MOTIVIC STRUCTURES IN NON-COMMUTATIVE GEOMETRY DMITRY KALEDIN

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1. November 24: De Rham cohomology in characteristic p and Dieudonné modules

The general title is motivic structures in non-commutative geometry.I'll explain what this is more or less today. Today's lecture is more or less about motivic structures and what I mean by this, it's something down to earth. This will be in the usual algebraic geometry. Today there will be nothing non-commutative. The algebraic geometry won't be too advanced.

The point is that some of the things I'll be talking about today make sense and are even more natural in non-commutative geometry, so at least for me the non-commutative setting clarifies what happens in commutative geometry. The second lecture is about de Rham type cohomology in the non-commutative setting (or that is, cyclic homology). Then we'll see how it goes. Today I start with motivic structures.

So I start with an algebraic variety X, smooth over some $\mathbf{k}$ which for now can be a field. I want to explain what I mean by motivic structures. Usually we have just one cohomology in geometry. In *algebraic* geometry we have several competing cohomology theories. Let me list the main ones

(1) algebraic K-theory of X and related things, Chow groups of cycles. It's not easy to compute but it's completely intrinsic. You define algebraic subvarieties and so on. You take cycles like what you did in algebraic topology.
(2) Étale cohomology, this is something, there are versions, but $H^*_{et}(X \otimes_{\mathbf{k}} \bar{\mathbf{k}}, \mathbb{Q}_\ell)$, there's something very nontrivial due to Grothiendieck that you can do that gives you something that looks like cohomology.
(3) Another way to extract some cohomology theory is some kind of de Rham cohomology. This has some disadvantages, I can't take coefficients in $\mathbb{Z}$ or $\mathbb{Q}$, but I can do ℓ-adic coefficients for étale cohomology. But here my coefficients are in $\mathbf{k}$. The definition is to repeat the usual definition almost literally. You just take the cohomology of X with coefficients in the sheaf Ω^* of differential forms. In the usual de Rham forms this sheaf has no cohomology so you just get something global but here your sheaf might have cohomology so you have to write $H^{\cdot}(X, \Omega^{\cdot})$ intead.
(4) To circumvent some difficulties Grothiendieck invented cristalline cohomology $H^{\cdot}_{cris}(X)$. If the characteristic is zero then this is just the de Rham cohomology. If $\mathbf{k}$ is $\mathbb{F}_q$ then cristalline cohomloogy is a module over $W(k)$. This lets you compute points in a way that you can't using de Rham.

In these theories (not K-theory) what happens is that, well, cohomology is different than K-theory, it would be interesting to express one in terms of the other. The first step toward this is to notice that these cohomology groups have additional structures. These structures are what is called "motivic" structures. The easiest example is étale cohomology. The structure is the following. What happens is that we have the Galois group of $\bar{\mathbf{k}}/\mathbf{k}$ which acts naturally on $H_{et}(X \otimes \bar{\mathbf{k}}, \mathbb{Q}_\ell)$. One can define étale cohomology of X without going to the algebraic closure, so-called "absolute étale cohomology" which is $H^{\cdot}(X, \mathbb{Q}_\ell)$, and there will be a spectral sequence converging to this which starts with $H^{\cdot}(Gal(\bar{\mathbf{k}}/\mathbf{k}), H^{\cdot}_{et}(X \otimes \bar{\mathbf{k}}, \mathbb{Q}_\ell))$.

Let me give an analogy. You have a map of topological spaces $X \to Y$. I can compute $H^*(X)$ directly or I can compute it in two steps. There is the direct image functor on sheaves $X \to Y$, so I can compute $H^{\cdot}(Y, R^{\cdot}\pi_* A)$, then if π is a fibration with some fiber F, then these things $R^{\cdot}\pi_* A$ are a local system on Y. Then you get a spectral sequence but if we have a fibration, you get the Leray-Serre spectral sequence which starts from $H^{\cdot}(Y, H^{\cdot}(F))$ and converges to $H^{\cdot}(X)$.

Here $Y = Spec\ \mathbf{k}$. This is a stacky-point, a point modulo the Galois group. Then $H^*(Y, A) = H^*(\pi_1^{et}(Y), A)$ and the $\pi_1^{et}(Y)$ is $Gal(\bar{\mathbf{k}}/\mathbf{k})$.

We have this point which has its own cohomology. Anyway, this thing is not geometric, but it's closer to K-theory. We have a regulator map $K_*(X) \to H^*_{et}(X, \quad)$, I can take a representation valued in roots of unity so I can twist by powers of this and get a regulator map

$$K_i(X) \to \bigoplus H^{2j-i}_{et}(X, \mathbb{Q}_\ell(j))$$

[example]

So there's this story for étale cohomology. What happens for de Rham cohomology and cristalline cohomology?

For de Rham theory you have $X/\mathbf{k}$, let's say that $\mathbf{k} = \mathbb{C}$, it was a great insight of Deligne who discovered that there is such a story, the category of mixed Hodge structures. Right, so we have our de Rham cohomology, defined completely algebraically, we can consider the usual cohomology in the topological sense, so there's this comparison theorem, $H^{\cdot}_{dR}(X) \cong H^{\cdot}(X_{an}, \mathbb{C})$. On the right the additional structure is the following. We can take any coefficients, we can take a sub-ring inside $\mathbb{R}$, then the right hand side is $H^{\cdot}(X_{an}, A) \otimes_A \mathbb{C}$. So you have an automorphism lifting complex conjugation. Algebraically you don't see this at all. On the left you have the Hodge filtration $F^{\cdot}$, and the definition is as follows. As I said, $H^{\cdot}_{dR}(X) = H^{\cdot}(X, \Omega^{\cdot}_X)$. You can consider what Deligne called the "stupid filtration." The term $F^i\Omega^j(X)$ is Ω^j_X if $j \geq i$ and 0 otherwise. You take your de Rham complex, choose some level, and take everything at that level and above. This is compatible with differentials for a trivial reason. Whenever you have a filtered complex and you do something like take cohomology, you get a natural spectral sequence. This induces a spectral sequence which starts with cohomology of X with coefficients in Ω^j_X and converges to $H_{dR}(X)$. Then there is this observation that at least for X smooth and also projective, this spectral sequence degenerates. The hardest part of this is the Hodge theory from the 50s. This is a repackaging of that difficult analytic statement. Later on Deligne and Illusie came up with a proof that is also algebraic which works for proper varieties that may not be projective. The target of the spectral sequence, then, gets a filtration, $F^i H_{dR}(X)$. This is the Hodge filtration.

So we have two structures, complex conjugation and this filtration.

Definition 1.1. *Fix $A \subset \mathbb{R} \subset \mathbb{C}$. An A-*mixed Hodge structure *is the following triple: $\langle V_A, W.V_A, F^{\cdot}V_{\mathbb{C}}\rangle$ where V_A is an A-module, $W.$ is an increasing filtration, and a decreasing* Hodge filtration *$F^{\cdot}V_{\mathbb{C}}$ where $V_{\mathbb{C}} = V_A \otimes_A \mathbb{C}$.*

For each i we can take the associated groded $gr_i^W V_{\mathbb{C}}$ and F induces something here but we also have complex conjugation. The condition is that

1. $F^j \cap \bar{F}^{i+1-j} = 0$.
2. $gr_i^W V_{\mathbb{C}} = \bigoplus F^j \cap F^{i-j}$.

We're used to a Hodge grading. Then $F^j = \bigoplus_{p \geq j} H^{p,q}$ and $\bar{F}^{i-j} = \bigoplus_{q \geq i-j} H^{p,q}$.

Proposition 1.1. *A-mixed Hodge structures form an Abelian category of homological dimension one. You can get exts between them. Even for pure Hodge structures it's interesting to check the dimensions of Ext groups. If A does not contain $\mathbb{Q}$ you get dimension two for stupid reasons.*

Why is this surprising? Filtered vector spaces do not form an Abelian category. The basic thing that goes wrong is, take $\mathbf{k}$ with filtration $F^1\mathbf{k} = 0$ and $F^0\mathbf{k} = \mathbf{k}$. Then renumber this, $k[1], F^0\mathbf{k} = 0, F^{-1}\mathbf{k} = \mathbf{k}$. There is a map $k[1] \to k$. This is an isomorphism of vector spaces and has no reasonable kernel or cokernel. In an Abelian category I expect this to be an isomorphism, but it's not because an inverse would have to do forbidden things.

This is typically what happens with filtered vector spaces. But with this condition, maps like what we said are not allowed any more. The maps have to preserve both filtrations. It's not so difficult to prove but the proof is not instructive, really.

Theorem 1.1. *(Deligne) For $X/\mathbb{C}$ quasiprojective, $H^{\cdot}(X, A)$ carries a natural A-mixed Hodge structure.*

As I said, this is an Abelian category. One can observe slightly more. We can take tensor products and one checks without much difficulty that the tensor product gets the same conditions. There is a distinguished invertible object, $A(1)$, which is $\langle A, W., F^{\cdot}\rangle$ where is A is concentrated in weight -1 and $F^{-1}A_{\mathbb{C}} = \mathbb{C}$ while $F^0 A_{\mathbb{C}} = 0$.

