INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS DMITRY KALEDIN LECTURES ON HOMOLOGICAL METHODS IN NON-COMMUTATIVE GEOMETRY

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1. November 4

Let X be an algebraic variety. Consider the deried category of coherent sheaves on X. Can we recover X from this category? Sometimes the answer is yes, from a result from 95 if X is Fano or general type. In general the answer is no. If X and Y are varieties and $\mathbb{F}$ is in $\mathbb{D}^b_{coh}(X \times Y)$ with some conditions, then $\Phi_{\mathcal{F}}$ is a map from $\mathcal{D}^b_{coh}(X)$ to $\mathcal{D}^b_{coh}(Y)$ given by $R\pi_{2*}(\mathcal{F} \overset{L}{\otimes} L\pi_1^*\mathcal{E})$

This functor between categories is called a Fourier-Mukai functor. It may happen that $\Phi_{\mathcal{F}}$ is an equivalence even if $X \neq Y$. For example, if X and Y are K_3 surfaces or if X is an Abelian variety and F is its dual.

Definition 1.1. *X and Y are derived Morita equivalent if and only if there exists an $\mathcal{F}$ such that $\Phi_{\mathcal{F}}$ is an equivalence*

Question 1.1. *What invariants of X are derived Morita invariant?*

Answer 1.1.
(1) *Smoothness and properness*
(2) *Differential forms and polyvector fields*
(3) *Algebraic K-theory*
(4) *the de Rham differential and the Lie bracket of vector fields*
(5) *Hodge theory (the algebraic part of it). This doesn't require any analysis.*
(6) *Cristalline cohomology with a Frobenius action (this is characteristic p)*

I'd like to start with the derived category and come up with all of these invariants. Let A be an associative algebra over $\mathbf{k}$. I'll forget that the algebra of functions is commutative. I can go from this to an Abelian category (say the category of A-modules or something else). I could instead go a different way and consider a differential graded algebra. This is an algebra with a differential that satisfies the Leibniz rule. I could also consider the derived categories of either of these two generalizations. I can start with an associative algebra and do these things. This part kind of only gives you the affine case. If we go all the way to the end to a triangulated category we can't do much but what we need is some kind of enhancement. There is no simple notion of an enhancement. These days the most developed thing is differential graded categories. You want something like the complex of maps from $\mathcal{E}$ to $\mathcal{F}$. But you run into trouble. $Hom*(\mathcal{E},\mathcal{E})$ should be a dg algebra but it only recovers the homology algebra. That's too little. We start with the *dg* category then, but then you also want a notion of equivalence. An object there is considered as some sort of non-Abelian variety. It's important to start somewhere centered. Next time i'll go to Hochschild and differential forms, etc. [missed].

Let's start doing something. Let A be a differential graded algebra over $\mathbf{k}$, which we will take to be a commutative ring. I want to assume that A is flat as a $\mathbf{k}$ module (in each degree). Then we have an Abelian cotegory of left differential graded modules over A. Formally, the definition of the derived category is very easy. It's just formed by taking the category of A-modules and inverting quasiisomorphisms.

There's a general stupid categorical construction when you have $\mathcal{C}$ and a class W of maps in $\mathcal{C}$. Then $\mathcal{C}[W^{-1}]$ is the universal category under $\mathcal{C}$ so that if $\mathcal{D}$ is a category then whenever $w \in W$, $F(w)$ in D is invertible.

Objects are usually unchanged but morphisms are presented by dagrams of the following shapes (zig-zag) with the property that all maps in the wrong direction lie in W.

This is useless, how do you work with this?

Definition 1.2. *A dg module M is cofibrant if for any surjective quasiisomorphism $N_1 \to N_2$ of dg modules, the map $Hom(M, N_1) \to Hom(M, N_2)$ is surjective.*

Lemma 1.1. *A is in degree 0. Then a complex of projective A-modules bonded from above is cofibrant.*

Proof. The proof is by construction. I want to lift as follows

$$\begin{tikzcd} & N_1 \arrow[d] \\ P \arrow[r] \arrow[ur, dotted] & N_2 \end{tikzcd}$$

So by induction, let's see what we have:

$$\begin{tikzcd} P^i \arrow[r, "g"] & N_1^i \arrow[r, "h"] & N_2^i \\ P^{i-1} \arrow[u, "d"] \arrow[rr, bend right, "f_1"] & N_1^{i-1} \arrow[u, "d"] \arrow[r] & N_2^{i-1} \arrow[u] \end{tikzcd}$$

We can create f_1 by projectivity but we don't know if $d \circ f_1 = g \circ d$. However, we do know that $d \circ f_1 - g \circ d$ is in the kernels of h and d.

But because h is a quasiisomorphism, the complex $Ker\ h$ with d has no homology (by the long exact sequence). So we have a map into the kernel of the differential. So this is also the image of the differential, and we can lift that map to $Ker\ h^{i-1}$. There is a map f_2 so that $d \circ f_2 = d \circ f_1 - g \circ d$. Then I can switch to $f_1 - f_2$. □

For rings we don't need to do this but for modules we do.

A cofibrant replacement of M is a cofibrant P equipped with a surjective quasi-isomorphism to M

Lemma 1.2. (1) *for all M there exist a cofibrant replacement and*
(2) *any two such are equivalent.*

It's a little hard to show some of this.

If M is cofibrant then it is enough to consider maps from $M \to M'$ because you can lift back and forth.

[break]

[some missed]

A map f of algebras gives a map f_* from $\mathcal{D}(A_2) \to \mathcal{D}(A_1)$ which has a left adjoint f^*.

Now $M \to A_2 \otimes_{A_1} M$ does not necessarily preserve quasiisomorphism, but it does for cofibrant M, so $f^*(M) = A_2 \otimes_{A_1} P$ where P is a cofibrant replacement for M.

Corollary 1.1. *If f is a quasiisomorphism then f^* is an equivalence of categories.*

Another application is homotopy colimits. Consider a small category I and diagrams of shape I in $\mathcal{C}$, denoted $\mathcal{C}^I$. Then W_I maps in $\mathcal{C}^I$ pointwise in W (so-called). If F_1 and F_2 are two diagrams, then a map of diagrams is a bunch of maps $F_1(i) \to F_2(i)$ making everything commute. a map is a weak equivalence if and only if every constituent map is a weak equivalence. We have a pullback that embeds $\mathcal{C}[W^{-1}]$ into $\mathcal{C}^I[W^{-1}]$ and the homotopy colimit is left adjoint to the pullback.

Lemma 1.3. *For $\mathcal{C} = A - mod$, and W all quasiisomorphism, every homotopy colimit exists.*

This is particularly easy when I Is a directed partially ordered set.

In this case the homotopy colimit is the colimit.

One point that is important is that $\mathcal{C}^I[W^{-1}]$ is typically very different from $\mathcal{C}[W^{-1}]^I$. So if you only have the derived category it's not enough.

For many purposes it's important to consider small subcategories. For instance you often want to pick out only coherent sheaves within quasicoherent sheaves. In this case, these are called perfect dg modules.

Exercise 1.1. *A set is finite if and only if for any directed poset and any functor σ from I to Sets, the natural map $\lim Maps(S, \sigma(i)) \to Maps(S, \lim(\sigma(i))$ is an isomorphism.*

This is a very general phenomenon. If we put in vector spaces instead we get a notion of finite dimensional vector spaces.

Definition 1.3. *A dg module M is perfect if for all directed I and directed inductive system N_i, we have $limHom(M, N_i) \to Hom(M, hocolimN)$ is a quasiisomorphism.*

Definition 1.4. *A module is a* finite cell module *if M is concentrated in degrees* 1 *through n and the cone on the map M_i to M_{i+1} is a free A-module (possibly shifted).*

Lemma 1.4. *M is perfect if and only if it is a retract of a finite cell module M_1.*

A thing is a retract, well, if you have an idempotent $p : C \to C$ if you have an object C' (which is unique if it exists) such that p factorizes as $a \circ b$ where $b \circ a = id_{C'}$. This is totally categorical and very common.

