of the fact that this invariant of Floer homology is refined by spectra. This is generally for N immersed in T^*N. The theorem works generally to find immersion classes with no embedding. When you have a Lagrangian, there is a K class in the loop space over M. There is a map to $\mathbb{Z} \times BG$ which classifies bundles, and that has to be the zero class. A lot of these K-classes that you can get from immersions. The dimension restrictions come from the homotopy groups of spheres, they're where the map on homotopy groups isn't zero.

So maybe I should not talk too much about day four, but write down a theorem and say there is another theorem, parameterized homotopy theory leads to this theorem of mine from 2011:

Theorem 2.2. *Any closed exact Lagrangian in T^*N is (after lifting to a finite cover of N) a homology equivalence*

Abouzaid had a similar result, but a homotopy equivalence, which is stronger, but he relied on Maslow index zero. So this is not as strong but more general. We combined these results to get an even better one but this is what I'll talk about here.

Now I'm going to start with some actual proper definitions and build up the theory.

3. Conley Indices

This is from a survey or lecture notes by Conley. I won't have all my pictures in my lecture notes but I'll hand them out later today. What is this and how is it defined?

Let M be a manifold without boundary. Let f be a smooth function. A *pseudo-gradient* X is a vector field $M \to TM$ such that $X(f) \geq 0$ and $X(f) = 0$ only at critical points. It looks very much like a gradient but it doesn't come from a Riemannian structure on the manifold. It's flexible to do this. One can usually almost do that instead. Almost any of these guys come from a metric. Since we might be putting on other metrics, this is more convenient for us.

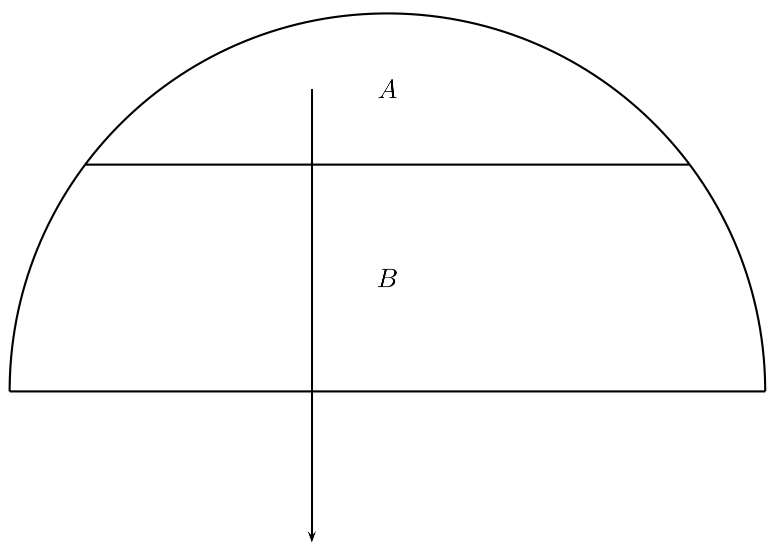
So we let this be fixed and define ψ_t as the flow of $-X$ (negative gradient flow). Note that for a flow line of ψ_t at a point x in M, we have that $\frac{\partial}{\partial t}$ of f is given by $-X(f)$. If you integrate a vector field to get a local diffeomorphism, it changes things by the derivative. This is almost by definition.

I want to define, let $a < b$ be regular values and assume that critical points with values in $[a, b]$ form a compact set. If you do Morse homology you do this as well. Most of the time this will be taken care of automatically.

So the most important notion in this entire lecture series is index pairs or Conley index pairs. Let me write that down.

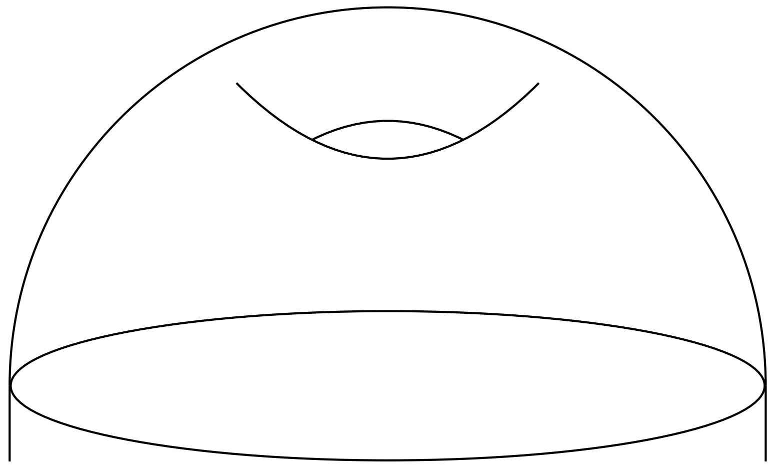
Definition 3.1. *A pair of spaces (A, B) with $B \subset A \subset M$ (this is usual in topology, this notation) is called an* index pair *if*

- I1 *The set $A - B$ is contained in $f^{-1}(]a, b])$*
- I2 *The sets A and B are compact*
- I3 *The set $int(A) - B$ contains all the critical points, and the most important:*
- I4 *For any $x \in A$, the two sets*
$$\{t \geq 0 | \psi_t(x) \in A\} \supset \{t \geq 0 | \psi_t(x) \in B\}$$
are either $\mathbb{R}_{\geq 0}$ and $\emptyset$ or two closed intervals with the same maximum. So you either flow for all time and avoid b or you leave A and never come back, leaving B at the same time.