Definition 1.2. *The* absolute Hodge cohomology *of X*

$$H^{\cdot}_{AH}(X, A(j)) = RHom^{\cdot}(A(-j), H^{\cdot}(X, A))$$

This is homological dimension one, so this has Hom and a first Ext. This is completely parallel to Galois cohomology. There I consider representations of the Galois group. My absolute Galois cohomology with expressed as cohomology of the group which is like an Ext from the trivial representation. So this, there is also Deligne cohomology which is close to this but doesn't quite coincide. Beilinson corrected this to the given definition.

Even if X is just smooth and proper, still, it's important to know that you have these mixed Hodge structures with nontrivial Ext^1.

Theorem 1.2. *(Beilinson) There is a regulator map*

$$K_i(X) \otimes \mathbb{R} \to \bigoplus_j H^{2j-i}_{AH}(X, \mathbb{R}(j)).$$

There are conjectures that predict very precisely the behavior of this map, but for $i \geq 2$ this is expected to be an isomorphism for X of the form $X_{\mathbb{Q}} \otimes \mathbb{C}$ for X smooth projective.

It's well-known that there is no chance that this is an isomorphism if X is not defined over $\mathbb{Q}$. The group of cycles will be huge. There are some conjectures about surjectivity but for injectivity there's no chance. If your thing is arithmetic in origin, then, maybe. It's more interesting K_1 and K_0 but there the statement is more complicated.

Do I do a break?

Let me present a third version of the story, not well-known thirty years ago although it existed, now it's better known, about cristalline cohomology. Actually, I now start with a finite field $\mathbf{k} = \mathbb{F}_q$, I want to start not over $\mathbf{k}$ but over the p-adic version, the Witt vectors $W(k)$, the unramified extension of $\mathbb{Z}_p$ of the same degree as $\mathbf{k}$. I start with X defined over $W(k)$. You can think of something given by equations with coefficients in this ring, but I think of a scheme mapping to $Spec(W(\mathbf{k}))$. Sitting inside $Spec(W(\mathbf{k}))$ is $Spec(\mathbf{k})$ and over this is the special fiber X_0. I want everything to be smooth.

As I said cristalline cohomology has a high-tech definition that I don't want to reproduce here, but we have a comparison theorem says that this is $H^{\cdot}_{dR}(X)$, defined naively, exterior powers of a tangent sheaf, you should symmetrize at some point, take the quotient and not the invariants. Then for X_0 you have the cristalline cohomology, and the comparison theorem says $H^{\cdot}_{dR}(X) \cong H^{\cdot}_{cris}(X_0)$. There are different lifts but this doesn't depend on the lift. When you have different deformations the cohomology should be the same, and this is a realization of this idea.

You should think of this comparison theorem as being analagous to the situation in de Rham cohomology because we have two structures on the two sides and the interplay is what is interesting. On the left you have $F^{\cdot}$ the Hodge filtration, which degenerates in the smooth case. On X_0 you have the Frobenius automorphism. This map is very easy. On points it's identical and every function is raised to the pth power. So it turns out that the interplay between the two is the interesting thing.

Let me give you right away the version of this mixed Hodge structure here.

Definition 1.3. *A* filtered Dieudonné module *(FDM for short) is a collection of the following data:*

1. *M a finitely generated $W(\mathbf{k})$-module,*
2. *a decreasing filtration $F^i(M)$,*
3. *a collection of maps $\varphi^i : F^i(M) \to M$ which are Frobenius semi-linear. What does this mean? There is a lift of Fr on $\mathbf{k}$ to $W(\mathbf{k})$. This means that $\varphi^i(am) = Fr(a)\varphi^i(m)$. There is also a condition $\varphi^i|_{F^{i+1}} = p\varphi^{i+1}$.*
4. *$\bigoplus \varphi^i : \bigoplus F^i M \to M$ is surjective.*

Proposition 1.2. *FDM form an Abelian category (of homological dimension 2).*

For example, assume we have M which is torsion, it's annihilated by p, so $pM = 0$. This says that $\varphi^i|_{F^{i+1}} = 0$. Then $\varphi^i : gr^i_F(M) \to M$. Then the surjective map factors through this, $gr^{\cdot}_F(M) \to M$. Now M is annihilated by p so it's a $\mathbf{k}$-vector space. Since this is finitely generated and surjective it's injective so it's an isomorphism. Take $\tilde{M} = \bigoplus F^i M/(t-p)$, this is called the Rees object. There's a map $t : F^i \to F^{i-1}$, an embedding. There is a natural map from $\tilde{M} \to M$ which will be an isomorphism.

It's kind of hard to keep track of all this. Filtered vector spaces on the geometric side should be the same as $\mathbb{G}_m$-equivariant sheaves on $\mathbb{A}^1$. This is a graded module over the base field with one variable of degree one. In this language the associated graded quotient corresponds to the fiber at $0 \in \mathbb{A}^1$ and M is the fiber at $1 \in \mathbb{A}^1$. Sometimes this is called normal cone degeneration. The point is that $\tilde{M}$ is the fiber at $p \in \mathbb{A}^1$.

I take such a guy and I want an isomorphism between the fiber at 1 and the fiber at p.

Anyway, this is a way to memorize all those things.

Now there is a theorem, this definition was introduced by Fontaine-Lafeulle (1982) and

Theorem 1.3. *(Faltings) If X is smooth and proper over $W(\mathbf{k})$ then $H^{\cdot}_{dR}(X)$ has a natural FDM-structure.*

What does Dieudonné have to do with this? He considers modules over $W(k)$ which has a Frobenius kind of automorphism. It was known that you had this on cristalline cohomology, this is a semisimple category, all possible skew fields over $\mathbb{Z}_p$ occur (the automorphisms of irreducibles are skew fields). The novelty here is the filtration.

This is not p-adic Hodge theory. You have the generic fiber X_η and you can try to compute its étale cohomology. Normally you want ℓ to be different that p. But here we have something in characteristic zero, so you can take $H^{\cdot}_{et}(X_\eta, \mathbb{Q}_p)$ or you can take $H^{\cdot}_{cris}(X_0) \otimes \mathbb{Q}_p$. Then p-adic Hodge theory relates these two to one another. They are not equal as is. One answer is that in large coefficients, you have some ring B_{dR}, and tensoring with this you get a canonical automorphism.

There is a more refined story, if you consider the latter as a FDM, you get the former as a Galois group representation. There is also that story. But I don't want to go into this at all. My main interest is not commutative. This is a complicated story because $Gal(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ is complicated.

Sometimes you want something like this, having this isomorphism somewhere can be valuable. If you have two different reductions to characteristic zero, are these constructions related? We have no other relationship.

Let me show how this comes about under some assumptions. Let me assume I'm in the situation when the Frobenius of X_0 lifts to a Frobenius of X. This is rare for projective varieties but you always have it locally. Then the structure is easy to see. The lift $\widetilde{Fr}$ acts on de Rham cohomology, on the the complex. If you compute the differential, you have $d(f^p) = pf^{p-1} = 0$. So on functions there's a map, but for one-forms, you have $\tilde{Fr}^*(\alpha)$ is divisible by p for any one-form α. Mod p it's $Fr^*(\alpha)$ which is 0 by this argument. So you can refine this map by saying the following. Ω^1 is a flat module over $\mathbb{Z}_p$. Define the new map C^{-1} which is $\frac{1}{p^i}\tilde{Fr}^*$ on Ω^i. Then I want the differential on the domain to be pd. Then this map is a quasiisomorphism. Since these are complexes of sheaves, you can do it at a point, so it's a computation in local coordinates.

Then one observes that this is equivalent to doing, well, $\langle \Omega^{\cdot}_X, pd \rangle$ is the same as $\widetilde{\langle \Omega, d, F \rangle}$.

So the left hand side is the same as if I allow 1-forms with $\frac{1}{p}$, 2-forms with $\frac{1}{p^2}$, and so on. Then C^{-1} induces an isomorphism between $\widetilde{H^{\cdot}_{dR}(X)}$ and $H^{\cdot}_{dR}(X)$.

You have a little more. In this case $\varphi^i : F^iM \to M$ actually lands in F^iM. In general this is not true and you can express the obstructions in terms of the failure of φ^i to respect the filtration.

Now consider the general case, considering only de Rham cohomology modulo p^2. We can choose an affine covering $\bigcup U_\alpha$ of X, and locally you have a lifting of Frobenius $\widetilde{Fr}_\alpha$, and they will not match on intersections. On $U_\alpha \cap U_\beta$, they do not have to agree. What is the difference between them? In general it's hard but modulo p^2 it's a derivation, $\xi_{\alpha\beta}$ which is a vector field on the intersection, in $H^0(U_\alpha \cap U_\beta, T \otimes Fr_*(?))$. So $\xi_{\alpha\beta}(fg)$ in this twisted setting is $f^p\xi_{\alpha\beta}g + g^p\xi_{\alpha\beta}f$.

