INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS IRREGULAR SEMINAR

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1. March 4: Hiro Tanaka: Factorization homology and stratified manifolds

Thanks to Gabriel for extending the invitation, everyone has been very nice so far. The title of my talk, more or less, will be "Factorization homology and stratified spaces." I'll talk about work joint with David Ayala and John Francis.

What is factorization homology? There's the word homology in there. It's a generalization of the usual homology. In usual homology, you fix an Abelian group A and out of this we get an invariant of topological spaces $H_*(—, A)$, and this forms an invariant of topological spaces. It's an invariant you can compute with a local to global sequence. That's one reason we really like homology. If you're studying manifolds that's not a strong invariant. There exist homotopy equivalent manifolds that are not diffeomorphic so ordinary homology can't tell the difference.

Factorization homology lets you pass again from local to global data. Now we want to construct a homology theory for manifolds. The local data for an n-dimensional manifold is $\mathbb{R}^n$, so we want to put algebraic data to put on that and it'll be an E_n-algebra A. Then we'll construct an invariant of n-manifolds. The kinds of manifolds I'll consider will be non-compact. So A is what you'd assign to $\mathbb{R}^n$, which is not compact.

The first thing I should explain. Everyone knows what an Abelian group is. Not everyone knows what an E_n algebra is. The definition I'll give isn't most standard but it's most useful.

Definition 1.1. *A* framed manifold *is a smooth manifold X with a trivialization $\phi_X : T_X \cong X \times \mathbb{R}^n$ of its tangent bundle. Every smooth manifold comes with a tangent bundle but not every one admits a framing. A framed manifold, I mean it should come equipped with one. For example, the space of maps $X \to GL_n$ acts on the space of framings.*

Definition 1.2. *An embedding of framed manifolds is a pair (j, h) where j is a smooth embedding between two manifolds and h is a homotopy from the framing of Y pulled back to X, $j^*\phi_Y$ to ϕ_X that is the identity on the base.*

Let's give some dumb examples.

Let's say that your manifold is $\mathbb{R}$ with a choice of trivialization $\phi_X : T\mathbb{R} \to \mathbb{R} \times \mathbb{R}$ and that Y is the same with a trivialization ϕ_Y. Let's look at the space of framed embeddings $Emb^{fr}(X, Y)$. So I want to point out that for a manifold as simple as $\mathbb{R}$ an orientation is the same as a framing. I mean the space of framings is contractible. What is this space of embeddings? It's homotopy equivalent to a point.

More generally, $Emb^{fr} \amalg^k X, Y) \cong \Sigma_k$ (as a torsor, this has no group structure). The picture of this is that you have k disjoint intervals sitting in $\mathbb{R}$.

Now let me make a definition of a category that you can define now.

Definition 1.3. *$Disk_n^{fr}$ is the category whose objects are of the form $\amalg^k X$ where k is a finite number (possibly zero) and $X = \mathbb{R}^n, \phi$. There are a natural number worth of objects. The morphisms are framed embeddings.*

This category has a symmetric monoidal structure given by disjoint union.

Another thing I want to mention, this doesn't just have a set of morphisms. There's actually a space of morphisms. There's a topology here.

Now I can define what an E_n algebra is. Fix $(\mathcal{C}, \otimes)$, a category enriched over spaces with a symmetric monoidal structure. An E_n-algebra in $\mathcal{C}$ is a symmetric monoidal functor $(Disk_n^{fr}, \amalg) \to (\mathcal{C}, \otimes)$.

The symmetric monoidal functor should respect the topology of morphism spaces.

Let's do an example. It might seem a little abstract. Let's let $n = 1$ and take $\mathcal{C}$ to be the category of vector spaces over $\mathbf{k}$. How is this enriched over spaces? Think of $hom(V, V')$ to be given the discrete topology. Now this won't be interesting after $n = 2$ but it's instructive to see $n = 1$ and $n = 2$. The empty manifold has to be sent to the monoidal unit $\mathbf{k}$. We have an object $\mathbb{R}, \phi$, which is sent to A. Any k disjoint copies of $\mathbb{R}$ goes to $A^{\otimes k}$. There should also be a map from the space $Emb^{fr}(\mathbb{R}^{\amalg k}, \mathbb{R}) \to hom(A \otimes k, A)$ If $k = 2$, the space of framed embeddings is homotopy equivalent to two points. You could embed two intervals in order or out of order. The first one gives you a multiplication map. Associativity will follow from looking at what happens when you include three disjoint intervals into one. There are two ways I can factor three intervals including into one, depending on how I gather things. This is diagram illustrating maps of disjoint intervals. The two compositions are isotopic. What does the functor do to this diagram? It says

$$\begin{tikzcd}
 & A & \\
A^{\otimes 2} \arrow[ur, "m"] & & A^{\otimes 2} \arrow[ul, "m"'] \\
 & A^{\otimes 3} \arrow[ul, "m \otimes id"] \arrow[ur, "id \otimes m"'] &
\end{tikzcd}$$