If you know from beforehand what Conley indices are, they're more general, but this makes sure that critical flow stays in A.

Here's an example. Let M be compact with boundary but define $f : int(M) \to \mathbb{R}$ by using a collar neighborhood $\partial M \times [0, 1)$ with $f(x, S) = \frac{-1}{S}$. Somehow smoothly extend to all of M.



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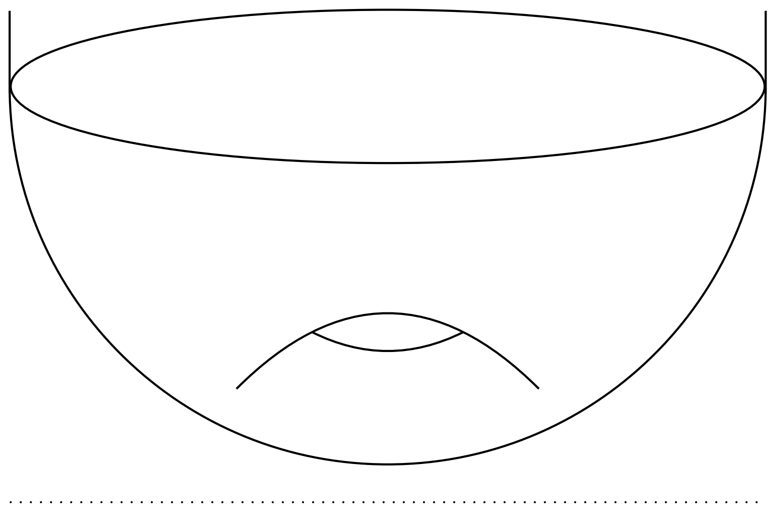
We see that $A = f^{-1}([a, \infty[)$ and $B = f^{-1}(a)$ is an index pair.

For an index pair (A, B) define the Conoly index $I_a^b(f, X) = A/B$. In our example, we have that the index pair of f, X is A/B, which is $M/\partial M$.

Let's take another example, using a height function that is reversed, $g(x, s) = \frac{1}{s}$. I'll let $A = f^{-1}(] - \infty, a])$ and $B = f^{-1}(a) = \emptyset$ below all the stuff. What does this mean?



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So $I(g, X) = A/B = A/\emptyset = A \sqcup \{-\infty\} \cong M \sqcup \{-\infty\}$.

Let me show the relation to Morse homology before showing this is homotopy invariant. We have $A/B = Z$, where Z is a CW complex with a basepoint $[B]$ and one cell per critical point of f in A. So then the reduced homology

$\tilde{H}_*(Z) = MH_*(f_{[a,b]})$. I won't write this down explicitly, but I'll prove that there's an isomorphism from this to a chain complex and then refer to a paper that explains that this complex gives you the Morse homology MH_*.

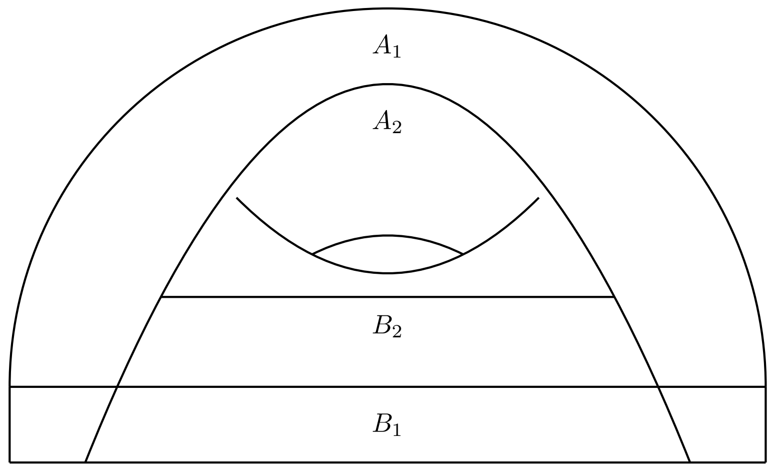
Let me prove this important lemma:

Lemma 3.1. *$I_a^b(f, X)$ (as a based space) does not depend on the choice of index pair up to (essentially unique) homotopy equivalence.*

What do I mean by essentially unique? Well, think of it this way. If I were not constructing spaces, I would want not just an Abelian group but also some sort of choice of the presentation. The proof is kind of long so let's take a break.