Then $\widetilde{Fr}^*_\alpha = \widetilde{Fr}_\alpha + \mathcal{L}_{\xi_{\alpha\beta}}$ where $\mathcal{L}_{\xi\alpha\beta} = dh_{\alpha\beta} + h_{\alpha\beta}d$ where $h_{\alpha\beta}$ is just $i_{\xi_{\alpha\beta}}$. So there is a canonical choice of homotopy between the two things that don't match up on the intersection. This gives a genuine map of Cech complexes. You can express the nontriviality of this in terms of the failure to respect the filtration. There's a longer story but this is not going in the direction I want to go anyway.

That's it for today. Next time we'll talk about cyclic homology, after proving that one of the categories is Abelian. The general goal is that thinking about cyclic homology seriously you get some structure like a FDM fairly naturally. There is no Frobenius map. In the general construction there is no place where commutativity simplifies this.

2. November 26

First one proof from last time. Let me recall the statement. Let $\mathbf{k}$ be a field and $W(k)$ the Witt vectors. If you don't know what that is, let $\mathbf{k} = \mathbb{F}_p$ and then $W(k) = \mathbb{Z}_p$.

Definition 2.1. *A filtered Dieudonné module is*

1. *a module M finitely generated over $W(k)$,*
2. *a decreasing filtration $F^{\cdot}M$*
3. *maps $\varphi^i : F^iM \to M$, Frobenius semilinear (linear in the simplified situation)*

which satisfies

- $\varphi^i|_{F^{i+1}} = p\varphi^{i+1}$,
- $\bigoplus \widetilde{\varphi^i} : \bigoplus F^iM \to M$ *is surjective.*

These form an additive category FDM.

Theorem 2.1. *(Fontaine-Lafeille) The subcategory FDM_{tors} of torsion modules (finite length) form an Abelian category.*

Proof. First let me embed it into some other category which is obviously Abelian and show that it is closed under kernels and cokernels.

So $M^{\cdot}$ is naturally a graded module over $W(k)[t]$ with t of degree -1, where $t : F^{i+1}M \to F^iM$. I can take all the φ^i together by letting $\tilde{M} = M^{\cdot}/(t-p)$. Then the conditions are equivalent to saying that there is a natural map $\tilde{\varphi} : \tilde{M} \to M$, also surjective.

In this language, by the way, I can take different quotients, M itself is $M^{\cdot}/(1-t)$ and $M^{\cdot}/t$ is the associated graded $gr^{\cdot}_F M$.

Consider these graded things, dropping all assumptions, so let $\widetilde{FDM}$ be the category of graded $W(t)$-modules equipped with a map $\tilde{\varphi} : \tilde{M} \to M$. Then $FDM \subset \widetilde{FDM}$ and the image consists of guys with the condition $(M^{\cdot}, \tilde{\varphi})$ is in FDM if and only if

- t is injective, so that this graded thing comes from a filtered thing, and
- φ is injective.

I can put a torsion condition on $\widetilde{FDM}$ as well. We can say $M^{\cdot} \in \widetilde{FDM}_{\text{tors}}$ if all M^i are of finite length, for $n \gg 0$ all M^n are zero, and for $n \ll 0$, $t : M^n \to M^{n-1}$ is an isomorphism.

The category $\widetilde{FDM}$ is obviously Abelian. I take kernels and cokernels as graded modules and there is no problem here. Then what I need to check is that this subcategory is closed under kernels and cokernels. For kernels it's pretty easy, but for cokernels, that's the reason that filtered modules don't form an Abelian category, you might have failure of injectivity for t.

Lemma 2.1. *$M \in \widetilde{FDM}_{\text{tors}}$ is inside FDM_{tors} if and only if $\tilde{\varphi}$ is an isomorphism.*

But this finishes things because this is stable under taking cokernels. □

Proof of the lemma. Consider $t - a : M^{\cdot} \to M^{\cdot}$ where a is an integer. It's very easy to see that under these assumptions, the kernel and cokernel have finite length and moreover, we can define like an index between the two, $\text{Index}(t-a) = \text{len}(\text{coker})(t-a) - \ln(\ker(t-a))$ does not depend on a. You can see this by setting up a spectral sequence that relates this, seen as a two term complex, to one with $a = 0$. Filter the two term complex by degree of $M^{\cdot}$. The first term of the spectral sequence is obtained by the same complex without a. You start with $M^{\cdot} \xrightarrow{t} M^{\cdot}$ and then a affects higher differenitals. In positive degrees we have zero and in negative degrees, t is an isomorphism so you also get zero. So you get finite length here and the index doesn't change under the further differentials.

Now $t-1$ is always injective; t is injective if and only if $t-p$ is injective. If there is something annihilated by t you can find something annihilated by t and p. This is enough; I want to show that t being injective and $\tilde{\varphi}$ surjective is equivalent to $\tilde{\varphi}$ being an isomorphism.

Since t is injective, so is $t-p$, there is no kernel for $t-p$. Then the length of the cokernel $\tilde{M}$, that is the same as the length of M, the cokrenl of $t-1$. Then since the map $\tilde{M}$ to M is surjective it is also injective.

In the other direction, if φ is an isomorphism, then the lengths of $\tilde{M}$ and M are the same, so there is no kernel, so that t is injective. □

This is roughly how it works. The moral of the story is,

(1) the first moral is that when we work with filtered things, it's kind of useful to pass to these kind of Rees objects, and
(2) even when things are not finitely generated, we can consider a triangulated category, a kind of derived version of FDM, consider $M^{\cdot}/W(k)[t]$ plus an isomorphism $\tilde{M} \to M$. But now instead of taking the cokernel we take the cone of the map $p - t$. This is obviously a triangulated category, since that property is preserved under cones and shifts.

To get this you can consider the case without isomorphisms and then check that being an isomorphism there is preserved, so you can compute in either the small category or the big category. I'll get to a point where I don't know how to phrase the condition, but it doesn't really matter because I can just work in the big category.

So my goal is to have a geometric construction of the same category, and I start by just considering filtered modules. Let me start with some simple linear algebra and then I'll introduce the cyclic category after a break.

Fix a ground ring R and I want to consider the filtered derived category of R-modules. This has two equivalent definitions.

(1) Take complexes of filtered R-modules and invert filtered quasi-isomorphisms. By definition these are maps which induce an isomorphism of associated graded quotients
(2) or equivalently, take graded modules over $R(t)$ with degree of t -1, and take the usual derived category of these guys. My claim is that both are the same. There is an embedding from filtered to graded objects. Not every graded object comes from a filtered object. But we can always replace it with a resolution.

The functor sends $(M, F^\cdot)$ to $\bigoplus F^iM$ with $t : F^iM \to F^{i-1}M$

[discussion of the history of this notion]

Now there is also a third equivalent construction of the same thing.

3. Maybe people have heard of Koszul duality. The simplest example is the symmetric algebra in one variable. Apply Koszul duality and see that the filtered derived category of R is equivalent to the derived category of an exterior algebra in one variable ϵ of degree 1 graded degree -1.

If you spell out the definition, it's the same thing as a bicomplex. So objects in the category of modules over $R[\epsilon]$ are bicomplexes. TO get a filtered object from a bicomplex, a bicomplex, a bicomplex corresponds to its total complex, take the stupid filtration. As a reminder I draw a vertical line, I put zeroes to the left and keep everything to the right.

When I take the total complex, you can take sum or product, this is delicate. I can impose conditions on my filtrations, separated, complete, let me ignore them, which may not be a good idea but let me ignore them. If your complex is finite, then it's all irrelevant, but this can't be the case for cyclic homology, typically one wants things that are periodic.

Let me say, in the Chern character story too, you get a sort of twisted periodic version of this filtered derived category. Namely,

Definition 2.2. $\mathcal{D}F^{\mathrm{per}}(R)$ *is the category of*

(1) M *is a filtered complex of* R*-modules and*
(2) $M_\cdot = M_\cdot[2](1)$*, that is,* F_iM_j *should be identified with* $F^{i+1}M_{j+2}$*. This filtered category is a basic version of this motivic category that I had on Monday.*

Let me mention how this appears in terms of bicomplexes as well. In term of bicomplexes periodic filtered complexes correspond to "mixed complexes," that is, complexes (V, d) with another map B in the wrong direction, if d goes down then B goes up. This anticommutes with d and squares to zero.

This means all my columns are the same and the differential ϵ gives me this B. This gives me a periodic bicomplex and the only invariant is what is in the zero column, and the differential.

So I get a triangulated category $D^{\mathrm{mix}}(R)$ which is equivalent to $\mathcal{D}F^{\mathrm{per}}(R)$. Now we'll have a break and then introduce the cyclic category.

So I am going to do some category theory. I want to use Connes' category Λ. Let me use a presentation that's slightly less standard than the usual one that will give me some things I'll need later. Assume we're given a category I and a functor $F : I \to \mathbf{Set}$. Then $\mathrm{Tot}(F)$ is a category whose objects are $(i \in I, f \in F(i))$. Morphisms are morphisms $i \to i'$ so that $f(\alpha)(f(i)) = f(i')$. This is in SGAI, chapter six. This maps to I and this is a "discrete cofibration." More restrictively, you could look at F which lands in the isomorphisms. Then $Tot(F) \to I$ is a "discrete bifibration." A map is a discrete bifibration if and only if for each $i \to i'$ and each object that projects to i, there is a unique map lifting $i \to i'$. The same should be true in the target. You could also formally invert all maps in I, getting a groupoid, which is equivalent to, I can take the geometric realization of the nerve, and then this groupoid is $\pi_1(|I|)$. If this thing is connected, then a functor from this guy to sets is the same thing as a set with an action of $\pi_1(|I|)$. These are a very naive analogue of coverings.