Projective modules are retracts of free things, for instance.

Proof. One can show that if we drop the finiteness condition, then every object can be realized as the homotopy colimit of finite cell complexes over some directed set. You take M and want to represent it as something with shifts. The index set can be very large, but it's possible. The map from M to the big homotopy colimit factors into one of the terms. So M is a retract as desired. $\square$

The last thing I want to do today, let $D^{pf}(A)$ be the full subcategory of perfect modules. Then assume we have a directed inductive system of dg algebras. Take its limit. Then the category of prefect modules over A is the limit of the categories of perfect modules over A_i. A corollary of the characterization of perfect modules is that the transport functor takes perfect modules to perfect modules.

For any M in $D^{pf}(A)$, M is of the form $A \otimes A_i M_i$ for some perfect A_i-module. Second, for any two M_1, M_2 that are perfect A-modules, the space of maps [missed end of lemma and sketch of proof].

2. November 6

So last time, I was considering A a differential graded algebra over $\mathbf{k}$. This means associative and usually unital, but almost never commutative. For such a guy I defined a derived category $\mathcal{D}(A)$. For a map $f : A_1 \to A_2$ I defined a pair of adjoint functors between the derived categories $f^* : \mathcal{D}(A_1) \to \mathcal{D}(A_2)$ and $f_*\mathcal{D}(A_1) \leftarrow \mathcal{D}(A_2)$. If f is a quasiisomorphism then f^* and f_* are equivalences. It makes sense then to consider algebras only up to quasiisomorphism.

More precisely, I want to localize the category of dg algebras at quasiisomorphisms. We want to do the same thing for algebras. I used cofibrant replacement for modules that let me construct the functor I wanted. In fact, there is an axiomatic setup for things like this. It's the notion of a model category. I want to give some brief introduction to this not proving much but just giving some idea. This is due to Quillen ("Homotopical algebra" in 67 or something, one of the first lecture notes). He says that certain settings come up often, this is my axiomatization, it's ad hoc but it'll do and we're still using it 50 years later.

Let $\mathcal{C}$ be a category. A model structure on $\mathcal{C}$ is the collection of three classes of morphisms C, W, and F (cofibrations, weak equivalences, and fibrations) such that:

(1) These classes are closed under composition and retract. Retract means the following. You have a map and some projections with maps back

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, dotted, bend right] & B \arrow[d, dotted, bend right] \\ A' \arrow[u] \arrow[r, "f'"'] & B'] \arrow[u] \end{tikzcd}$$

Here you have f' a retract of f if there is a one sided inverse like this.

(2) The two out of three axiom; if $f, g, f \circ g$ have two out of three weak equivalences, then the other is as well.

(3) If you have maps in the following square:

$$\begin{tikzcd} M \arrow[r] \arrow[d, "a"] & M' \arrow[d, "b"'] \\ N \arrow[r] & N' \end{tikzcd}$$

If a is in $C \cap W$ and $b \in F$ or $a \in C$ and $b \in F \cap W$, then you can lift from $N \to M'$ so that the whole thing commutes.

(4) Any map $f : M \to M'$ decomposes as $M \to^{C \cap W} N \to^F M'$ and $M \to^C N' \to^{F \cap W} M'$

Definition 2.1. *A model category is a category with all finite limits and colimits and equipped with a model structure.*

I don't think there's any convenient way to lose the requirement about finite limits and colimits. We could define a fibrant object to be one where the map to the terminal object is a fibration and a cofibrant object to be one where the map from the initial object is a cofibration.

Example 2.1.
(1) *Topological spaces (good). Cofibrations are closed embeddings. Weak equivalences are weak homotopy equivalences, and fibrations are defined by the lifting axiom. They're all maps that have the lifting property with respect to trivial cofibrations.*
(2) *Let A be a ring. Then let $\mathcal{C}$ be complexes of A-modules. W will be quasiisomorphisms. We can have two categories, one where fibrations are surjective maps and cofibrations are defined by the lifting property. Coifbrant objects are complexes of projective modules. I should set some sort of finiteness condition. Alternatively, I could say that cofibrations are injective maps and fibrations are defined by the lifting property and fibrant objects are complexes of injective modules.*

Let me give you some yoga about this.

Lemma 2.1. *b has a lifting property with respect to $a \in C \cap W$ if and only if b is a fibration.*

Proof. Factorize b as a trivial cofibration followed by a fibration

$$\begin{tikzcd} M \arrow[r, "id"] \arrow[d, "C\cap W"'] & M \arrow[d, "b"] \\ M' \arrow[r, "F"] \arrow[ur, dotted] & N \end{tikzcd}$$

Then b is a retract of b'. □

Analogously, anything with the lifting property with respect to trivial fibrations is a cofibration.

This has a corollary that says fibrations are closed under pullbacks. If you have a pullback square

$$\begin{tikzcd} M \arrow[r] \arrow[d, "b'"] & N \arrow[d, "b"] \\ M' \arrow[r] & N' \end{tikzcd}$$

So if b is a fibration then so is b'. Dually, cofibrations are closed under pushouts.

Here is one of two main results that Quillen has in his book.

Proposition 2.1. *(Quillen) Let $\mathcal{C}$ be a model category. Maps in $\mathcal{C}[W^{-1}]$, called $Ho\mathcal{C}$, can be represented in the following way:*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & I \\ M & N \arrow[u] \end{tikzcd}$$

where P is cofibrant replacement for M, which is a factorization of the map $0 \to M$ into a cofibration followed by a trivial fibration. Likewise I is a fibrant replacement for N.

We didn't have I in our example, but everything is fibrant in our example, so we can take $N \to I$ to be the identity.

Let me show you how to compose two such maps.

We have

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & I & P' \arrow[r] \arrow[dll] & I' & \\
M & & M \arrow[ul] & & M''
\end{tikzcd}$$

We have that $P' \to I$ is a weak equivalence which we can factorize as a trivial cofibration followed by a trivial fibration. We get the following:

$$\begin{tikzcd}
& {} & N \arrow[dl] & & \\
P \arrow[r] \arrow[d] & I & P' \arrow[r] \arrow[dll] \arrow[ul] & I' & \\
M & & M \arrow[ul] & & M''
\end{tikzcd}$$

We can pull back on one side and push out on the other and get

$$\begin{tikzcd}
Q \arrow[rr] \arrow[d, "F\cap W"] & & N \arrow[rr] \arrow[dl] & & Q \\
P \arrow[r] \arrow[d] & I & P' \arrow[r] \arrow[dll] \arrow[ul] & I' \arrow[ur, "C\cap W"] & \\
M & & M \arrow[ul] & & M''
\end{tikzcd}$$

Then we erase everything and write just

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] & Q' \\
M & M'' \arrow[u]
\end{tikzcd}$$

This is one proposition in this story. Then the other has to do with constructing adjoint functors at the level of homotopy categories. This shows that if you have adjoint functors on the categories, then the derived functors will be adjoint at the level of homotopy categories. Apply the functor to the cofibrant replacement. It works in complete generality once you have something satisfying the axioms.

There is still some difficulty but the difficulty has been moved to constructing the model structure. Let me give you the statements about model structures.

Proposition 2.2. *This is rather old. The category of differential graded algebras has a model structure whose weak equivalences and fibrations are the same as for complexes, quasiisomorphims and surjective maps.*

Such a model structure is unique, if it exists you get uniqueness of cofibration.

Definition 2.2. *A cell algebra is $k\langle x_i\rangle$, freely generated by x_i where I is a filtered partially ordered set. There is a differential. The condition is that dx_i (a linear combination of monomials) sits inside the algebra generated by variables with index less than i.*

It's an easy exercise to show that you can always find an algebra that is equivalent to a given algebra and has this form. Algebras like this are cofibrant.

You need to show that everything has a cofibrant replacement, but that's easy.