This needs to commute and the upshot is that an E_1 algebar is a unital associative alegbra. The empty manifold is sent to the base field and you get the unital conditions by chasing diagrams.

A fancy way to say that you get nothing else is that you an E_1 operad is just a model for the A_∞ operad.

Now let's look at the case $n = 2$. For those following the script, I've claimed that every associative algebra gives you an invariant of 1-manifolds. You can daydream about the circle.

The claim is that an E_2-algebra in vector spaces with the usual tensor product will be a unital commutative algebra.

Consider two copies of $\mathbb{R}^2$, labeled one and two, and one embedding would be embed them into $\mathbb{R}^2$ at some random location. I could shove them in the same location with their labels swapped. You notice that in $\mathbb{R}^2$ these embeddings of $\mathbb{R}^2$

are isotopic. In other words, applying the functor, if we have an E_2 algebra, this diagrams becomes that the following commutes:

$$\begin{tikzcd} A^{\otimes 2} \arrow[r, "m"] \arrow[d, "\sigma"'] & A \arrow[d, equal] \\ A^{\otimes 2} \arrow[r] & A \end{tikzcd}$$

The same argument as before shows this is associative. You can do the same thing for $n \geq 2$.

So the takeway is that $Vect$ has too dumb morphism spaces to see the topology of embeddings of high dimensional disks. So we should consider categories with nicer morphism spaces. For instance, we could use spaces.

Say $n = 1$ and let the category be the category of spaces with $\times$. Then we fix a space X with a basepoint x_0. Let A be the space of maps $(D^1, \partial D^1) \to (X, x_0)$. These are maps from $D^1 \to X$ so that f restricted to the boundary goes to the basepoint. So I get a loop in X. This is the based loop space of X, ΩX. We learned early on that this space has an interesting structure if you look at the connected components, π_0 of this is the fundamental group. This has a product but it's only associative up to homotopy. That's what an E_1 algebra is in spaces. It has a product which is associative up to homotopy.

Now for $n = 2$ and to be adventurous you could imagine $n = 18$ or whatever you like. Let A be $Maps(D^n, \partial D^n), (X, x_0))$. The components of this are π_n. Let me make the algebra maps explicit. Given an embedding of many copies of $\mathbb{R}^2$ into $\mathbb{R}^2$, we are supposed to get a map from $A^3 \to A$. If I'm given some tuplet of functions from the n-disk into X, I can construct a single map as follows. I'm given this embedding and can put in the functions on this tiny disk. On the boundary I get x_0 and I can send the rest to x_0. So A is the n-fold loop space of X, $\Omega^n X$.

There's another example I won't do in detail that comes from work of Kevin Costello and Owen Gwilliam. Given a Lie algebra $\ggg$, you can shove it into a black box, I could tell you it some other time, and you get a universal enveloping E_n algebra of $\ggg$. To transition into the rest of my talk, factorization homology will give global quantum observables on your framed manifold X of some n-dimensional topological field theory given by your Lie algebra $\ggg$. I won't talk about this for the rest of my talk, if this didn't make sense.

Now I can get to factorization homology. First I'll give a slightly abstract definition, using some properties that I can use to actually compute. The first observation that we make, how did I define an E_n-algebra? You might have interrupted me and asked about other framed manifolds. The first observation is that $Disk_n^{fr}$ is a subcategory of $Mfld_n^{fr}$, whose objects are framed n-manifolds, possibly non-compact, and the morphosms are the space of framed embeddings. How could we construct an invariant of framed n-manifolds. If I have an E_n-algebra A, I have this natural inclusion to $Mfld_n^{fr}$ and this functor to $\mathcal{C}$. If I could extend this to manifolds universally, that would be great.

If you're a category theorist, the answer is yes, it's Kan extension. This functor $\int_{\_} A$ is the left Kan extension of A along the inclusion of framed n-disks into framed n-manifolds.