Now denote by $[1]_\Lambda$ the category with one object and endomorphisms equal to nonnegative integers. You can think you have a quiver with a single object with an arrow to itself and take the path category of the quiver. If I invert all morphisms, I get all integers, and π_1 is just $\mathbb{Z}$, the group of integers. So for every $n \geq 1$, I can consider $\mathbb{Z}$-sets, and amond those I can consider $[\mathbb{Z}/n\mathbb{Z}]$ as having an action of $\mathbb{Z}$ by left multiplication, so this corresponds to a discrete bifibration on $[1]_\Lambda$, which I will denote by $[n]_\Lambda$. This is a category which the path category of a wheel quiver with n vertices. Objects are residues mod n and morphisms $a \to b$ are nonnegative integers ℓ so that $a + \ell = b \mod n$. For every point there is an endomorphism n. The geometric realization is still a circle. In fact, if you take geometric realization, this guy is forced to be an n-fold cover.

For every functor $f : [n]_\Lambda \to [n']_\Lambda$ induces a map on π_1, so by multiplication by some number, which I call the degree of f. If I normalize the morphism so that going around the loop is the generator, then the degree is nonnegative.

Definition 2.3. *The map f is non-degenerate if* $\deg f > 0$*, it's horizontal if the degree is* 1*, and it's vertical if it's a discrete bifibration.*

Definition 2.4. *The* cyclotomic category ΛR *(I don't know if there is standard notation) has objects positive integers* $[n]$ *and the maps are non-degenerate functors* $f : [n]_\Lambda \to [n']_\Lambda$.

The cyclic category Λ *is, I can take the subcategory of horizontal maps.*

You can ask what if I only consider vertical maps. Then what matters, the category disappears and what matters is the action of $\mathbb{Z}$, this is the category of finite $\mathbb{Z}$-orbits. In fact, this might be a fancy way to introduce this stuff but it makes the combinatorics cleaner.

Lemma 2.2. *Every map f in ΛR factorizes uniquely as $f = V \circ h$.*

Proof. Say that ℓ is the degree of f. For every degree I can take the degree ℓ discrete bifibration $[n'\ell]_\Lambda \to [n']_\Lambda$. I can take a pullback in some category

$$\begin{tikzcd} I \arrow[r] \arrow[d] & {[n'\ell]_\Lambda} \arrow[d, "\text{discrete bifibration}"] \\ {[n]_\Lambda} \arrow[r, "f"] & {[n']_\Lambda} \end{tikzcd}$$

It's easy to see that pulling back a discrete bifibration is a discrete bifibration. On fundamental groups, this sholud be multiplication by ℓ. so the map $I \to [n]_\Lambda$ is degree 1 so it splits. Then I can take the map to $[n'\ell]_\Lambda$ as h. $\square$

Exercise 2.1. *Any horizontal functor $f : [n]_\Lambda \to [n']_\Lambda$ has a left adjoint $f^\# : [n']_\Lambda \to [n]_\Lambda$.*

Corollary 2.1. *The category Λ is equivalent to its opposite by sending f to $f^\#$*

One third thing which is important for applications, we have the category Δ which is non-empty finite totally ordered sets.

It has a natural embedding into Λ. Consider the category $\Lambda/[1]$, which is a pair $[n]$ with a map $[n] \to [1]$. So consider the diagram of categories with the pullback

$$\begin{tikzcd} {[n]_\Delta} \arrow[r] \arrow[d] & {[n]_\Lambda} \arrow[d] \\ \mathrm{pt} \arrow[r] & {[1]_\Lambda}. \end{tikzcd}$$

I claim that $[n]_\Delta$ is a finite totally ordered set. One of the maps in $[n]_\Lambda$ goes to the generator in $[1]_\Lambda$ and the others go to the point.

The category $\Lambda/[1]$ forgets to Λ so this gives a map $\Delta \to \Lambda$.

Let me compute the homology and the cohomology of Λ.

For any small category I, I can consider $Fun(I, R)$, the Abelian category of functors $I \to R-\text{mod}$, and by definition homology and cohomology are the derived functors of direct and inverse limits.

Definition 2.5.

$$H_{\cdot}(I, -) = L^{\cdot} \lim_{\substack{\to \\ I}}$$

$$H^{\cdot}(I, -) = R^{\cdot} \lim_{\substack{\leftarrow \\ I}}$$

I could take the constant functor.

Proposition 2.1. (1)

$$H^{\cdot}(\Lambda, R) = R[u]$$

with u in degree 2 for a constant functor.

(2)

$$H^{\cdot}(\Lambda, E)$$

can be computed by a bicomplex, $E_i = E[i+1]$.

I can make this a simplicial R-module, with the standard differential b, an alternating sum of face maps $\sum_{i=0}^{n}(-1)^i d_i$. But I can also take $b' = \sum_{i=0}^{n-1}(-1)^{d_i}$ which is acyclic but still a differential. So my columns are E with differential b alternating with differential b'.

These have actions of the cyclic group since $\mathrm{Aut}([n])$ is $\mathbb{Z}/n\mathbb{Z}$.

I have $\sigma' = (-1)^{i+1}\sigma$, and then my horizontal differentials are $1 - \sigma'$ alternating with $(1 + \sigma' + \cdots \sigma'^{n-1})$. Composition is zero.

$$\begin{tikzcd}
{} \arrow[r] & E_0 \arrow[r, "1"] & E_0 \arrow[r, "0"] & E_0 \\
{} \arrow[r] & E_1 \arrow[u, "b"] \arrow[r, "1+\sigma'"] & E_1 \arrow[u, "b'"] \arrow[r, "1-\sigma'"] & E_1 \arrow[u, "b"] \\
{} \arrow[r] & E_2 \arrow[u, "b"] \arrow[r, "1+\sigma'+(\sigma')^2"] & E_2 \arrow[u, "b'"] \arrow[r, "1-\sigma'"] & E_2 \arrow[u, "b"] \\
{} \arrow[r] & E_n \arrow[u, "b"] \arrow[r, "1+\sigma'+\cdots+(\sigma')^n"] & E_n \arrow[u, "b'"] \arrow[r, "1-\sigma'"] & E_n \arrow[u, "b"]
\end{tikzcd}$$

Now $Fun(\Lambda, R)$ is generated by $R_{[n]}([n']) = R[\Lambda([n],[n'])]$. It is enough to consider $E = R_{[n]}$. So then $H.(\mathbb{Z}/n'\mathbb{Z}, R[\Lambda([n],[n'])])$, the action of $\mathbb{Z}/n\mathbb{Z}$ is free so the only cohomology is at degree 0. There we have the R-module generated by the quotient $\Lambda([n],[n'])/\mathbb{Z}/n'\mathbb{Z}$. So you have the space of all maps, you act by rotation, and you want to describe the quotient. The answer is that you can see where any object goes, and every map is uniquely a composition of a map that sends a prescribed guy to another prescribed guy followed by a rotation.

I can see that $\Delta^{\mathrm{op}} \to \Lambda$ are the maps which send a prescribed object to a prescribed object.

So in the end computing cohomology in the horizontal direction, all that is left is the standard chain complex of $\Delta_{n'}$.

This was a bit heavy but I can give you the positive side of this.

You can also produce a mixed complex out of this guy, I take the two columns and take the total complex of, call that $K.(E)$, then I get a map from the other differential $B : K.(E) \to K.(E)[-1]$, which together gives me a mixed complex. This construction has a sort of converse.

Definition 2.6. *I had this notation for a category of functors. Let $\mathcal{D}(I,R)$ be the derived category of $Fun(I,R)$. I say that $E \in \mathcal{D}(I,R)$ is* locally constant *if for any morphism $\alpha : i \to i'$, the corresponding map $E(i) \to E(i')$ is a quasiisomorphism.*

Denote by $\mathcal{D}_{\mathrm{lc}}(I,R)$ the full subcategory spanned by locally constant objects.

This is clearly a triangulated subcategory. The cone between locally constant things is also locally constant

Proposition 2.2.
$$\mathcal{D}_{\mathrm{lc}}(\Lambda, R) = \mathcal{D}F^{\mathrm{per}}(R) = \mathcal{D}^{mix}(R)$$

Such a functor inverts all maps, so it's a map from the fundamental groupoid. So since $\mathbb{CP}^\infty$ has no π_1, well, that [unintelligible]

The whole category is generated by the constant R and maps to itself are like $R[u]$ so you find a generator and show that maps match. I write that it's a proposition because it's conceptually important but the proof is trivial.

The rough program for next week is, I introduced ΛR. In a nutshell, these extra maps in characteristic p give you φ. I have something filtered that is locally constant, but I need these extra maps φ, and the procedure of doing this, you need more than just maps from ΛR. The something more is some homological phenomenon which is interesting. There will be two lectures. The first one will be on Mackey functors. I also have vertical maps ΛR_V. There's a story for any group

G. There is an interesting category of functor associated to this. On Wednesday I'll put things back together to work on Λ.