I don't want to go into full detail, but just a remark. The story is different if your algebras do or don't have a unit. There's a functorial way to do cofibrant replacement for a dg algebra which is associative but without unit. It's roughly speaking called the double bar construction. If you have some A, we can interpret the multiplication which is a map from $A \otimes A \to A$ as, well, consider BA, the tensor coalgebra of A shifted by 1, so this is $\bigoplus A^{\otimes n}[n]$. A very pleasant exercise is that the associativity on A corresponds to the coderivation extending m on BA squaring to zero. Professor Oh's lectures, he did this in great detail. So this guy with the coderivation becomes a differential graded coalgebra. You do it once again. Consider BBA, this is coalgebra, so we do the opposite shift, this is $T(BA[-1])$. If you want to do this explicitly, this is a sum over an integer which has been partitioned of $A^{\otimes n}$ which we think of as being partititioned. All of the integers involved are at least one. Then again, the coalgebra induces a codifferential. You need to check that the differential squares to zero. This is a triple complex with three different differentials. They all commute and all square to zero. We get the total complex. This is an algebra. Since it's free, it's easy to check that it's an algebra of the cell algebra kind. There is an algebra map from $BBA \to A$ and there is an inverse map on complexes that embeds A as the component where $n = 1$. It's very easy to show that it's a quasiisomorphism. The differential is acyclic on all terms with $n \geq 2$. Just combinatorially, the number of partitions of n is 2^{n-1}. The differential, if you take fixed n, you take this sum, this will be $A^{\otimes n} \otimes k[\epsilon_1, \dots \epsilon_{n-1}]$. These are all acyclic. This will generally be a tensor product of $n-1$ copies of $k \to k$. So the map is a quasiisomorphism. We get a canonical quasiisomorphism.

This is useful because you can modify the construction. There are nontrivial coderivations that do not come from products. The general form is an A_∞ structure on A. You still get a quasiisomorphism and this will be an A_∞ quasiisomorphism. Every A_∞ algebra is canonically isomorphic to a dg algebra. You say there is a model category structure on A_∞ algebras and a functor and the homotopy categories are exactly the same. It's absolutely equivalent how you work if you're interested in the homotopy category. All the theorems I have for dg algebras apply too for A_∞ algebras. You can always get rid of higher corrections. If you do this with units it doesn't work.

For units you have to work separately. There are several notions and you have to work separately. If you want to be very conceptual you say there is a notion of algebras over operads, that quasiisomorphic operads give you the same homotopy category of algebras. You say there is a model category on operads and that the A_∞ operad is a cofibrant replacement for the associative operad.

Let me turn now to dg categories. It can turn out that two different dg algebras have the same derived category. I need to generalize the notion of an algebra. The slogan is that this should be a dg algebra with many objects. It's a collection, a set of objects S. This is over $\mathbf{k}$. For any two objects we have a complex of morphisms and then have compositions and identity maps satisfying all the usual axioms. If S is just one point, this is just a unital algebra.

For any dg category A we have a genuine, a **k**-linear category $H^0(A)$, which has the same objects and morphisms are zero homology groups of our morphisms. A dg module is like a functor to the category of complexes.

A module is a collection of objects M_s and actions $A(s', s) \otimes M_s \to M_{s'}$ (up to my confusion of right and left). An A-module has a projective model structure and the rest of the story. We have a derived category $\mathcal{D}(A)$ and we also have a Yoneda embedding from $H^0(A)$ to $\mathcal{D}(A)$. Every object s goes to M_s whose value at s' is $A(s, s')$. I get confused about left and right modules, I might have that reversed.

A dg functor $A_1 \to A_2$ is again defined in the obvious way. We have a map on the sets of objects. Then for every pair of objects you have a map on morphisms $F : A_1(s, s') \to A_2(F(s), F(s'))$ This should be compatible with composition. Every F induces a pair of adjoint functors as in the algebra case between $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$. The pullback is trivial again and the the pushforward requires adjunction.

Definition 2.3. *F is a quasiequivalence if and only if $H_0(A_1) \to H^0(A_2)$ is an equivalence of categories and it is fully faithful in the dg sense. For any s and s', the map on morphisms is a quasiisomorphism.*

If F is a quasiequivalence then F^* and F_* are equivalences.

Theorem 2.1. *(Tabuada) There exists a model structure on dg categories whose weak equivalences are quasiequivalences.*

Definition 2.4. *F is a Morita equivalence if and only if f^* and f_* are equivalences.*

Theorem 2.2. *There is another model structure whose weak equivalences are Morita equivalences.*

For many applications you never go to the gritty business of actually working in the model categories, so the details are irrelevant, unless you want to prove it or something. This gives us what we actually want: noncommutative algebraic varieties are the same thing as objects in the homotopy category of dg categories with weak equivalences Morita equivalences.

We want to work with small dg categories up to Morita equivalence in this sense. Let me give you an example. Assume that you have A and B both dg categories. You can take $A^{opp} \otimes B$, where objects are pairs and morphisms are tensor products. We can consider $\mathcal{D}(A^{opp} \otimes B)$. Consider P living there.

Definition 2.5. *We call P* pseudoperfect *if it is perfect over B.*

For example, if we have a functor $F : A \to B$ then we can consider the graph of it, which is B, with an action of B on the right and A on the left. This is pseudoperfect.

For every guy who lives where P lives gives a functor $\mathcal{D}(A) \to \mathcal{D}(B)$ where $M \mapsto P \otimes^L_A M$. This has to be a derived tensor product. Then if P is pseudoperfect, then the functor sends perfect modules over A to perfect modules over B. This is if and only if.

A typical perfect module over A will be A and inserting that means that I'll get P back. The same is true for shifts and finite extensions so we're done.

Theorem 2.3. *(Töen) In the homotopy category, $Maps(A, B)$ are pseudoperfect [missed]*

One final remark. As I said, a dg algebra is an example of a dg category. Here is a question. Up to quasiequivalence, there are dg categories which are not equivalent to dg algebras. So when is a dg category Morita equivalent to a dg algebra? The answer is:

(1) If the set of objects is finite, then this is obviously the case. You take the sum of all objects of the morphism algebras. This has to be finite to get a unit here.

Definition 2.6. *An object in a triangulated category $\mathcal{D}$ is a generator if when $Hom(\mathcal{E}, \mathcal{F}) = 0$ then $\mathcal{F} = 0$.*

Then the claim is that, assuming we have a dg category A, and assuming it has a compact generator, an $\mathcal{E}$ which is a generator in $\mathcal{D}(A)$ which is perfect.

Perfect objects are built out of representable functors, shifts, finite sequences of cones of shifts, retracts of those. You can make an $RHom$ algebra of morphisms from this to itself and A is Morita equivalent to $RHom(\mathcal{E})$. I told you I won't tell you about the model structure, let me tell you one thing.

Recall that we have the embedding $H^0(A)$ to $D^{pf}(A)$. It's called pretriangulated if this is an equivalence. We can ask for the image to be closed under cones and shifts and then direct sums, projectives. It's easy to do something for the first case. The point is that if A is cofibrant with respect to the Morita model structure then A is automatically pretriangulated. This is more or less an equivalence. Roughly speaking it's true. Anyway, when you do cofibrant replacement, you get something pretriangulated. Working up to Morita equivalence, you can see then that cofibrant Morita things have a generator and so you have a dg algebra that the category is equivalent to.

Theorem 2.4. *(Bondal, Von den Bergh) A smooth quasiprojective X, the derived category of bounded coherent sheaves has a compact generator.*

This is, roughly speaking, Morita equivalent to a dg algebra. It's important to know that our objects are dg algebras. The maps are really given by bimodules. If you want those maps you have to go to that category first.

This is where I stop today.

3. November 11

Definition 3.1. *A is of finite type (or homotopically finitely presented) if for any filtered inductive system of dg algebras B_i, we can consider $Hom(A, hocolim B_i)$ or $lim Hom(A, B_i)$. So it's finite type if this is an isomorphism.*

So what are examples of algebras of finite type. If you remember when I discussed cofibrant replacement I had the example of a cell algebra, which is obtained by attaching generators one by one, which are ordered, and the differential lies in the subalgebra spanned by the previous ones. You can see that a finite cell algebra satisfies the condition. In this case, the condition is that $dx_j \in k\langle x_i, \dots x_{j-1}\rangle$.