Let (A_i, B_i) for $i = 1, 2$ be two index pairs. Let me note that if we take $(A, B) = (A_1 \cap A_2, (B_1 \cup B_2) \cap (A_1 \cup A_2))$, then we still have an index pair but we have reduced the problem to the case where $A_2 \subset A_1$ and $B_1 \cap A_2 \subset B_2$. The difficult thing to check is I4, but either it stays in a critical point and never hits either B, or otherwise it hits both.

So now we want to define a homotopy equivalence back and forth between the two.



Let $d : A_1 \to \mathbb{R}$ be

$$d(x) = inf\{t \geq 0 | \psi_t(x) \in A_2 \cup B_1\}$$

The map d is bounded by compactness. $-X(f) < -k < 0$ on the set $A_1 - (A_2 \cup B_1)$. If you flow for time $(b - a)^{-1}k$ then either you reach $A_2 \cup B_1$ or $f(\phi_t(x)) < k$. Similarly, we define $e : B_2 \to \mathbb{R}$ in almost the exact same way:

$$e(x) = inf\{t \geq 0 | \psi_t(x) \in B_1\}.$$

Again we can bound e. Let c be an upper bound on e and d. Then $h_t^{12} : A_1/B_1 \to A_2/B_2$ and $h_t^{21} : A_2/B_2 \to A_1/B_1$ is defined by

$$h_t^{12}(x) = \begin{cases} \psi_t(x) & \psi_t(x) \in A_2 - B_2, x \in A_1 - B_1 \\ B_2 & \text{otherwise} \end{cases}$$

the other direction is:

$$h_t^{21}(x) = \begin{cases} \psi_t(x) & \psi_t(x) \in A_1 - B_1, x \in A_2 - B_2 \\ B_1 & \text{otherwise} \end{cases}$$

This is well defined if $t \geq c$. Because we're flowing for enough time, we get into $A_2 - B_2$. As an exercise, this is well defined and continuous in t. You have to check the quotient sets. When you compose $h_t^{12} \circ h_t^{12}$ you get $\psi_{2t}(x)$ if $\psi_{2t}(x) \in A_1 - B_1, x \in A_1 - B_1$ or $[B_1]$ otherwise. This is homotopic to the identity because it is defined for all $t \geq 0$ and continouous in t as well. Similarly, you have the same for the other composition.

If you do this for other ones, you put this together and get a discrete category with contractible morphism spaces. It doesn't really matter too much. This is for people who are really into homotopy theory and topology. This won't matter for the result that this is essentially unique. This speaks to some sort of naturality.

Let me give you a class of examples where everything is much easier.

Definition 3.2. *A function and pseudogradient (f, X) is called completely bounded (CB) if*

- C1 *The flow ψ_t is defined for all $x \in M$ and all $t \in \mathbb{R}$. This might be called a complete flow.*
- C2 *The function $X(f)$, which is only zero at critical points, is bounded from below by a positive constant on the complement of some compact set. If you remove a small neighborhood of each critical point, you get a uniform lower bound.*

Lemma 3.2. *If (f, X) is CB then index pairs exist for any $a < b$ regular.*

Proof. I'll just write down a formula. Let $K \subset M$ be compact such that $X(f) > k > 0$ on the complement of K. This is C2 from the definition of completely bounded. Define $K_T = \psi_{[0,T]}(K) = \cup \psi_t(K)$. This is a compact set, because flow is defined for all finite time (C1).

Define $A = f^{-1}([a, \infty[) \cap (f \circ \psi_{-T})^{-1}(]-\infty, b])$. If you have your function, you have a big manifold and you are flowing, you have b and a, then $f^{-1}(b)$, $f^{-1}(a)$, if you flow backwards and then take f, you get something below b, maybe above, maybe below a. But if $T \geq (b-a)k^{-1}$ you'll get something compact. Why? Fix an $x \in A - K_T$. Then flow for $t \in [-T, 0]$, the flow $\psi_t(x) \notin K$. Then $\frac{\partial}{\partial t} f(\psi_t(x)) > k$ for $t \in [-T, 0]$ so flowing gives you $f(\psi_{-T}(x)) - f(\psi_0(x)) > (b - a)$. So x being in $A - K_T$ gives us $f(\psi_-T(x)) \leq b$ and we get $x \notin A$.

Define $B = f^{-1}(a) \cap A$.

Let me just say at the end today then how one can create a complex similar to the Morse homology complex out of this data. Maybe I won't have time for more than the example I need first. So, example. $q : \mathbb{R}^n \to \mathbb{R}$ is a non-degenerate quadratic form. We use the usual ∇q as our pseudogradient. Then to make this easier we realize that we can transform this function without changing the gradient if we change coordinates in a particular way. By change of coordinate from $O(n)$ we may assume that $q(x_1, \ldots, x_n)$ is just a sum of $\lambda_i x_i^2$. This is an isometry. So the gradient of q at the point $(x_1, \ldots, x_n)$ is $2(\lambda_1 x_1, \ldots, \lambda_n x_n)$.