This is probably a term that you don't need if you're not a category theorist. I could describe what it is in part two. Explicitly, you start with an E_n-algebra A

and then apply the left Kan extension. This relies on some assumptions on $\mathcal{C}$. It should have small colimits and also for every object in $\mathcal{C}$, $-\otimes\mathcal{C}$ preserves filtered colimits and geometric realizations.

These conditions are probably not meaningful so let me just give some examples. You could choose chain complexes over $\mathbf{k}$ with the derived tensor product. Another example is spaces with product. Another example is spectra with smash product.

Most categories you're happy dealing with can be recipients of factorization homology.

Let me give some consequences of the definition. Two are obvious and the third is a theorem that characterizes, more or less, factorization homology.

As I go through these, the chef recommends that you have in your mind the usual notion of homology.

- Because $-\otimes C$ commutes with geometric realization, $\int_{\_} A$ lifts to be a symmetric monoidal functor. If you take the tensor product to be the direct sum, then in the classical case, you get disjoint union going to direct sum.
- I was doing a Kan extension where we have morphism spaces. $\int_{\_} A$, this functor, respects topology of hom spaces. What does that mean? If two embeddings are isotopic, then they are sent to homotopic maps in $\mathcal{C}$. That's something we have in usual homology theory. You have to prove that in a first semester graduate course.
- There's one property missing, the local to global method. This is a theorem.

In my mind this proves that this is an interesting invariant. Let me make an observation first. If $N = V\times\mathbb{R}$ as framed manifolds then N has an E_1-algebra structure. You can draw a picture or you can say that there is a functor from framed 1-disks to framed n-manifolds by sending an object to that object cross V.

There are also module structures. If M_0 is a manifold with a collar, I mean it has a diffeomorphism of framed manifolds of the region near the boundary with $V\times\mathbb{R}$. You might have seen this in cobordism theory. You may need the boundary to look like a product to compose. Then M_0 is a module over $V\times\mathbb{R}$. Let me draw a picture. A module means that you get a map $M_0\sqcup V\times\mathbb{R}\to M_0$. Such a map, I hope, is obvious.

If I have a functor out of manifolds which respects disjoint union, well, if I have two things that are like a right module and a left module, I can tensor them and see what I get, I can start doing algebra. After I state the theorem let's take a break or put part two another time.

Theorem 1.1. *(Francis, Ayala-Francis-Tanaka) Let M be a framed manifold $M_0\cup M_1$ with intersection $V\times\mathbb{R}$. Then $\int_M A\cong\int_{M_0} A\otimes_{\int_{V\times\mathbb{R}} A}\int_{M_1} A$*

Let me give an example and then we can call it quits. So for $n=3$, let's fix a Lie algebra $\ggg$ (the same as in Chern-Simons theory). Let X be a framed copy of the circle, $S^1\times D^2$, and Y be S^3. Can we detect $\pi_0 Emb^{fr}(X,Y)$? This is one of the fundamental questions of knot theory. There's a framing lying around. So the answer is yes. I'll attribute this to Costello-Francis. Kevin's machinery outputs an E_3-algebra. So first create one of these out of $\ggg$. What do I mean? I mean in the category of chain complexes over a field of characteristic zero.

If you're given two embeddings from X into Y, what effect do j_1 and j_2 have on the homology $H_*\int_X A$ and $H_*\int_Y A$? I get two different chain complexes and I can ask about the maps on homology. I can compute whether j_1 and j_2 are isotopic by seeing if they give the same maps on homology. They can recover Rezhitikhin-Turaev invariants doing this.

Let me do an easier example. So $n = 3$ is a bit far. For $n = 1$ everyone knows what an associative algebra is. I'll consider just the circle. What is factorization homology of a circle with coefficients in an associative algebra? You can think of factorization homology as giving you an invariant of associative algebras. By excision, I can write the circle as a union of= three manifolds. If I have a framing on the circle I can decompose it as two hemispheres where the intersection is two disjoint lines. By excision, this is the tensor product of two copies of A. The tensoring is over $A \otimes A$ but because of the framings this is $A \otimes A^{op}$. So this is $A \otimes_{A \otimes A^{op}} A$, which is the Hochschild homology of A. This is one reason this is called topological chiral homology or higher Hochschild homology.

[Why does this care about smooth structure?]

Whatever invariants you compute are only as strong as point configurations in X. Using different structure groups you might get something, it's an open question how sensitive that is to diffeomorphism.