There's also the notion of retract of finite type algebras. You have some endomorphism that squares to itself. There is a lemma due to Töen which says that the homotopy category of dg algebras has images of idempotents. The problem is that you want to find something $A \to B \to A$ so that going in this direction you get your idempotent, but the other direction is the identity. A factorization like this is unique if it exists. There is no notion of a general image so it's not clear it exists.

Then there is a proposition, just the same as for perfect modules.

Proposition 3.1. *A in the homotopy category of dg algebras is finite type if and only if it is a retract of a finite cell algebra.*

The proof is easy. You represent A as a certain possibly infinite cell algebra and then as usual it's a direct limit of its finitely generated subalgebras. You have a map that induces an isomorphism, and that gives you the retraction. Informally, this means that an algebra of finite type is defined by a finite amount of data. You need to assign degrees to the generators. You need differentials, you need the retraction. That's all finite data. If you have some algebra of finite type over a large ring, it's actually defined over a finitely generated subring.

If you have a quasiprojective variety defined over some ring, it's defined over a finite dimensional subring. This is true because we work over Noetherian rings in algebraic geometry. You can use Noetherian in a noncommutative setting but it's too restrictive when you lose commutativity.

This is bad maybe because a commutative ring in degree zero may be of finite type as a commutative ring but not of finite type homologically. In the case of commutative algebra over $\mathbf{k}$ this would imply that the algebra was smooth.

Now there is another notion that is useful.

Definition 3.2. *A dg algebra A is called* compact *or homologically compact if A is perfect over* $\mathbf{k}$ *and* homologically smooth *if A is perfect over $A^{op} \otimes A$.*

It's always perfect over itself, but it's also a module over these other rings where it doesn't have to be perfect. The first is that there should only be finitely many homology groups each of which is finite over the base field. You have A a bimodule over itself. Then there should be a finite length resolution of things of finite size. So this implies that A has finite homological dimension, which is a strong condition often.

Theorem 3.1. *If A is Morita equivalent to an algebraic variety X. We say if A is Morita equivalent to an algebraic varity X then A is compact if and only if X is proper and A is smooth if and only if X is smooth over* $\mathbf{k}$.

For example, $k' \supset k$ is a finite extension of a field. Then k' is smooth over k if and only if k' is a separable extension. If the extension is not separable then $k' \otimes_k k'$ is never of finite type, it's some kind of nilpotents.

These are conditions which are geometric. What is the relation between compactness, smoothness, and finite type?

If something is smooth and compact then it is automatically of finite type, and if something is of finite type then it is automatically smooth (but maybe not compact). Both conclusions are sharp.

Let me explain. In order to do this, I need some reformulation of the definition. I have maps between these things represented as bimodules. You have a bimodule $M \in \mathcal{D}(A^{op} \otimes B)$ for A and B dg algebras. We call M pseudoperfect by definition if it is perfect over B.

If you have a pseudoperfect guy, then the corresponding functor $\otimes M : \mathcal{D}^{pf}(A) \to \mathcal{D}^{pf}(B)$, it sends perfect modules to perfect modules.

Lemma 3.1. *A is compact if and only if for any B, we can consider pseudoperfect and perfect modules. We can ask about the relation. The answer is that A is* *compact if and only if every perfect guy is pseudoperfect . A is smooth if and only if every pseudoperfect guy is perfect.*

This looks intimidating because there's lots to check but in fact it's quite easy. To check the first part, take $B = \mathbf{k}$ then A^{op} is obviously perfect so it should be pseudoperfect. That's the definition of being compact. In the other direction, if A is perfect over $\mathbf{k}$ and B over itself then $A^{op} \otimes B$ is perfect over $\mathbf{k} \otimes B$.

The other direction is pretty straightforward as well. Take $B = A$. Then A is pseudoperfect so it's perfect over $A \otimes A^{op}$. So it's smooth. In the other direction, take a pseudoperfect module P. Then $P = P \otimes_A A$ which is a perfect module.

Now let's assume A is smooth and compact and conclude that it's of finite type. We have to check this system, take this inductive system B_i. We have f, and then A acts on one side by f and B on the other side (of $g(f) = B$). If you forget one side of this then we have B as a B-module. A map is the same as a bimodule plus a trivialization. We also have an A-action, so we have a map $A^{op} \to End_B(g(f))$. It should be clear that $g(f)$ is pseudoperfect. Then it's also perfect since A is smooth. The product algebra is $hocolim(A^{op} \otimes B_i)$. We saw that this is a property of perfect modules, that something like this comes from one of the factors. This is isomorphic to $M_i \otimes_{B_i} B$ for perfect module M_i.

Since A is compact, now, we can conclude that M_i is perfect over B_i. Now we use the same statement but apply it to modules over B_i. For some $j > i$, the map $\tau : g(f) \cong B$ comes from an isomorphism of B_j-modules $\tau_j : M_i \otimes_{B_i} B_j = B_j$. Already at this level we have such a bimodule $M_i \otimes_{B_i} B_j \in \mathcal{D}(A^{op} \otimes B_j)$ but such a pair must come from a map $A \to B_j$. We can't say anything about the map, but we can replace the map with a bimodule and an isomorphism. The map must appear at some finite level, so we're done. It looks strange, lots of playing with definitions. I don't know any direct proof.

[Can you check compactness and smoothness?] Compactness is usually easy because you have a presentation. Smoothness, you have to build up a resolution. Homologically, finite type is also building a resolution. If you have, say, a stack, you can look at coherent sheaves on the stack, and if the stack is smooth, locally a quotient of a smooth thing by a finite group, as long as the group doesn't add homology, you're okay. If you have control of the homological dimension of your category for some reason. If you know the Hochschild homology is nice, you have reason to expect it.

This is finiteness conditions for dg algebras. Let's talk about Hochschild homology and then at some point we'll have a little break.

3.1. **Hochschild homology.** We have an associative unital algebra A over $\mathbf{k}$, I'll assume it's flat, and we have M which is an A-bimodule. I'll treat this as a derived functor. I start by defining $HH_0(A, M)$. That's easy to define. You take M and the quotient by the commutators, $M/[am - ma]$. You can also write it down as $A \otimes_{A^{op} \otimes A} M$.

Definition 3.3. *$HH_i(A, M)$ is the derived functor of $HH_0(A, M)$.*

A couple of general remarks. First, the Hochschild Kostant Rosenberg theorem. Let me give the statement.

Theorem 3.2. *HKR If A is commutative and finitely generated over $\mathbf{k}$ (in the commutative sense) and smooth ($X = Spec\ A$ is smooth) then there is an equivalence between $HH_i(A, A)$ and $H^0(X, \Omega^i_X)$*

Let me give a sketch of a proof. So modules over $A^{op} \otimes A$, this is commutative, so think of sheaves on $X \times X$. You have the diagonal. You have to compute $\mathcal{O}_\Delta \otimes^L_{\mathcal{O}_{X\times X}} \mathcal{O}_\Delta$. You need to choose a resolution which we can do locally. We can use the second projection to X and you have, you start with $\mathcal{O}_\Delta$, we have the kernel $\mathcal{J}$ of $\mathcal{O}_{X\times X} \to \mathcal{O}_\Delta$. We can find a map from the pullback of $\Omega^1_X \to \mathcal{J}$ which is surjective on the diagonal. It doesn't have to be surjective everywhere but it's surjective on a Zariski open neighborhood, remove the rest. We extend the resolution by taking the exterior algebra of $\pi_2^*\Omega^1_X$. The differentials are induced by the map ϵ from that to the kernel.

In a Zariski neighborhood of Δ this is a resolution of $\mathcal{O}_\Delta$. When computing this, you can restrict to such a neighborhood, and we get exactly what we want.