The most important thing here is that the gradient flow splits on each coordinate. Solutions to this are of the type $\gamma(t) = (a_1 e^{\lambda_1 t}, \ldots, a_n e^{\lambda_n t})$. Whether this contracts or expands depends on the size of these guys. So what you see is that, well, let

$\lambda_1 < \cdots < \lambda_n$ and m the number of negative λ_i, then $A = D^m \times D^{n-m}$ and $B = S^{n-1} \times D^{n-m}$ is an index pair. In $\mathbb{R}^2$ the flow is a hyperbolic foliation and B cuts off the sides of a square in the plane. The point is that this is homotopy equivalent to $D^m \times D^{n-m}/S^{m-1} \times D^{n-m} = S^m \wedge (D_+^{n-m}) \cong S^m$ with the usual basepoint. The pair of the square and its sides deformation retracts onto the interval and its endpoints.

□

4. October 1

[Missed first section]

Lemma 4.1. *Let γ^s, λ^s be a smooth compact family of deformations with pseudogradients on M which has a good index pair with respect to $a < b$. Then $I_a^b(f^{s_1}, X^{s_1}) \cong I_a^b(f^{s_2}, X^{s_2})$ for all s_1 and s_2 in S. In fact the Conley indices $I_a^b(X^*, f^*)$ defines a based fibration over S.*

...

Example: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x+iy) = -x^4+x^2-y^2$. Then $I_a^b(f) \cong S^2 \wedge S^2$. If you try to do parallel transport you get holonomy on the reduced homology which switches the two factors. Have an essentially unique homotopy equivalence between two fibers means that you have a way to choose paths. So if the $s \in S$ in the lemma comes from contractible S then these are essentially unique.

A sketch of the proof of the lemma. Let $s_0 \in S$ and (A_{s_0}, B_{s_0}) be a good index pair for (f^{s_0}, X^{s_0}). Then claim, this is a good index pair for (f^s, X^s) close to s_0. So the Conley index is locally constant. Why is this true and why did I assume they were good index pairs?

If you had an index pair then you start with something where you know the flow exits and never returns. Before going far down, the flow might escape to ∞. There might be a close but not too close flow that comes in from ∞ but this is not the same flow line. If you perturb things a little bit they might connect up. A lot can happen out at ∞, it's difficult to control this. So that might be a flow line after perturbing and reenter A. This is always only the problem when you deal with Conley indices of this type. If the manifold is closed, compact, then everything works out fine. So this can go wrong. But our pair doesn't look like this, so this will never happen. The values of s go down, so once you escape from B there's no way to come back. The value of f is important, but another thing that's important, for s close to s_0, we see that X^s is still transversal to the defining equations. Transversality is an open condition. It'll still point in and out at the same points. Combining with the fact that $B \subset f^{-1}(a)$ gives you what you want.

Okay. This was why these good pairs were very nice. Now we can talk about Conley indices' invariance when changing data.

These structures that I'm going to define now are closely related to Floer homology. These are inclusions and quotients. To make this discussion as easy as possible, let $a < b < c$ be regular for $f : M \to \mathbb{R}$, X a pseudogradient as usual, with good pairs for $I_a^c(f, X)$. I'll sketch this because I have a drawing I'm going to change a little bit. We have three pairs, how do they relate?

Let $f_a^d = f^{-1}([a,d]) \cap A$. We had $I_a^c(f, X) = f_a^c/f_a^a$ and $I_a^b = f_a^b/f_a^a$. Finally, $I_b^c(f, x) = f_a^c/f_a^b$. These are associated to the triple f_a^c, f_a^b, f_a^a, so there is a long exact sequence for the homology of these guys, using reduced homology because they're based.

$$\cdots \to \tilde{H}_*(I_a^b(f,X)) \to \tilde{H}_*(I_a^c(f,X)) \to \tilde{H}_*(I_b^c(f,X)) \to \tilde{H}_{*-1}(I_a^b(f,X)) \to \cdots$$

The maps are inclusion, quotient, and a connecting homomorphism.

Now I've got some properties that make it easier to connect this to Morse homology. Let me just write something here. Without going into detail, just recall that Morse homology was a chain complex for a Morse-Smale system (f, X) where the most important part is that you get some homology coming out of this data. How does it fit into this picture precisely? This system has a generator per critical point of the Morse function. Can we see that there's a complex like this given its homology? I'll show you that this homology here, I'll motivate why it's homomorphic to Morse homology. I said a little bit about it with CW structures last time but let me give you a more precise and more direct description.