3. December 1

So the topic today is not related to anything last week and the topics will come together in the last lecture on Wednesday. Today the topic is Mackey functors. You can treat this as an exercise in homological algebra.

One starts with a group, let's say a finite group G, and I denote by Γ_G the category of finite G-sets. I can consider the opposite category. Fix a coefficient ring R, and then

Definition 3.1. *A functor $E \in Fun(\Gamma_G^o, R)$ is additive iff $E(S \sqcup S') \to E(S) \oplus E(S')$ is an isomorphism.*

This says that the value of the functor is determined by its action on sets with transitive action, also known as orbits, these are sets of the form $[G/H]$ where H is some subgroup. Then the category of additive functors is equivalent to the category of all functors on G-orbits:

$$Fun_{\mathrm{add}}(\Gamma_G^o, R) \cong Fun(O_G^o, R).$$

Now consider the category $Q\Gamma_G$, whose objects are again finite G-sets but where the morphisms from S_1 to S_2 are some kind of correspondences, they are isomorphism classes $S_1 \leftarrow S \to S_2$ in Γ_G.

Definition 3.2. *A G-Mackey functor is a functor $E \in Fun(Q\Gamma_G, R)$ such that $E|_{\Gamma_G^o}$ is additive.*

I should have said, I have natural embeddings $\Gamma_G \to Q\Gamma_G$ and $\Gamma_G^o \to Q\Gamma_G$.

Denote the category of G-Mackey functors by $M(G, R)$. So Mackey had very little to do with this definition. This was introduced in 1973 by Dress, and then further clarified. This definition is due to Lindner. People use this in group theory, pure algebra, and also in algebraic topology, the motivation comes from stable homotopy theory, G-equivariant stable homotopy theory. You have some X a topological space with an action of G, and you consider maps between these guys up to equivariant homotopies. Then for any subgroup H, the space of H-fixed points are homotopy invariant. Naively you think you get a homological action of G, but you get more. You can think of this as $Map_G([G/H], X)$. So then $C_*(X^H, R)$ for a complex in $Fun(\Gamma_G^o, R)$.

But then when we go to the stable homotopy category, it's kind of a general principle that whenever I have a finite cover f, in the stable homotopy category I get a "transfer map" in the opposite direction. So in the stable theory you should get maps in both directions. In fact, you get $Q\Gamma_G$. This is kind of a long story and it goes in kind of the opposite direction of the pure algebra I want to do, but as motivation it's good to keep in mind.

This category has some structure. If you have $H \subset G$, we can consider $[G/H]$, and look at $Aut_G[G/H]$. This is N_H/H, which we'll denote W, for Weyl. Then we have a functor $\varphi^H : \Gamma_G \to \Gamma_W$ where $X \mapsto X^H$. If you have a fiber product, and you take fixed points, it's still a fiber product. So φ^H preserves fibered products. So we can extend this to $Q(\varphi^H) : Q\Gamma_G \to Q\Gamma_W$. This induces adjoint functors between the categories of Mackey functors. We have $M(G, R)$ and $M(W, R)$, the functor from $M(W, R)$ to $M(G, R)$ is called inflation, and it is pullback with respect to $Q(\varphi^H)$. The adjoint functor Φ^H, called geometric fixed points, which is Kan extension $Q(\varphi^H)!$. You have to make sure this preserves additivity.

What is nice about this functor, there is a lemma

Lemma 3.1. *Consider $N \subset G$ a normal subgroup with $W = G/N$. Then $Infl^N$ is fully faithful, and $E \to Infl^N \Phi^N E$ is surjective. So a Mackey functor gets a filtration indexed by normal subgroups of G.*

There is another structure that maybe I don't even need.

Lemma 3.2. *We have this embedding $M(G,R) \subset Fun(Q\Gamma_G, R)$, and this has a left adjoint additivization, I can just apply Add and make a functor additive.*

You can take the product of two G-sets and this gives a functor $m : \Gamma_G \times \Gamma_G \to \Gamma_G$, and this lifts to $Q(m) : Q\Gamma_G \times Q\Gamma_G \to Q\Gamma_G$. This allows us to define a tensor product.

Definition 3.3. *If R is commutative, then*

$$E_1 \circ E_2 = AddQ(m)!(E_1 \boxtimes E_2)$$

Then $\circ$ makes $M(G,R)$ into a symmetric monoidal category. It actually has a unit object. It's nice because it doesn't come directly from representation theory. The unit object $\mathcal{A} \in M(G,\mathbb{Z})$ is called the Burnside Mackey functor. Its values are as follows, $\mathcal{A}([G/H]) = A^H$, the Burnside ring of H. Take the isomorphism classes of finite H sets, generate the free Abelian group, and mod out by the relation that $S \sqcup S' = [S] + [S']$. The product is induced by a product of sets. You have some matrix coefficients for the product. This was introduced by Burnside a long time ago. This turns out to be the unit object in this category (and indeed be a Mackey functor).

As I said this is an Abelian category and if I want to work homologically I can consider its derived category. This Abelian category has been around for 40 years, but it turns out that the derived category $\mathcal{D}(M(G,R))$ is not the right thing to consider. Some nice theorems start to fail at this level, for instance that the inflation functor is fully faithful. Also it's clear why this is the wrong thing to consider, in $Q(\Gamma)$ you get isomorphism classes of morphisms, but you expect that automorphism groups should act and there should be higher group homology. So the reason is that $Q\Gamma_G$ is really a 2-category. I want to say that the category of derived Mackey functors sits inside the derived category $\mathcal{D}(\mathcal{Q}\Gamma_G, R)$ (this is the 2-category). There are multiple ways to do this but I'll use nerves.

Recall that, well, begin with a 1-category. Recall that the nerve $N(\mathcal{C})$ of a small category $\mathcal{C}$ is a simplicial set, a functor $\Delta^o \to Sets$, one usually says that objects in Δ^o are sets $[n]$, and then $[n]$ is evaluated by the nerve to the diagrams like this $\{c_0 \to \cdots \to c_n\}$ in $\mathcal{C}$. So it's useful to do a Grothiendieck construction like last time here, we want to define another category $\mathcal{N}(\mathcal{C})$, which is a pair $[n] \in \Delta$ and then a diagram c as before. Morphisms are maps from $([n'], c') \to ([n], c)$, the map $g : [n'] \to [n]$ should satisfy that $g^*c = c'$.

I also have a natural functor $\mathcal{N}(\mathcal{C}) \to \mathcal{C}$ which sends $([n], c)$ to c_n. I have a pullback

$$q^\# : \mathcal{D}(\mathcal{C}, R) \to \mathcal{D}(N(\mathcal{C}), R)$$

Say that a map f in $\mathcal{N}(\mathcal{C})$ is special if $f(n') = n$. An object is special if $E(f)$ is an isomorphism for all special f. Then it turn sout that $q^\#$ is an equivalence to the full subcategory spanned by special objects.

This looks pretty useless. But the observation is that if $\mathcal{C}$ is a 2-category, this is still a perfectly good 1-category. Let me write this down. The functor $N(\mathcal{C})$ is a functor to categories, not sets. But $\mathcal{N}(\mathcal{C})$ is still a 1-category. Objects are pairs $\langle [n], c.\rangle$ and morphisms are almost as before, a morphism is given by a map $g : [n'] \to [n]$ between ordinals along with a map $\alpha : c'_. \to g^*c.$. The rest goes through, with special maps. Then you consider the same subcategory and you get a repleacement for the derived category.

Definition 3.4.

$$\mathcal{D}(\mathcal{C}, r) := \mathcal{D}_{\mathrm{sp}}(\mathcal{N}(\mathcal{C}), R)$$

Why is this right? You can take an alternative cumbersome approach, take the geometric realization of the nerve of $\mathbb{Z}[\mathcal{C}(c, c')]$ for $\mathcal{C}$ a 2-category. If you do things carefully enough, you can make this an A_∞ category, and then prove that this gives the same answer.

This is unpleasant to work with because you need all the A_∞ relations. Another motivation is that in the literature, people always work like this, Segal spaces or whatever. This is the only thing that you could conceivably do.

I want not just any 2-category, I want $Q\Gamma_G$. We can consider $\mathcal{N}(Q\Gamma_G)$, but then diagrams will be like

$$\begin{tikzcd}
 & S_{01} \arrow[dl] \arrow[dr] & & {} \arrow[dl] & \cdots & S_{n-1,n} \arrow[dl] \arrow[dr] & \\
S_0 & & S_1 & & \cdots & & S_n
\end{tikzcd}$$

We can also use a smaller thing, $\mathcal{C}$ has fibered products, so you can do this

Definition 3.5. *$S\mathcal{C}$ has objects $c_0 \to \cdots \to c_n$ and morphisms are $\langle g, \alpha$ where $g : [n'] \to [n]$ and $\alpha : g^*c. \to c'_.$.*

So you get something like this:

$$\begin{tikzcd}
c'_0 \arrow[r] & \cdots \arrow[r] & c'_{n'} \\
c_{g(0)} \arrow[u, "\alpha"] \arrow[r] & \cdots \arrow[r] & c_{g(n')} \arrow[u, "\alpha"']
\end{tikzcd}$$

So you get a bunch of squares and the condition is that the diagram

$$\begin{tikzcd}
c'_i \arrow[r] & c'_{j'} \\
c_{g(i)}c_{g(j')} \arrow[u, "\alpha"] \arrow[r] & {} \arrow[u, "\alpha"']
\end{tikzcd}$$

is Cartesian so they're all determined by the last one.