I didn't use characteristic anywhere in the proof, but this can be made slightly better in characteristic zero. Maybe not now, but later maybe I'll introduce some formulas and there will be a map.

Now, this motivates some interest. Let me now build up to a theory of this kind of homology.

One obvious property which I'll call a trace property. Assume you have A and B algebras. Then you have some M which is a module on which A acts on the right and B on the left. Then N is the other way around.

Claim 3.1. *You have a canonical isomorphism between $HH_0(A, M \otimes_B N)$ and $HH_0(B, N \otimes_A M)$.*

I don't write that this is a lemma because it's very obvious. Yu take $M \otimes N$ over $\mathbf{k}$. You impose two types of relations. You mod out by $mb \otimes n - m \otimes bn$ to make it a tensor product over B. Then you need to impose $am \otimes n - m \otimes na$. If you look at this, this is symmetric, the relations, they just come up in a different order. The corollary that you don't have to prove because it comes from the general machinery of derived functors that if you take the derived tensor product you get the same thing in higher degrees

$$HH_i(A, M \otimes^L_B N) \cong HH_i(B, N \otimes^L_A M).$$

Now I want to extend the whole business to dg algebras. This is the statement I want to keep. Now it's a good time for a break.

I want now to extend this story to differential graded algebras and bimodules. You can do the same to dg algebras but you can also do this more explicitly. When you do Hochschild homology, there is a completely canonical complex you can look at.

So Hochschild complex. When I take this theorem, I chose a resolution that was convenient for me. I'll resolve either A or M. I'll take the bar resolution, which will be a resolution for M which is flat as a derived A-module. The terms are $A^{\otimes n} \otimes M$. There is a certain differential b' which works as follows. You think of the terms geometrically. Put $n+1$ marked points on an interval. At each point you put either A or M. You put M in the last point and over the others you put A. If you take an interval here, what we can do is contract it. The contraction operation gives us an interval with one less cell. To each such operation I define a map $m_e : A^{\otimes n} \otimes M \to A^{\otimes n-1} \otimes M$ which works as follows. It's a tensor product of identity maps but the edge, we go to one point. Depending on where this was sitting, we have either two copies of A going to A or one copy of A and one of M. There is a natural multiplication map $A \otimes M \to M$ and $A \otimes A \to A$. Also there is a map in the other direction. So what I want to do is take a map which is a product of all identities and one multiplication. Then this b' is just the alternating sum over all edges $(-1)^i m_i$. These are all A-bimodules. So this is a complex, b' is a map of A-bimodules. It's easy to show that the whole thing is actually acyclic. There is a map h which sends $a_0 \otimes \cdots \otimes m$ to $1 \otimes a_0 \otimes \cdots \otimes m$. So if I start at $A \otimes M$ then I get just a resolution of M. So then we get a canonical complex to compute Hochschild homology $CH(A, M)$ which has the following shape.

Its terms are exactly the same because if we apply the functor $HH_0(A, A \otimes M')$, you get M'. You get the same terms $HC_i(A, M) = A^{\otimes i} \otimes M$ for $i \geq 0$. You get a new differential b which is similar but with a different graphical picture. You draw a circle instead of an interval. You mark one point specially with M. You can contract edges as before, getting something with fewer copies, and the b is the alternating sum of thoes m_ℓ. The difference is just in the one term. You had an interval with two ends at the bottom, and you had one contraction, so $b'(a \otimes m)$ was am. In this complex, you have to think about the circle. To get from one to the other, there are two edges you can contract. We have two terms and there's a question about the order of the product which comes from the order of the circle. They'll be multiplied in the opposite order. So here we get $am - ma$. The 0 Hochschild homology was the cokernel of this map, so it's not surprising.

If you now have dg algebras, for a dg algebra, you can define

Definition 3.4. *$HH_i(A, M)$ is the homology of the total complex $CH(A, M)$. There are potentially an infinite number of terms, so take the sum total.*

This thing is invariant under quasiisomorphism with the sum total. You have a map of pairs and the maps are quasiisomorphisms of algebras and modules.

Now you want a version for categories. This has been worked out by Keller. You just have many objects.

If A is a small dg category with the set of objects S and M is an $A^{op} \otimes A$-bimodule, then we need to say the complex

Definition 3.5.

$$CH_n(A, M) = \bigoplus_{s_0, \ldots, s_n \in S} A(s_0, s_1) \otimes \cdots M(s_n, s_0)$$

For example, in degree 0, I put $\bigoplus M(s, s)$. In degree one I'd have two objects and a morphism from A and another from M. The differential is exactly as in the picture.

The main theorem,

Theorem 3.3. *$HH(A)$ is derived Morita invariant.*

This is not obvious. Already for quasiequivalent categories you have something to prove here. You get different complexes. But my trick is to first use the lemma that says that Hochschild homology has this tracelike property:

Lemma 3.2.

$$HH(A, M \otimes^L_B N) = HH(B, N \otimes^L_A N)$$

This is easy to prove, it's the same as for the algebra case. So how does this follow? Assume I have two dg categories A and B and a dg functor $f : A \to B$. This generates a pair of adjoint functors $f^* : \mathcal{D}(A) \to \mathcal{D}(B)$ and f_* in the other direction. It's very convenient to have these represented by bimodules. I can define two bimodules naturally. I can consider $g(f)$ (for graph of f) which is a bimodule over $A^{op} \otimes B$. I get an object and the associated complex to it. $g(f)(s_1, s_2) = B(s_1, f(s_2))$. We can also look at $g(f)^\vee$ which is in $\mathcal{D}(B^{op} \otimes A)$ which is $B(f(s_1), s_2)$.

An observation or an exercise. I realized in preparing the lecture that I wasn't sure but then I got it so I leave it as an interesting exercise. These give you exactly f^* and f_*. That is,

$$M \otimes_A g(f) \cong f^*(M); N \otimes_B g(f)^\vee = f_*(N)$$

Proof of the theorem. If $f : A \to B$ is a Morita invariance then the Hochschild homologies are the same. So f^* and f_* are mutual inverses. So $HH_*(A, A)$, we have $A = g(f) \otimes_B g(f)^\vee$ and $B = g(f)^\vee \otimes_A g(f)$. So this first one is $HH(A, g(f) \otimes_B g(f)^\vee)$ and the second is $HH(B, g(f)^\vee \otimes_A g(f))$. Of course, all these tensor products are derived. My tracial property proves the invariance. □

There is actually more which allows us to compute Hochschild homology where things are not affine.

Definition 3.6. *A sequence, say I have three dg categories A, B, and C, and two maps between them. Then the sequence is called a* localization sequence *if $f_* \circ f^* \cong id$ and $g^* \circ f_* = id$. Also*

$$g(f) \otimes g(f)^\vee \to B \to g(g)^\vee \otimes g(g)$$

is a distinguished triangle in the category of $B^{opp} - B$ derived modules. This is like a long exact sequence in homology.

Here's an example. You have X containing closed Z and the open complement U. Then you have the category of coherent sheaves on X, let me choose a dg enhancement for this. There's the category of coherent sheaves on U. There's a functor from coherent sheaves on X to coherent sheaves on U which is restriction, and the kernel is coherent sheaves on X supported on Z. Then this is a localization sequence.

The theorem due to Keller is that

Theorem 3.4. *For a localization sequence we have a long exact sequence $HH(A) \to HH(B) \to HH(C)$.*

For example, if f is a derived Morita equivalence then the first map is an equivalence so C has to be zero.