Let $a < b$ be regular and Morse and let (A, B) be a good index pair for $a < b$. I want to construct something that looks like Morse homology. First perturb f so it takes distinct critical values. Look around a critical point, you can change the value without changing that it's a critical point. Next, pick a parttion $a = s_0 < \cdots < s_n = b$ so thateach interval has one critical value. Then you inductively look at the long exact sequence. Say we have constructed a $MC_*^i(f, X)$ so that

$$MC_*^i(f,X) \to \hat{C}_*(I_a^{s_i}(f,X))$$

which is a homotopy equivalence, with one generator on the left per critical point. We can make a short exact sequence

$$0 \to \tilde{C}_*(I_a^{s_i}(f,X)) \to \tilde{C}_*(I_a^{s_{i+1}}(f,X)) \to \tilde{C}_n(I_{s_i}^{s_{i+1}}(f,X)) \to 0$$

and you can look at the one on the right and see it's $\mathbb{Z}[m_i]$ and things of this sort are easy to work with. It's not difficult, you can work this out so you get the next step. The reason I'm saying it, one reason is, this is not unique. But Morse homology is not unique either. Much of that nonuniqueness is also found here in these choices. I'm not actually sure if it's actually the same choices. You probably have to be in dimension at least six to do this. So you need h-cobordism theorems that work in dimension six and higher. The homology of these Conley indices is a way of reconstructing Morse homology. If you want to see a complete proof, you will get the notes later today or tomorrow morning and there's a reference there.

So much for Morse homology, but let me just remind you that last time I made this picture that said if you have A and B then you can construct a CW complex by attaching cells. If you go from s_i to s_{i+1} then you're adding a handle. If you didn't know this, then one way of constructing Morse homology is to think of this as the reduced CW complex of some complex you construct. You count degrees of attaching maps for these cells onto each other for Morse homology. Morse homology is really CW homology in the finite world. It's just thought of in a different perspective.

4.1. **Thom Spaces.** Let (A, B) be a pair and let $E \to A$ be a metric vector bundle over A. Then define

$$(A,B)^E = (DE_A, SE_A \cup DE_B)$$

and define

$$(A,B)^E/ = (DE_A/SE_A \cup DE_B.$$

Sometimes people define this differently with one-point compactifications. This doesn't work on noncompact spaces. Why is this important? Let $f, X, a, b, (A, B)$ be as before, maybe the pair doesn't need to be good. Let $E \to M$ be any vector bundle, metric again. Let q be a function on this vector bundle, $q(x, v) = -||v||^2$. This generalizes to a quadratic form. Fiberwise this is just a negative definite quadratic form. I'll define some stuff and leave some checking and an exercise. First define $f + q = \pi^* f + q$ which means $(f + q)(x, v) = f(x) - ||v||^2$. I've defined a function on a bigger manifold. I want to relate the Conley index here to the Conley index on the lower manifold. I need a pseudogradient for this. Define $X^1/X \oplus \nabla^f q$. The $\oplus$ is a splitting associated to some compatible connection. I mean this is compatible with the metric. It gives us a way to define horizontal vectors. The inner product of the norm is preserved so the norm is preserved. You can check as an exercise that X' is a pseudogradient for $f + q$ and $(A, B)^E$ is an index pair. I'm going to draw this and geometrically motivate it. It's not difficult to do the exercise after seeing these pictures.

[at this point I had to leave]

5. October 2

So first I just wanted to set this record straight. Somehow there's direct limits, also called, well, they sit in colimits, and there are also inverse limits, which are a kind of limit. There's a general thing in category theory and limits lead to products while colimits lead to coproducts. The reason why these are upside down is because, well, they're easier to handle. Direct limits make sense. The names are also used, but it's colimit and limit. I changed the notes to say limits, which I shouldn't have done. Either change them in your notes or wait for them to be changed in the whole set.

I said I'd begin today with finite dimensional approximations. Because people might not have experience with spectra or like these limits, I've decided it's probably better to do more of these examples of spectra today. I'll say a little more before continuing, starting with an example that's an extension of an example from yesterday. Let X be a based space. Define the spectrum $\Sigma^\infty X$ called the suspension spectrum of X by letting $(\Sigma^\infty X)_k = \Sigma^k X$. I have to give you a sequence of (structure) maps $\sigma_k : \Sigma\Sigma^k X \to \Sigma^{k+1} X$, and these are equal, so I'll just use the identity map. This also means that we get that $H^*(\Sigma^\infty X) = \lim \tilde{H}_*(X) \to \tilde{H}_{*+1}(\Sigma X) \to \cdots)$ but when you have a limit of isomorphisms, you have $\tilde{H}_*(X)$.