I say that a map g, α is special if $g(n') = n$ and α is an isomorphism.

Definition 3.6. *I let $\mathcal{D}S(\mathcal{C}, R)$ in $\mathcal{D}(S\mathcal{C}, R)$ is the full subcategory spanned by special objects*

Lemma 3.3. *$\mathcal{D}S(\mathcal{C}, R) \cong \mathcal{D}_{\mathrm{sp}}(\mathcal{N}(\mathcal{QC}), R)$.*

If those diagrams had no automorphisms, I could take the one-categorical version but in general you can't do that.

Let me give the definition of derived Mackey functors and then we'll take a break

Definition 3.7. *A* derived Mackey functor *is an object $E \in \mathcal{D}S(\Gamma_G, R)$ such that restricting it to $\Gamma_G^o \subset S\Gamma_G$, $e^*E \in \mathcal{D}(\Gamma_G^o, R)$ is additive.*

Okay, so example. Let's take the simplest group one can think of, $\mathbb{Z}/p$ where p is prime. There are only two orbits. They are $[G/G]$ and $[G/e]$, where e is the trivial subgroup. What is a G-Mackey functor? It should have an R-module E^G and an R-module E^e which is an $R[G]$-module. Let σ be a generator. Then there is an automorphism σ of E^e of order p. There is one map $V : (E^e)S\sigma \to E^G$, and a map the other direction $F : E^G \to (E^e)^\sigma$. There is a compatibility because the composition of V and F will decompose as a union of p copies of itself, so $F \circ V = Id + \sigma + \cdots + \sigma^{p-1}$. For every finite group there is a map coinvariants to invariants given by averaging over the group. That's this map.

What about derived Mackey functors? I won't prove this but let me give an answer. Now $E^e_\cdot$ and $E^G_\cdot$ are complexes, and now instead of invariants and coinvariants we get

$$C_\cdot(\mathbb{Z}/p\mathbb{Z}, E^e_\cdot) \xrightarrow{V} E^G_\cdot \xrightarrow{F} C^\cdot(\mathbb{Z}/p\mathbb{Z}, E^e_\cdot)$$

with the same condition. You can choose a model for $E^e_\cdot$ which is injective or projective, so that one of the two sides of this collapses, but you can't find a simultaneous choice so that both of them collapse. This is the reason that the categories are different.

Let me do the following trick, giving a different description of the same data. As my objects I chose these two guys and got these structure maps, but I can instead look at $\bar{E}^G_\cdot$, which is $Cone(V)$. SO then I can take $\langle E^e_\cdot, \bar{E}^G_\cdot$, and now what I need is $\varphi : \bar{E}^G_\cdot \to \check{C}_\cdot(G, E_c dot^e)$, the Tate cohomology complex, the cone of the trace map $Id + \sigma + \cdots + \sigma^{p-1}$.

This is somehow more effective because the Tate cohomology is often zero.

This can be rephrased as follows. One way to compute Tate cohomology, take a projective resolution of $\mathbb{Z}$ in $\mathbb{Z}[G] - mod$, let $\tilde{P}$ be the cone of this, $\cdots \to P_1 \to P_0 \to \mathbb{Z}$. Then $\check{C}_\cdot(G, E) = \lim C^\cdot C^\cdot(G, E \otimes F^\ell \tilde{P})$. This means that $\mathcal{D}M(G, R)$ can be described as dg-modules over:

$$\begin{pmatrix} R[G] & \tilde{P}_\cdot \\ 0 & R \end{pmatrix}$$

This $R[G]$ is finite dimensional, but its modules have infinite dimension, this means that you get things that are acyclic but not contractible. You could take a field, dual numbers, and the same thing. You'll get two things that are quasiisomorphic but with different categories of dg-modules.

Morally this next part is known but I couldn't find it in the literature

Maximal Tate cohomology. Let G be a finite group and let $\mathcal{D}_{\text{ind}}(G, \mathbb{Z}) \subset \mathcal{D}^b(G, \mathbb{Z})$ be the smallest triangulated subcategory containing $\text{ind}_G^H(E_\cdot)$ for $E_\cdot \in \mathcal{D}^b_{fg}(G, H)$, $H \subsetneq G$.

Consider the quotient $\bar{\mathcal{D}}(G, \mathbb{Z}) = \mathcal{D}^b(G, \mathbb{Z})/\mathcal{D}_{\text{ind}}(G, \mathbb{Z})$. The observation is that sometimes this is not the whole thing.

Definition 3.8. *The maximal Tate cohomology groups $\check{H}^\cdot_{max}(G, E)$ are $Hom^\cdot_{\overline{\mathcal{D}}(G,\mathbb{Z})}(\mathbb{Z}, E\cdot)$.*

Definition 3.9. *A complex $\tilde{P}$ is* maximally adapted*, $\mathbb{P}_0 = \mathbb{Z}$ if*

(1) *P_i is induced for all i, and*
(2) *$\tilde{P}_\cdot|_H$ is contractible.*

For example, if H is just the trivial group, then P is projective and contractible. The first condition becomes weaker and the second stronger as you add subgroups.

Then $\check{H}^{\cdot}_{max}(G,E) = \lim H^{\cdot}(G, E\otimes F^{\ell}\tilde{P})$ for a maximally adapted $\tilde{P}$.

Definition 3.10. *For E a complex of $R[G]$-modules, $\check{C}_{max}(G,E) = \lim C^{\cdot}(G, E\otimes F^{\ell}\tilde{P})$.*

Consider $\mathcal{D}M(G,R)$, the statement is true as in the ordinary Mackey functor case, that Infl^N is fully faithful for N a normal subgroup of G.

Before I had Φ^N, the fixed points, this was a W-Mackey functor. I can take its value on the free orbit. Let me define $\bar{\Phi}^N(E)$ as $\Phi^N(E)([W/e])$, which sits in $\mathcal{D}(R[W])$. One shows it has a right adjoint $\overline{\mathrm{Infl}}: \mathcal{D}(R(W)) \to \mathcal{D}M(G)$, and this guy is also fully faithful.

For commutative groups where all subgroups are normal, the images of these generate the whole derived category. If the group is non-commutative, I can define $\bar{\Phi}^H$ and its adjoint is still fully faithful, the whole category has a filtration by this.

I can also descibe the extension data. Being fully faithful is equivalent to saying that if I take $\bar{\Phi}^H \circ \overline{\mathrm{Infl}}^H \cong Id$. What if I take fixed points with respect to one group but inflate with respect to another? What's $\bar{\Phi}^G \circ \overline{\mathrm{Infl}}^e : \mathcal{D}(R[G]) \to \mathcal{D}(R)$. The claim is

Proposition 3.1. *This is isomorphic to $\check{C}^{\cdot}_{\max}(G,E)$.*

I gave you this category. Aside from general interest, I believe this is useful to give counterexamples and so on.

One final thing I want to say today that I will need for applications on Wednesday. There are two. We'll need the group $\mathbb{Z}$ and everything I've said has been about finite groups. I'll talk about that next time. Now let me make some comments about $\mathbb{Z}/n\mathbb{Z}$ for some n. Let's consider the arbitrary case.

It turns out I can have a rather explicit description of the category. It's pretty similar to this coalgebra. Roughly speaking, it's difficult because you have a coalgebra with an upper triangular shape but with arbitrary coefficients. The maximal Tate cohomology is zero unless you have prime coefficients. You turn out not to need anything A_∞.

Denote by I_n the groupoid of finite G-orbits and isomorphisms between them. For any L, let me denote by pt_ℓ the point with automorphisms $\mathbb{Z}/\ell\mathbb{Z}$. Then I_n is the disjoint union of these over divisors.

For p let I_n^p bet the subcategory where $p|\ell$ (with embedding i). We have two projections, we can go $\pi : I_n^p \to I_n$ where $pt_{p\ell} \to pt_\ell$, since there's a quotient map $\mathbb{Z}/p\ell \to \mathbb{Z}/\ell$. Let $I_n^{\cdot} = \sqcup_p I_n^p$. Then there are two projections, i or π to I_n. We can consider the category of functors $(I_n^{\cdot}, \mathbb{Z})$, this is just a collection of representations of cyclic groups. Fix a projective resolution $P.$ of $\mathbb{Z} \in Fun(I_n^{\cdot}, \mathbb{Z})$, and as before let $\tilde{P}$ be the cone of this augmentation map.