Proof. $HH(B, \quad)$ preserves triangles. You compute the other two parts of the triangle by the trace property. We get the long exact sequence and identify terms. □

Then one application is the following. My favorite example, you have $U_1 \cup U_2 = X$, then there is a localization sequence of coherent sheaves on U_1 times coherent sheaves on U_2 which goes to, how does it work, it should be coherent sheaves on the intersection. It's not a localization sequence but it gives us a Meyer-Vietoris. $Coh_{Z_1 \cup Z_2} = Coh_{Z_1}(X) \times Coh_{Z_2}(X)$. If we apply this localization sequence to U_1 and U_2, we get Meyer-Vietoris, but here we just have the sums. The corollary is that you get a long exact sequence of Hochschild homology

$$\cdots \to HH(Coh(X)) \to HH(Coh(U_1)) \oplus HH(Coh(U_2)) \to HH(Coh(U_1 \cap U_2)) \to \cdots$$

So by induction you can do it for a finite covering by several opens. You can choose an affine covering, for each affine covering we know that the Hochschild homology groups are just forms. So Hochschild homology of coherent sheaves on X are isomorphic to $\bigoplus H^p(X, \Omega^{p+i})$. We need vanishing differential to get this result for the splitting and for this we need stronger Hochschild Kostant Rosenberg. For this then we should assume we have characteristic zero. In general we have a spectral sequence. We can get things here then for varieties that are not affine. This is it for today. I'll do cyclic homology and the de Rham differential in the last lecture.

4. November 13

Let me try to go fast because there is now a time limit. Let me make a couple of remarks about Hochschild homology.

(1) First, if you have some algebraic variety, and assume it's Morita equivalent to some dg algebra, and then look at $HH_*(X)$ which we define as $HH_*(A)$, you can compute it using Hochschild Kostant Rosenberg. Let me give you a general statement. You have the diagonal embedding $i : X \to X \times X$. You can pull back along i and $HH_*(X) = H^*(X, Li^* i_* \mathcal{O})$. In the smooth case you can compute the individual pieces of this complex, and the nth derived functor $L^n i^* i_* \mathcal{O} = \Omega^n_X$. There can be some sort of nontrivial extension, this doesn't have to be the sum of its pieces. If the characteristic is zero or bigger than the dimension of X then this whole thing is isomorphic to the sum of $\Omega^n_X[n]$, and this is the sum of Hodge homology that I wrote last time.

But an observation it's easy to check using this identification, this is local with respect to the Zariski topology. It's also local with respect to étale topology (in characteristic zero) but not in the flat topology (with respect to flat morphisms). For example if Y has a group action of a non-discrete group G, then $HH(Y/G) \neq H(G, HH(Y))$.

(2) Now, remark number two is some propreties. There's a Künneth isomorphism. Say you have $A \otimes B$ with the bimodule $M \otimes N$ where M is an A-bimodule and N a B-bimodule. Then

$$HH(A \otimes B, M \otimes N) \cong HH(A, M) \otimes HH(B, N)$$

and you can show this in characteristic zero.

We also have functoriality. If you have a map $f : \langle A, M\rangle \to \langle B, N\rangle$. Then f_* takes $HH(A, M) \to HH(B, N)$.

We can construct Chern classes. Say we have M a perfect A-module. Then M is a bimodule over A and $\mathbf{k}$. This defines a functor $\mathbf{k} \to A - mod$ by i_M. Now $\mathbf{k} = HH(\mathbf{k})$ so this maps to $HH(A)$. Define the Chern class to be the image of 1. Now if you have a short exact sequence then $ch(M_2) = ch(M_1) + ch(M_3)$ for the sequence $0 \to M_1 \to M_2 \to M_3 \to 0$.

Let me denote the algebra of upper triangular two by two matrices by B. A B-module is the same as a collection $V_1 \to V_2$. You could look at this, $M_1 \to M_2$, which can be thought of as a module over $B^{op} \otimes A$. This gives a functor from B-modules to A-modules. The category of B-modules, you have $\mathbf{k} \to 0$, $\mathbf{k} \to \mathbf{k}$, and $0 \to \mathbf{k}$, which form an exact sequence. It's enough to check that in Hochschild homology of B, the class of $\mathbf{k} \to \mathbf{k}$ is the sum of the classes of the others.

(3) There's something that takes a little work but I want to present it because it is useful in applications. Let's think that on the left I'm acting by $A^{op} \otimes A$ and on the right by $\mathbf{k}$. This means that tensoring with this guy induces a functor from $\mathcal{D}(A^{op} \otimes A) \to \mathcal{D}(\mathbf{k})$, which is just the Hochschild homology functor. Assume A is compact so it's perfect over $\mathbf{k}$, this is pseudoperfect and we get a map at the level of small dg categories. This induces a map on Hochschild homology, $HH(A^{op} \otimes A) \to HH(\mathbf{k})$. The complex for $HH(A^{op})$ is the same as for $HH(A)$, and so by Künneth we get the pairing

$$HH(A) \otimes HH(A) \to \mathbf{k}$$

WE can give this explicitly. Let $M = End_{\mathbf{k}}(A)$, then we have a map $A^{op} \otimes A \to M$. If A is compact, this is Morita equivalent to $\mathbf{k}$ via the trace. This pairing can be written down explicitly as the trace. So for finite dimensional algebras, then the pairing $([a],[b])$ (considering these as zero chains) is $Tr\ f_{a,b}$ where $f_{a,b}(c) = acb$. In the commutative case it's the trace of the product, but still you can take the trace in the non-commutative case.

What is nontrivial is that if A is also smooth then the pairing is nondegenerate. It's obvious I guess that $HH(A)$ is perfect over $\mathbf{k}$, but it's not obvious that the pairing is perfect.

Proof. If the algebra is compact, you have the map $\mathcal{D}(A^{op} \otimes A) \to \mathcal{D}(\mathbf{k})$. If it's also smooth then it's pseudoperfect in $\mathcal{D}(fieldk^{op} \otimes (A^{op} \otimes A))$. So we get a map in the other direction. The image $HH(\mathbf{k}) \to HH(A^{\otimes 2})$ is the Chern class, essentially. Then you have to check that the two things are compatible. Gabriel gave a field theory in higher degrees, but this is degree one. Let me denote this pairing by ρ and the Chern class by λ. The compatibility is that $(\rho \otimes id) \circ (id \otimes \lambda)$ is the identity. Perhaps I should write a diagram.

$$\begin{tikzcd} HH(A) \arrow[r, "id\otimes\lambda"] \arrow[dr, "id"'] & HH(A)^{\otimes 3} \arrow[d, "\rho\otimes id"] \\ & HH(A) \end{tikzcd}$$

□

Now it's an exercise that if I have $\lambda \in V^{\otimes 2}$ and $\rho : V^{\otimes 2} \to \mathbf{k}$ and this holds. Then V is finite dimensional and ρ is nondegenerate. A field theory in dimension one is not just a vector space, but a finite dimensional one equipped with a nondegenerate pairing. To see why this happens, let me prove this. You have λ, which is a finite linear combination $\sum v_i' \otimes v_i$. Now for any e, you have $e = \sum \rho(e, v_i')v_i$.

Automatically this sits inside $\mathbf{k}\langle v_i \rangle \subset V$ so since e was arbitrary you kow that V is finite dimensional. The rest I leave as an exercise.

These are all the things I wanted to say in generality about Hochschild homology.

4.1. **Cyclic homology.** To compute Hochschild homology, I can use the standard complex $A^{\otimes n} \to \cdots \to A^{\otimes 2} \to A$. The picture was a circle with some marked points and one special marked point. If you erase the special one you get more symmetry and you get a miracle. In characteristic zero you can take the cyclic quotient of these guys, $(A^{\otimes n})_\sigma$ where σ is $(-1)^n$ times the longest permutation. In general you should take the homology of the cyclic group, which you can compute using a very well-known complex. The terms are $A^{\otimes n} \to A^{\otimes n} \to A^{\otimes n}$ where the differentials are given by summing over σ^i and $(1-\sigma)$. Then you repeat the same differentials. This doesn't commute with b but if you use b in the even cases and b' (the acyclic differential) in the odd cases, then it commutes and you get a bicomplex.