This is a functor from based spaces to spectra, Σ^∞ and this fits into a diagram using homology to graded Abelian groups

$$\begin{tikzcd} \text{based spaces} \arrow[rr, "\Sigma^\infty"] \arrow[dr, "\tilde{H}_*"'] & & \text{spectra} \arrow[dl, "H_*"] \\ & \text{graded Abelian groups} & \end{tikzcd}$$

Let me note a couple of points. For a spectrum $X = [X_k, \sigma_k]$, forgetting the first i spaces does not change the equivalence class. One way to see this is that if you start with your sequence $X_0, X_1, \ldots, X_i, X_{i+1}, \ldots$, you're just replacing the first i with points $\{*\}, \{*\}, \ldots, \{*\}, X_{i+1}, \ldots$. Remember that an equivalence was a map that commuted, and these are just the inclusion of basepoints. Only remembering a subsequence X_{k_i} and the maps $\sigma_{k_i}^{k_{i+1}} : \Sigma^{k_{i+1}-k_i} X_{k_i} \to X_{k_{i+1}}$ is enough to reconstruct the spectrum uniquely. I can write two sequences, you can give basepoints until X_{k_0} and then $\Sigma^i X_{k_0}$ until I get to X_{k_1}. Then you continue with the idea. So you fill things up with the suspension of what went before. We won't define every single space in the sequence, only the nth space in the sequence. The inclusions that make the diagram commute are $\sigma_{k_0}^{k_0+i}$. Remember I assumed these maps were inclusions.

It was also mentioned yesterday about formally inverting suspensions. So what you do is set $X = [X_k, \sigma_k]$ and define $X_k = *$ when $k < 0$. So you could think of them as starting in negative numbers. So then you could think of $X = [\Sigma^d X_{k-d}, \Sigma^d \sigma_{k-d}]$. The reduced suspension of the basepoint is the basepoint. This is the example $d = 2$:

$$\begin{tikzcd}
\{*\} \arrow[d] & \{*\} \arrow[d] & \Sigma^2 X \arrow[d] & \\
X_0 & X_1 & X_2 & X_3
\end{tikzcd}$$

Here's an example, let $X_k = \wedge_{i=0}^k S^i$. So $k = 3$ it'll be a 1-sphere, a 2-sphere, and the 3-sphere. Then all of the spheres go one up in dimension and the span of the homology gets wider and wider, so H_* of this guy is in fact $\mathbb{Z}$ when $* \leq 0$ and 0 otherwise.

So define Σ^{-d} as the equivalence class of $[X_{k-d}, \sigma_{k-d}]$ and define $\Sigma^d X$ as $[X_{k+d}, \sigma_{k+d}]$. So if you pass to homology complexes it's just a grading shift.

And now for something, I've talked about spectra and their homology, but I want a tool that tells me that they are better than their homology.

Definition 5.1. *Let $X = [X_k, \sigma_k]$. Then $\pi_*(X)$ is the limit of homotopy groups*

$$\pi_*(X) = \mathrm{colim} \pi_{*+k}(X_k)$$

So what maps do I use? $\pi_{*+k}(X_k) \to \pi_{*+k+1}(\Sigma X_k)$. What is this map? If $f : S^n \to X_k$, then $\Sigma f : S^{n+1} \to \Sigma X_k$. So the map sends $[f]$ to $[\Sigma f]$. It's not in general an isomorphism but we can use it to compute anyway. This limit is not as nice as the other one, but there's something similar to the isomorphisms we had. So $\pi_*(\Sigma^\infty X)$ is known as the stable homotopy groups of X. You've probably heard about this. They relate to unstable homotopy groups. On some groups $\pi_{*+k}(X_k) \to \pi_{*+k+1}(X_{k+1})$ will be an isomorphism. Let me concentrate on spheres, where $\pi_{n+k}(S^k) \to \pi_{n+k+1}(S^{k+1})$ is an isomorphism for $k > n + 1$. If you've computed stable ones, you've computed some unstable ones. They also show up here. They're easier to compute because the structures of the category of spectra almost look like an Abelian category.

If X_k become more and more connected, then this stabilizes, but the example with infinitely negative things never stabilize. You're not going to get anything. You could replace this with a low dimensional thing that doesn't stablize.

Let me define now what a map of spectra is.

A map of spectra $f : X \to Y$ is an equivalence class of

- Representatives for X and Y
- a sequence of maps $f_k : X_k \to Y_k$

such that the structure maps commute:

$$\begin{tikzcd} \Sigma X_k f_k \arrow[r, "\Sigma"] \arrow[d, "\sigma_k"'] & \Sigma Y_k \arrow[d, "k"', "\sigma"] \\ X_{k+1} \arrow[r, "f_{k+1}"'] & Y_{k+1} \end{tikzcd}$$

The equivalence relation is generated by restricting to smaller representatives $Y_k' \subset Y_k$ and $X_k' \subset X_k$ such that $f_k(X_k') \subset Y_k'$. This is why it's important for the subsequences to fill out the spectrum. If you have two different sequences and you try to restrict them to something smaller and they're equal.