Theorem 3.1. *The category $\mathcal{D}M(\mathbb{Z}/n\mathbb{Z}, R) \cong \{(E.,\alpha) | E. \in Fun(I_n, R); \alpha : i^*E. \to \pi^*E. \otimes \tilde{P}\}$, let me denote this by $\mathcal{D}^{\alpha}(I_n, P, R)$. It does not depend on the choice of P, and it's equivalent to the $\mathcal{D}M(\mathbb{Z}/n\mathbb{Z}, R)$.*

This is basically the same as what I was doing before. Normally for a coalgebra there should be a condition where composing with itself gives itself. Here the coalgebra is so trivial that you don't need this. The basic reason this works is that you can show that maximal Tate cohomology of $\mathbb{Z}/n\mathbb{Z}$ is trivial unless n is a prime. So we only get nontrivial gluing between pieces which differ by a prime.

In the next lecture I'll plug in the cyclic category. When the coefficient ring is p-local, we'll get something familiar. But there will be a version with $\mathbb{Z}$. That's for next time.

4. December 3

Today I want to bring things together, combine the Mackey functors from last time with the cyclic categories of last week. But first I should say what to do for Mackey functors for infinite groups. In the topological setting they normally consider compact Lie groups. That's kind of orthogonal, I want to consider infinite discrete groups.

Formally I don't need to do anything. The theory works. I could consider Γ_G, the category of finite G-sets. I need this because otherwise things become zero. But this replaces G with its profinite completion.

There is an alternative even here. Enlarge your category of G-sets. Say a G-set is admissible if

(1) for any $s \in S$, the stablizer of s is finite, and
(2) for all cofinite H, the fixed points of H is finite.

So S is a disjoint union of orbits $[G/H]$ where for N a normal cofinite subgroup, there are only a finite number of indices where H_i contains N.

Denote by $\hat{\Gamma}_G$ the category of admissible G-sets.

Definition 4.1. *E in $Fun(\hat{\Gamma}_G, R)$ is additive if*

$$E(\sqcup S_i) \to \prod E(S_i)$$

is an isomorphism.

I can still take fibered products, look at $Q\hat{\Gamma}_G$, and $\widehat{S\Gamma}_G$.

Definition 4.2. *A G-*Mackey profunctor *is a functor $E : Q\hat{\Gamma}_G, R)$ such that $E|_{\hat{\Gamma}_G^o}$ is additive. These are denoted $\hat{M}(G, R)$.*

A derived G-Mackey profunctor *is $E \in \mathcal{D}S(\hat{\Gamma}_G, R)$ such that $E|_{\hat{\Gamma}_G^o}$ is additive. These are denoted $\widehat{DM}(G, R)$.*

We have all the fixed point functors. Now for normal cofinite N we have this inflation functor $Infl^N : \mathcal{D}M(G/N, R) \to \widehat{DM}(G, R)$ which is fully faithful. For every M, we can take the limit with respect to normal subgroups. You need to derive this, this has to be defined in a precise way. This is $\lim Infl^N \Phi^N M$. If M is a bounded above as $\widehat{DM}(G, R)$ and G is finitely generated (really what you need is for any cardinality, there are only many finitely many quotients of that cardinality), then this is an isomorphism.

Strangely, you need to go to the derived version, you have a right exact, an exact, and something left exact. The composition is not a derived functor. I could maybe construct an example where if you remove the derived stuff this is not true. But that's a technicality. If you want to consider $\widehat{DM}^-(G, R)$, you might as well consider this system over normal subgroups.

To understand the difference between Γ_G and $\hat{\Gamma}_G$, let me consider $A^{\mathbb{Z}}$, which is generated by all orbits, which are numbered by integers, so this is $\mathbb{Z}[\epsilon_1, \epsilon_2, \dots]$

where ϵ_n corresponds to $\mathbb{Z}/n\mathbb{Z}$. Then $\epsilon_n\epsilon_m = \frac{nm}{\{n,m\}}\epsilon_{\{n,m\}}$ where $\{n,m\}$ is least common multiple.

Then $\hat{A}^{\mathbb{Z}} = \mathbb{Z}[[\epsilon_1, \epsilon_2, \ldots]]$. By definition this is spanned by admissible things modulo relations and [missed]. This is better than $A^{\mathbb{Z}}$ because for example, if I take R to be p-local for some prime, so everything prime to p is invertible, then elements $\frac{1}{n}\epsilon_n$ is idempotent in the Burnside ring. We have commuting idempotents. In $\hat{A}^{\mathbb{Z}}$ you can transfer them to orthogonal idempotents, for $p \nmid n$, they can be simultaneously diagonalized. The Burnside Mackey functor is in the center of the category and acts on all the objects in the category. In this case that gives:

Proposition 4.1. *If R is p-local, then $\widehat{M}(\mathbb{Z}, R) \cong \prod_{p\nmid n} \widehat{M}(\mathbb{Z}_p, R[\mathbb{Z}/n\mathbb{Z}])$.*

This can be called a "p-typical decomposition" because this $\hat{A}^{\mathbb{Z}}$ is the Witt vectors of $\mathbb{Z}$.

In the interest of full disclosure I should say this is lifted from p-adic representations.

Now let's do the plug in the cyclic part of things. When I defined the cyclic category I used a strange definition that people were unhappy with but now there's the payoff.

Definition 4.3. *A small category I is* admissible *if $I = \sqcup_S [n_s]_\Lambda$ with $[n_s] \geq 1$ such that for all $n \geq 1$ there is only a finite number of s such that $n = n_s$ (or divides or is less than, all of these give the same definition). When I have such a disjoint union, what is a functor between two of these? I should have a map $f : S \to S'$, and then for every s there is a map $f_s : [n_s]_\Lambda \to [n'_{f(s)}]_\Lambda$. Any functor is of this shape.*

Definition 4.4. *I say that $(f, \{f_s\})$ is* nondegenerate *if the degree of f_s is nonzero, that it is* horizontal *if the degree is 1 for every f_s and f is invertible. We say it is* vertical *if f_s is a discrete bifibration.*

Definition 4.5. *Let $\widehat{\Lambda R}$ be the category of admissible categories and non-degenerate functors.*

When I introduced the cyclotomic category last week I used the same thing but without disjoint union. So $\Lambda R \subset \widehat{\Lambda R}$. The vertical maps in ΛR_v are orbits. Now in $\widehat{\Lambda R}$ the vertical maps are $\hat{\Gamma}_{\mathbb{Z}}$.

Lemma 4.1. (1) *any f in $\widehat{\Lambda R}$ can be uniquely factorized as $f = v \circ h$.*
(2) *any diagram*

$$\begin{tikzcd} & I_2 \arrow[d, "v"] \\ I_1 \arrow[r, "f"] & I \end{tikzcd}$$

extends to a Cartesian diagram

$$\begin{tikzcd} I_k \arrow[r] \arrow[d, "v"] & I_2 \arrow[d, "v"] \\ I_1 \arrow[r, "f"] & I \end{tikzcd}$$

I don't really need the first thing but I need the second. Because of it, I can play the same game and define $Q\widehat{\Lambda R}$ to have the same objects but maps the isomorphism classes of diagrams

$$\begin{tikzcd} & I \arrow[dl, "v"'] \arrow[dr] & \\ I_1 & & I_2 \end{tikzcd}$$

Now let $S\widehat{\Lambda R}$ be the same as $S\mathcal{C}$ with objects compositions of vertical maps $I_0 \xrightarrow{v} I_1 \xrightarrow{v} \cdots \xrightarrow{v} I_n$

Definition 4.6. *A* cyclotomic derived Mackey profunctor *is an object $E \subset \mathcal{D}S(\widehat{\Lambda R}, R)$ such that $E(I)$ is isomorphic to $\prod_S E([n_s])$ for every I which is admissible.*

These for the category $\widehat{\mathcal{D}M}(\Lambda R, R)$.

This category is what will contain the filtered Dieudonné-modules. You take the subcategory where horizontal maps induce isomorphisms. I need some other model which will be much smaller. What I want to do here is the same thing I did with cyclic groups last time. The answer for what Mackey functors were had two answers, a Koszul dual description. The same description is possible here.

Let me explain what this is.

I need to start with something like the groupoid of all orbits.

Definition 4.7. *For any $n \geq 1$ let Λ_n be the category whose objects are vertical maps $[mn] \to [n]$ in ΛR of degree n and morphisms are diagrams*

$$\begin{tikzcd} {[m,n]} \arrow[r, "v"] \arrow[d, "h"] & {[m]} \arrow[d, "v"] \\ {[m',n]} \arrow[r, "v"] & {[m']} \end{tikzcd}$$

There are maps $\Lambda \xleftarrow{i} \Lambda_n \xrightarrow{\pi} \Lambda$, there are some extra automorphisms. So for example we could have $|\Lambda_n| = \mathbb{CP}^\infty$ and i is an isomorphism but π is induced by the n-fold cover of the circle. So the fiber of this map is the classifying space for a finite group. This realizes this picture categorically.

Okay, fine. If you consider only vertical maps you get groupoids of orbits and it's exactly the picture from last lecture.