Lemma 4.1. *This is a bicomplex. I have little idea about how to come up with this. It was discovered independently by Alain Connes and Boris Tsygan. Tsygan took cyclic quotients and looked at some sort of matrices, homology of them. Connes I have no idea. The bicomplex is in one quadrant so you have no issues of convergence. Then $HC*(A)$ is the homology of the total complex. Unital you don't need for the definition but I always use it.*

We can consider the spectral sequence where you first do vertical and then horizontal differential. You'll have something trivial in the even columns and $HH(A)$ in the odd columns, with a shift. So you have $HH(A)[u^{-1}]$ which converges to $HC(A)$, with degree of u equal to 2. So the first nontrivial differential is $B : HH_i(A) \to HH_{i+1}(A)$. In the Hochschild Kostant Rosenberg case where these are just forms, that's the de Rham differential.

You can do slightly more. You can fix the contracting homotopy for the odd columns. By pure algebra, you can erase those rows and change the complex to incorporate that, and get an equivalent complex. You can write down B explicitly. There's some formula I don't want to reproduce for B. Under Hochschild Kostant Rosenberg this gives the de Rham differential. This first appeared in a 1963 paper of George Rinehart. Already next year there was this interpretation of the de Rham differential. Somehow it was forgotten for twenty years. Then it was rediscovered.

In characteristic zero you can come up with an explicit map involving averaging that tells you that the spectral sequence collapses after this term. A priori there might be higher differentials in characteristic nonzero.

Now this is an affront. Specific complexes should be irrelevant. When you define a homology theory by writing down a complex that feels strange. There's a better construction invented by Connes, slightly less ad hoc. I don't know a construction that's actually natural.

So what I wrote, this is an alternating sum of some terms that we get often working with simplicial Abelian groups. Recall the category Δ, and then a simplicial object in a category is a functor to it from Δ^{op}. So we can look at $M : \Delta^{op} \to \mathbf{k} - Vect$. Now M_n is the value on the set of integers from 0 to n. Then there are lots of maps between them. Faces, degeneracies, and so on. You can write down an explicit complex. For M you have the standard complex whose terms are these vector spaces and whose differential is given by the alternating sum of those face maps. These are maps induced by specific maps of partially ordered sets.

Now what does this complex compute? There are different interpretations but one interpretation is the following. This complex computes homology of $H_*(\Delta^{op}, M)$, homology with coefficients in M. This is similar to homology of groups. If you have functors from the small category I into $Vect$, this functor category is an Abelian category. You can consider the colimit over I of such a functor. The homology of I with coefficients in a functor is the left derived functor of this. A functor if I has one object and only isomorphisms gives you a representation, and then this is just group homology.

So for cyclic homology, Connes wanted to change the category Δ^{op}. Connes' idea was to introduce a certain category Λ which contains Δ^{op} but is bigger. The objects are the same, they're positive integers, $[n]$ for $n \geq 1$. You think of these topologically as decompositions of the circle. So you have n points on the circle. The morphisms in Λ from m to n are homotopy classes of cellular maps from a circle with one decomposition to the other. Then, I want some conditions. That's continuous, that's implied, but I want them to be monotonous. The maps $f : S^1 \to S^1$ lifts to $\mathbb{R} \to \mathbb{R}$ and I want that to be order preserving. It should be degree one. I allow contraction of edges and I allow maps that forget points.

The case to understand is when you have one point and two points. From one to two there are two maps, depending on what you choose. In the other direction there are also two maps, depending on whether you contract one or the other. There are all sorts of combinatorics of Λ but let me stop at this.

Take an object with a fixed point, if you look at those, that's equivalent to Δ, if you fix this point. So this is an embedding $\Delta^{op} \to \Lambda$. One fact it's nice to know is that, for every simplicial set $X \subset \Delta^{op}Sets$ you have the geometric realization $|X|$. If X extends to a functor $\Lambda \to Sets$ then $|X|$ has a circle action. Let this embedding be J.

Let $\mathcal{E} : \Lambda \to Vect$. Then

Definition 4.1. *Define* $HH(\mathcal{E}) = H(\Delta^{Op}, j^*(\mathcal{E}))$ *and* $HC(\mathcal{E}) = H(\Lambda, j^*(\mathcal{E}))$

Lemma 4.2. *Theree is a spectral sequence* $HH(\mathcal{E})[u^{-1}] \to HC(\mathcal{E})$.

You can see this, see the reason for it, if you look at the nerve of Δ^{op} it's a point. If you look at Λ, it's $\mathbb{CP}^\infty$, which is $K(\mathbb{Z}, 2)$. Then the cohomology, the right derived functor of the limit, of Λ to $\mathbf{k}$, is $\mathbf{k}[u]$, with that in degree 2. So that's the kernel of the proof.

Let A be associative and unital. Then we can define $A_\# : \Delta^{op} \to Vect$ such that $CH(A)$ is its standard complex. The next observation is that $A_\#$ extends to $\Lambda \to Vect$. You extend this, you get copies of A with a number equal to the number of marked points and then the contraction is the product. The point is that this is what you have to check and the rest comes about by stupid linear algebra. So that's the story about cyclic homology and the de Rham differential. We should have a break now.

You can't quite get the de Rham complex this way because you have many copies. You have $u : HC(A) \to HC_{-2}(A)$. You can define the periodic cyclic homology, you can extend it to the right by periodicity. Then the comparison theorem says that $HP(A)$, limit over u of $HC(A)$. In the Hochschild Kostant Rosenberg case in characteristic zero, this becomes $HP(A) = H_{DR}(X)((u))$. The degree is two. So another way to say it is, you have something in all degrees, so $HP_{i+2} = HP_i$. There are only two nontrivial terms. We get de Rham cohomology but we forget the degree, get only the parity. So $HP_0(A)$ is the sum of all even degree de Rham cohomology while HP_1 is the sum of all odd degree. In the general associative situation, too, this is not an algebra.

4.2. **Hochschild cohomology.** This is a different but somehow complementary story.

Definition 4.2. *$HH^*(A)$ is $Ext_{A^{op}\otimes A}(A, A)$.*

You need to resolve A. If you choose the bar resolution you get a standard complex $CH^*(A)$. You have $A \to Hom_{\mathbf{k}}(A, A) \to Hom_{\mathbf{k}}(A^{\otimes 2}, A) \to \cdots$ Let me give the differential in low terms. $\delta(a)(b) = [a, b]$. The next differential $\delta(f)(a\otimes b) = f(ab) - af(b) - f(a)b$. You can write down a formula in higher degrees.

You can also obtain the same complex in a different way. Another interpretation, take $A[1]$, the complex, take its tensor coalgebra $T(A[1])$, and consider the space of coderivations on this. The space of coderivations is a Lie algebra, a graded Lie algebra. The differential δ is the bracket with m, the multiplication $A^{\otimes 2} \to A$. This is the formula for the differential. It's a standard exercise to check that m squares to zero if and only if m is associative. On the other hand on my original definition, this is an Ext so it has a multiplication, you can compose Ext. So the Hochschild cohmology has two structures. It has an associative multiplication. In fact, you can show that this is actually commutative. It also has this bracket but here there is a shift of degrees. You can see that the bracket is of degree one. So the zero point for the bracket is not $HH(0)$ but $HH(1)$. We know the Lie algebra, and the commutatator of derivations is derivations. This is derivations modulo inner derivations. Another place where this is useful is in deformation theory. You can check that $HH^2(A)$ controls deformations of A. I can't go into this in detail. You want to change the multiplication, change it to $m + \gamma$. This has to bracket to zero, $[m + \gamma, m + \gamma] = 0$. You get something called the Maurer Cartan equation:

$$\delta(\gamma) + \frac{1}{2}[\gamma, \gamma]$$

There is a nice formalism for this which you can read in the literature which gives you some reasonable control over the deformations.

You can also check that there is a compatibility between the product and the bracket that makes this into a kind of Poisson structure.

$$[ab, c] = \pm[a, c]b \pm [b, c]a$$

with some (important) signs that I don't remember. This is known as a Gerstenhaber algebra, with a product and Lie bracket of degree one, and compatibility. I think what Gerstenhaber invented was the bracket. He gave the bracket on this complex and it was named in honor of him. This structure has some very nice geometric interpretation too.