If you have a sequence of Abelian groups, it's defined, maps from this into something is easy, but it's hard to map into this. If you want to map between two of these, that's messy. It's not even maps that go over and forward. If you assume finitely generated, then each of the A_1 will be realized after a finite step. Then there's a B_{i_1} that accepts a map of A_1. This will be a map of limits in the finitely generated case. In the compact case the same is true for us. What you could have said instead was, take a different representative inside each of these. These represent the same limit if you put the right restriction on. That's why we use this equivalence class. Now the maps can go directly between these guys.

I've defined maps of spectra now and it makes perfect sense to define homotopy of maps which is just crossing every step of the way with an I and mapping in. Then you can use homotopy equivalences and so on, and these start to look like spaces. The thing that looks different is just that the objects and the maps are encoded in these sequences. Most of what works for spaces works for spectra as well. Let me give an example for why spectra are better than chain complexes. Let's look at $\Sigma^{-3}S \wedge \Sigma^4 S$. Let me concretely define this as

$$X_k = \begin{cases} S^{k+4} & k \leq 2 \\ S^{k-3} \wedge S^{k+4} & k \geq 3 \end{cases}$$

We see that $H_*(X)$ is $\mathbb{Z}$ in dimensions -3 and 4 and 0 in other degrees. The wedge product in spectra behaves like direct sum in chain complexes. It has the same formal properties.

A fact, you can think about this, is that any chain complex with this homology is chain homotopic to $\mathbb{Z}[-3] \oplus \mathbb{Z}[4]$ with zero differential. This means that all chain complexes that look like this are essentially equivalent. But now you'll see, I'll construct a different spectrum with this homology and you'll see it's different.

Define $Y = [Y_k, \sigma_k]$ by the following. I'll still have one cell of each dimension like this, but I'll do a non-trivial extension of the two. I put the four-cell on very early, but I need the other cell to put it on in a non-trivial way. Let me define Y_k as the basepoint for $k \leq 20$. Let me write the first k explicitly, it's $S^18 \cup_\phi D^25$ but this ϕ is going to attach, it's a map from $S^24 \to S^18$. Now these are classified up to homotopy, and $[\phi] \in \pi_{24}(S^18)$ is in the stable range, this is in $\pi_6(S)$ then. So we've attached a cell and we're in a stable homotopy group, you can do $S^{k-3} \cup_{\Sigma^{21-k}\phi} D^{k+4}$. What do we end up seeing? Beyond this level, when we suspend we go to the exact same homology. The limit of something that starts out being zero and then becomes this homology A_* at twenty-one, and we just get a sequence of isomorphisms. This is what is meant by stabilizing. So it's just going to be A_*. It has the right homology. How do we see that it doesn't have the right homotopy type? We can think of this as, if you take a long exact sequence of homology, you see that you don't detect it. But stable homotopy groups have long exact sequences, but they're nontrivial for spheres in many dimensions. Looking at the long exact sequence for stable homotopy groups, we see that $\pi_6(Y) \neq \pi_6(X)$ which in fact tells us we get this element of ϕ. It becomes a quotient modulo this image. The important thing is that homotopy groups detect that they're not the same. What does this say about Conley indices? If something is homotopy equivalent to the wedge of two spheres, there's nothing stopping you from moving the one critical value below the other. If the attaching map is nontrivial then you can't move the one below the other. This is one of the main motivations for using spaces instead of their homology.

One more theorem I want to talk about is the Hurewicz theorem, slightly reformulated. If $f : X \to Y$ is a map of spectra which induces a homology isomorphism, then it induces a homotopy group isomorphism. I won't prove this, but it looks so much nicer here than for spaces, where you need an assumption about π_1. Basically, it's because you can take $[X_k, \sigma_k]$ which is equal to $[\Sigma^2 X_{k-2}, \Sigma^2 \sigma_{k-2}]$ which has no fundamental group. What does this tell us? That X and Y as constructed above have no maps $X \to Y$ inducing the homology equivalence. This is a little more about how these maps are different. So having maps of spectra is something strong.

That's all I wanted to say about spectra now. Now I want to go to finite approximations, which is very different. The Morse theory will only show up at the very end of the discussion. Let me start in $\mathbb{R}^{2n}$ and let me shorten finite dimensional approximation to FDA.

First let me explain quickly what I want to approximate. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth Hamiltonian. Then we have an associated Hamiltonian flow ϕ_t, which is the flow of X_H, $\omega_0(X_H, \quad) = dH$ or $\pm J_0 \nabla H = X_H$ [ed: some discussion about the sign, some disagreement] Then you have an action on loops $S^1 \to \mathbb{R}^{2n}$:

$$A(\gamma) = \int_\gamma \lambda_0 - H dt$$

Here $\lambda_0 = pdq$ where $\omega_0 = -d\lambda_0$

[What is your notation for the standard symplectic form?] You have $x_i + iy_i$ as a basis for $\mathbb{R}^{2n} = \mathbb{C}^n$ and say this is $dx_i \wedge dy_i$ or $dq \wedge dp$.