Let $\Lambda = \sqcup_p \Lambda_p$ and choose a resolution $P.$ of the constant functor $\Lambda \to \mathbb{Z}$. Then

Definition 4.8. *A* cyclotomic complex *is a pair $(E., \alpha)$ where $E.$ is some complex in cyclic objects, E is a complex in $Fun(\Lambda, R)$. Then $\pi^* E \xrightarrow{\alpha} \tilde{P} \otimes i^* E$. Here $\tilde{P}$ is the cone of the augmentation $P \to \mathbb{Z}$.*

I have a feeling that last time I made a mistake and had this map going in the other direction, but this is how it should go.

These are cyclotomic complexes. Inverting quasi-isomorphisms, you obtain a category that should be denoted $\mathcal{D}\Lambda R(R)$ with some P but it doesn't depend on P.

Theorem 4.1.

$$\widehat{\mathcal{D}M}^{-}(\Lambda R, R) \cong \mathcal{D}\Lambda R(R).$$

This is again an exercise in Koszul duality. Now is a good time for a break.

Definition 4.9. *A* generalized filterd Dieudonné module over *R is* $\langle M, F^{\cdot}, \varphi^p_{i,j}$ *where* $\varphi^p_{i,j} : F^iM \to M/p^jM$ *for* $i, j \in \mathbb{Z}$ *and p a prime. I want the colimit of* $F^{-i}M$ *to be M and also the limit of* M/F^iM*.*

I should also have $\varphi^p_{i,j} = \varphi^p_{i,j+1}$ *modulo* p^j *and* $\varphi^p_{i,j}|_{F^{i+1}} = \varphi^p_{i+1,j}$*.*

Effectively you just replace these with $\varphi^p_i : M \to \hat{M}_p$ which is the limit of M/p^iM. This combines structures for all p that don't talk to each other. If M is p-local, then M/q^jM vanishes except for $q = p$. Then the only thing that survives is φ^p_i and I recover filtered Dieudonné modules.

There is a fine point. I should get derived 2-periodic filtered modules, not filtered modules on the nose. Denote by $\mathcal{D}^{\text{per}}(gFDM)$ the twisted 2-periodic derived category of $gFDM$. So we should have a complex of M with an equivalence $M. \cong M.[2](1)$. Here (1) is the twist.

Theorem 4.2. *The category* $\mathcal{D}^{per}(gFDM) \cong \mathcal{D}\Lambda R^b_{lc}(\mathbb{Z})$*. Here lc means locally constant and b means bounded.*

In order to do this I need to revisit the correspondence between these two and make it more precise.

I could use whatever resolution P I want but I'll choose one which makes things simple. Already if you consider $\mathbb{Z} \in Fun(\Lambda, \mathbb{Z})$, there is one kind of preferred one which is periodic. We have $u \in H^2(\Lambda, \mathbb{Z})$ and this can be represented by Yoneda by a certain complex

$$0 \to \mathbb{Z} \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

Think of a circle, think of $P_i([n]) = C_i(S^1[n], \mathbb{Z})$. There are n zero-cells and n-one cells. They are free $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-modules of rank one. The cohomology of S^1 does not depend on the decomposition.

In fact, I can cook up a complex

$$\to P_1 \to P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} P_0$$

This will be the standard 2-periodic resolution of the trivial representation, for every $[n]$.

Now let me observe one thing about this complex right away.

Lemma 4.2. *You have these maps* π *and* i *from* Λ_n *to* Λ*. I can take* $\pi_*i^*P_i$ *and the claim is that this is still* P_i*. The differential* d_1 *is still* d_1 *and* $\pi_*i^*d_0$ *is* nd_0*.*

The differentials are $1 - \sigma$ and $1 + \sigma + \cdots + \sigma^{m-1}$. You have $1 + \sigma + \cdots + \sigma^{mn-1}$ which is the same as $(1 + \sigma + \cdots + \sigma^{m-1})\underbrace{(1 + \sigma^m + \cdots)}_{n}$

Let me make it clear that α gives those maps φ_i. Consider periodic filtered complexes V, the direct limit of $F^{-i}V$ and also the inverse limit of the quotients V/F^iV. Periodic means I have this identification $V \cong V[2](1)$. It's convenient to consider Rees objects. We have $V_{\cdot,\cdot}/R[t]$ where the degree of t is one. I need to impose the inverse limit condition, which says that $V_{\cdot,\cdot}$ is t-adically complete. So $V_{\cdot,\cdot}/t^i$ is an isomorphism.

Okay. Now let me define a functor, cyclic expansion, from filtered modules to cyclic objects which works like this.

$$Exp(V_{\cdot,\cdot}) = (V_{0,\cdot} \otimes P_1)[1] \oplus (V_{0,\cdot} \otimes P_0)$$

The differential d is the sum of $d_V \otimes id$ and

$$\begin{cases} Id \otimes d_1 \\ Id \otimes d_0 \end{cases}$$

So if I took the stupid filtration,

Example 4.1. *Let $V = \mathbb{Z}$. Then $Exp(V.)$ is $\mathbb{Z} \in Fun(\Lambda, \mathbb{Z})$ represented by this resolution $P.$.*

Proposition 4.2. *The functor Exp is an equivalence between $\mathcal{D}F^{per}(R)$ and $\mathcal{D}_{lc}(\Lambda, R)$.*

The proof is basically the same as last week. All the locally constant functors are just constant, and you just need to check that you get an isomorphism on Ext.

The statements, now, are like this.

Definition 4.10. *For any filtered complex $V.$, with Rees object $V_{.,.}$, its n*th subdivision *$Div_n(V_1)$ is $V_{.,.}$ where t acts by nt.*

There is one twist. I can formally do this, but I want my guys to be t-adically complete and after multiplying by n this may not be true any more. So I need to take again the completion.

I'll give another definition and then an example.

Definition 4.11. *The* stabilized subdivision *$Stab_n(V.) = \xrightarrow{lim} Div_n(V)$.*

Example 4.2. *Take $V = \mathbb{Z}$ with $F^1 = 0$ and $F^0 = \mathbb{Z}$. Let $n = p$. The Rees object is*

$$0 \to 0 \to \mathbb{Z} \xrightarrow{t} \mathbb{Z} \xrightarrow{t} \mathbb{Z} \xrightarrow{t} \mathbb{Z}.$$

I multiply by p:

$$0 \to 0 \to \mathbb{Z} \xrightarrow{pt} \mathbb{Z} \xrightarrow{pt} \mathbb{Z} \xrightarrow{pt} \mathbb{Z}.$$

now I need to take the limit, and I get $Div_p(V) = \mathbb{Z}[\frac{1}{p}]$ and $Stab_p(V) = \mathbb{Q}_p$.

Inside here $F^i\mathbb{Q}_p = p^i\mathbb{Z}_p$

Lemma 4.3. *For any M, $Stab_p(M)$ is $\hat{M}_p \otimes \mathbb{Q}_p$ with filtration $F^i = p^i\hat{M}_p$.*

We're almost done, now let's compare this business with the definition of cyclotomic complexes.

Lemma 4.4. (1)

$$\pi_* i^* Exp(V.) = Exp.Div_p(V.)$$

(2)

$$Exp(Stab_p(V.)) = \lim \pi_*(i^*(V.) \otimes F^{2\ell} i^* \tilde{P}.)$$

This means that for every ℓ, we have $F^{2\ell}$ is just $\mathbb{Z}[2\ell]$ (this is the stupid filtration). This is the same as taking subdivision and twisting by ℓ.

Now this proves the theorem, to construct a functor. The lemma identifies $\mathcal{D}\Lambda R^b_{lc}(R)$ with $\mathcal{D}F^{\mathrm{per}}(R)$ via $\alpha : \pi^*E. \to i^*E. \otimes \tilde{P}$ which is the same as $\varphi : E \to \pi_*(i^*E \otimes \tilde{P})$. Then $\varphi : V. \to Stab_p(V.)$ with $\varphi^i : F^i(V.) \to (\hat{V.})_p$ with $\varphi^i|_{F^{i+1}} = p\varphi^{i+1}$.

Okay this is bizarre but this is how things work. Just some final observations.

I completely dropped the condition that things should be isomorphisms that give a nice category. I do this packaging differently than in normal FDMs. There

I had a map $\tilde{\varphi} : \hat{M} \to M$ which I wanted to be an isomorphism, but here I have $\varphi : M \to Stab_p M$. I don't know how this repackaging affects this being an isomorphism.

[example where you *don't* expect an isomorphism].

In this case $F^i\mathbb{Z}_p = p^i\mathbb{Z}_p$ and the maps φ^i are the identity map. Going one level down I should get divisibility by p. I can take the identity map at every step of the process. I cannot truncate because the collection is surjective. Each of the φ^i is an isomorphism but $\tilde{\varphi}$ is not an isomorphism, torsion but huge. This is an example of the things that come up. This corresponds to so-called topological Hochschild cohomology, cristalline cohomology, slightly bigger. There are two worlds, one where characteristic p things live and the other for things that lift to $\mathbb{Z}_p$. I'm going to think more about this.

My final remark was that all of this was in characteristic p. The packaging I would also like for Hodge structures. This looks not as bad as the original definition, you don't need to specify p, you don't need to guess. I'd like some singular notion that would incorporate Hodge structure but I don't know. It's not a well-posed question. In any case it's pure linear algebra, too simple, but still it would be nice to have.