There is this machinery of operads we heard about on Monday. Structures of this type cry out to be talked about in terms of operads. You want to have this space G_n for any n and maps $G_n \otimes HH^*(A)^{\otimes n} \to HH^*(A)$. G_n is the "space of operations." This can be made precise but let me just give you the answer. G_n has the following geometric interperetation. This is a graded vector space. One source of geometric graded vector spaces is homology.

Let $\mathcal{O}_n$ be the homotopy type of the configuration space of n marked distinct points on a unit disk D. In other words, we take $D^n\setminus\{\text{diagonals}\}$. Then G_n can naturally be identified with $H_*(\mathcal{O}_n, \mathbf{k})$.

This only works nicely if $\mathbf{k}$ is characteristic zero. If $\mathbf{k}$ does not contain $\mathbb{Q}$, then part of it works. You can define this operad and you have something like a map like this, but it describes, on HH^1 you don't get a Lie algebra but a restricted Lie algebra. This was discovered probably many times but I know it from a paper of Turchin (around 95). Let's assume we're in characteristic zero then.

How does this work in examples?

Example 4.1. *If $n = 1$ then $\mathcal{O}_n$ is a disk and this is contractible. Then we get the identity. If $n = 2$, you have two points on the disk, or the complex plane. You can normalize it so that the difference is one. Then you can move one point to be zero. So the other point can be anything nonzero. So in this case $\mathcal{O}_2$ is homotopy equivalent to a circle. It has* $G_2^i = \begin{cases} \mathbf{k} & i = 0 \\ \mathbf{k} & i = 1 \\ 0 & i \geq 2 \end{cases}$ *This is the product and the bracket. If you look at $H_{n-1}(\mathcal{O}_n)$, this gives you the Lie operad. This encodes the structure of the Lie algebra. H_0 gives you the associative structure, and everything in between gives you the relations.*

There was a question. What I have here is what happens on HH^*. There is a question asked by Deligne maybe in 1993. Assume now that we consider the complex itself. On the complex we can consider the chain complex. Is it true that the chain complex $C_*(\mathcal{O}_n, \mathbf{k})$ naturally acts on $CH^*(A)$. This has to be interpreted correctly. You have to choose specific a space interpreting this homotopy type. You need it homotopically. The answer if you interpret correctly is yes. This is sometimes known as Deligne's conjecture. There are many proofs, maybe ten proofs. All are somewhat unnatural. The reason there are many proofs, someone new invents a proof thinking existing ones are unnatural. There are no papers shorter than 50 pages. One way to make it precise is to make it into an operad and then make the operad act here. This doesn't want to be a cellular operad. There is some kind of mess.

People started to care when Tamarkin used this for Kontsevich formality.

The question is easier than one thought. There is a theorem, first also proved by Tamarkin, which is that this operad is formal. It's quasiisomorphic to its homology. That means that up to homotopy, you don't just have an action of this guy, you have a homotopy Gerstenhaber algebra. This statement is more non-trivial. If you want to do this, you need to pick some associator.

If you put $\mathbb{C}$ instead of the disk then it's an algebraic variety. In this case the mixed Hodge structure (due to Deligne) it's pure. The obstructions of formality come from a place that has to be zero. If you want to make this precise, you have to explain the Massey products and it turns out not to be worth it. You get a long paper if you want to write this down. Tamarkin observed that if you have a polynomial algebra $A = \mathbf{k}(x_1, \ldots, x_n)$, there's a version of Hochschild Kostant Rosenberg that gives $HH^*(A)$ is polyvector fields $H^0(X, \wedge^1 T_X)$. So why this is formal as a Lie algebra is not a trivial fact. Formality means that you have your differential graded thing and it's quasiisomorphic to its cohomology. You can phrase this in deformation theory. Tamarkin observed that if you look at this as a Gerstenhaber algebra, it's rigid, it has no deformations. Every Gerstenhaber algebra with this cohomology must be formal. What is a Gerstenhaber algebra? It's some kind of Poisson algebra. You have generators in degree zero. You have generators in degree one which anticommute. You have $HH(A) = \mathbf{k}(x_1, \ldots x_n, \epsilon_1, \ldots, \epsilon_n)$ and then you have an odd symplectic bracket. Then you know that symplectic things are all the same by Darboux. You have to prove that $CH^*(A)$ is a (homotopy) Gerstenhaber algebra. If you have a Poisson manifold, $\theta \in \wedge^2 T_X$ which squares to zero, you can form a Poisson complex $\mathcal{O}_X \to T_X^{[\theta,]} \to \wedge^2 T_X^{[\theta,]}$. [missed some]. The general scheme of the argument is like this.

I'm almost out of time and I'm almost finished. My final remarks are about combining the stories for homology and cohomology. There is homology $HH_*(A)$ which has this structure B, the Rinehart differential. There is cohomology $HH^*(A)$ which has the Gerstenhaber bracket. It's easy to show that $HH_*(A)$ is a module over the algebra $HH^*(A)$ and the same is true of the Lie structure. This should be thought of as polyvector fields acting on forms. You have the Cartan homotopy formula in the geometric case. That shows that the two actions are related in some way. In the general case there are people who worked on this. The one I know best is, there is the notion of a "noncommutative calculus" introduced by Tsygan and Tamarkin. This axiomatizes this situation. You have a Gerstenhaber algebra, a thing with a differential, and an action of one on the other. You want to work at the level of chains and get a formality statement. This is controlled by a version of an operadic formalism. For homology, they appear as the homology of something. The relevant configuration spaces are points on a disk with boundary, where we distinguish between points inside and on the boundary. This came up in physics all the time, although there you don't quite know the statements. With Getzler Jones it pays to be precise because it's easy to make mistakes.

The points on the boundary, you also have points inside. There's one way to do this. Another way that is even more useful, although less general. I said that homology is naturally an algebra over cohomology. Assume we have a class in Hochschild homology ω of degree d such that we get a map $HH^*(A) \to HH_*(A)[-d]$ by $\alpha \mapsto i_\alpha(\omega)$ Assume that this is an isomorphism. This is known as Calabi Yau of degree d. The class of dimension d is a d-form. Fix the volume form and then we will get such an isomorphism (trivializing the canonical bundle). In fact, quite often you want the volume form to be closed. It's better to refine this right away so that this is closed somehow. We said we have cyclic homology, periodic cyclic homology, and define a third term, negative cyclic homology $HC^-(A)$. There was a bicomplex that was periodic, I extended it to the right and that was HP and the new part is HC^-. This gives forms which are closed. A class in is a form which is closed if it lifts. Then you have this identification.

Then what happens in the end is that this table collapses, we have both kinds of structures on the same vector space by the isomorphism. In the Calabi-Yau case we have $HH^*(A)$ with a bracket and the image of the differential under the isomorphism, of degree -1. There are relations. It also has the product. This differential is usually notated Δ. The bracket then can be expressed in terms of the differential. The difference of Δ from being a derivation of the product is the bracket. You have

$$[a, b] = \pm\Delta(ab) \pm \Delta(a)b \pm \Delta(b)a$$

which is called a Batalin-Vilkovisky or BV algebra. It's also controlled by a very nice operad, the operad which is the configuration space of framed disks. When we consider points on the disk, we could have put small holes. Now we want to add a point on the boundary of each hole. Or with points you choose a nontrivial tangent vector. Already in degree one we have a nontrivial operation. We have a choice of a point on the boundary. We already have a circle in degree 0. This is Δ. You can read the rest of the structure from this. This is maybe more important than the Gerstenhaber structure, studied by many people.

You might think this is nice but only for the Calabi Yau case. But you can notice that $A \oplus A^*[d]$ is Calabi-Yau of degree d, where you get a product of zero on the right. If you consider the Hochschild homology with higher structures, you can actually recover pretty much all of the general formalism. So the Batalin Vilkovisky formalism is what you might have to study and you can generalize the normal case to this. This trick is stupid but on the other hand it works.