Okay assume H is equal to $H_0(z) = ||z||^2$ outside a compact set. We want something to keep the critical points from escaping.

Now look at critical points for A_H, these are 1-periodic orbits of ϕ_t. Now look at A_{H_0}. Now the flow ϕ_t associated with X_{H_0} is rotation with angle t, so multiplication in $\mathbb{C}^n$ by $e^{2\pi i t}$. It's not surprising that you integrate a vector field and get an exponential function. This means that 1-periodic orbits are in fact just given by $\gamma(s) = 0$. It's rotation, but if you take a rotation then you get the only fixed point as zero. One can check that this is a non-degenerate critical point and make sense of Floer homology, and then get to the conclusion that Floer homology of A_{H_0} is isomorphic to $\mathbb{Z}$ in degree 0 and 0 otherwise.

There's a big machinery to do this correctly, because things are infinite dimensional, you need Gromov compactness, transversality, perturbations, independence of perturbations. FDAs will let us avoid some of this machinery.

This is not a conclusion but something one can prove: this is well-defined for all such H. This means $FH_*(H) = FH_*(H_0)$ and so these have the aforementioned homology, so any H like this has a 1-periodic orbit. If it didn't have a periodic orbit then you'd already be Morse-Smale and then you don't have any critical points so you don't have a generator.

Now let me define a finite dimensional approximation of this action. Fix a large natural number $r \in \mathbb{N}$. We define $\Lambda_r\mathbb{R}^n = \{q_0, \dots, q_{r-1} \in \mathbb{R}^n\}$. The cotangent bundle of this is $\{(q_0, p_0), \dots, (q_{r-1}, p_{r-1}) \in T^*\mathbb{R}^n\}$. I'll shorten notation and have $\vec{q} = (q_0, \dots, q_{r-1})$, $\vec{p} = (p_0, \dots, p_{r-1})$, and $\vec{z} = (\vec{q}, \vec{p})$.

A paper from the early 80s does this (in a slightly different notation), let me just write it down. We have these points. I don't have the condition that they are close to each other. It's going to be a big mess when they're not close. Now define $\gamma_j : [0, \frac{1}{r}] \to \mathbb{R}^{2n}$ and $\gamma_j(t) = \phi_t(z_j)$, using Hamiltonian flow. Also define $z_j^- = \gamma_{j-1}(\frac{1}{r})$. Cyclically, we let $z_0^- = \gamma r - 1(\frac{1}{r})$.

Now I can write down a finite dimensional approximation $S_r : T^*\mathcal{L}_r\mathbb{R}^n \to \mathbb{R}$.

$$S_r(\vec{z}) = \sum_{j=0}^{r-1} \int_{\gamma_j} (\lambda_0 - H dt) + p_j^-(q_j - q_j^-)$$

We have these points and curves and we're integrating. It's unnatural for symplectic area to integrate something to give you symplectic area if it's not even closed. The other factors are there to make the area by closing up the curve in a particular canonical way, first real then imaginary. Why am I not just going directly? You don't get Conley index pairs. I'll explain that a different time. I'll be able to explain what goes wrong later.

Proposition 5.1. *For $r \gg 0$ (depending on H) the critical points are in correspondence with critical points of A_H with the same critical value. Further, $S_r, \nabla S_r$ is completely bounded (CB).*

This gives you good index pairs so you get the same thing up to homotopy.

I won't prove this because I only have six minutes left but let me draw a sketch and say something about the proof here. If you look at z_j, you have some flow line γ_j, then the connector that goes real, imaginary, and then the next flow line, think about it. If you want to take the gradient with respect to z_j, we only care about this local part. If you move the point z_j in some direction, you take the dual to get the gradient, the first thing that happens is that $z_{j+1}-$ moves. If r is large it moves in roughly the same direction. So the differential of $\phi_{1/r}$ is practically the identity. So approximately, $z_{j+1}^- = z_j + \frac{1}{r}X_H$. This picture looks almost like having these guys [gesture]. Let me say it like this. There's a calculation of the action of a curve like this. But we only care about the endpoints, which we want to push into where the symplectic value is bigger. It just wants to maximize the are underneath. Both of these have something to do about the symplectic area, the x and y motions too. The gradient, you get after a lot of work, is approximately equal to

$$\nabla_{z_j} S_r = (p_j^- - p_j, q_j - q_j^-) + o(||p_j - p_j^-||^2 + ||q_j - q_j^-$$

We get that critical points for S_r, eventually, are points where $p_j = p_j^-$ and likewise for q. This is the flow lines making a closed curve. So you get an actual flow curve. When $q_j = q_j^-$, in our integral we get the regular flow. The parts that don't come from the gradient look almost like a quadratic form, and those if notndegenerate are CB. In fact, in many cases this IS a quadratic form. If you scale this by a factor, you get the same flow factor, and the things change by the square. So that's sort of, this same argument works again for H_0. Let me stop there and say how to turn these things into spectra the day after tomorrow.