# INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS IRREGULAR SEMINAR

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## 1. September 23: Lino Jose Campos Amorim, Tensor product of $A_\infty$-algebras

Thanks, so, I was going to talk about tensor product of $A_\infty$ algebras. Let's start with the definition. For now, $A$ will be a graded vector space over a field $\mathbf{k}$. I could mean $\mathbb{Z}$-graded or $\mathbb{Z}/2$ graded. I'll pretend it's $\mathbb{Z}$-graded. There are a bunch of operations $m_k : A^{\otimes k} \to A$ of degree $[2-k]$. For now, $k \geq 1$. These are required to satisfy the following equation, for each $n$

$$\sum (-1)^* m_{n-j+1}(a_1, \ldots m_j(a_i, \ldots, i_{i+j-1}) \ldots a_n) = 0$$

Just for once I'll write down the sign, it's $\sum_{\ell=1}^{i-1} |a_\ell| + i - 1 = \sum ||a_\ell|| - \sum |a_\ell| - 1$.

The first equation says $m_1^2 = 0$. The second equation says that $m_1$ and $m_2$ satisfy the Leibniz rule (with strange signs), $m_1 m_2(-,-) \pm m_2(m_1 -, -) \pm m_2(-, m_1 -) = 0$ The next one shows that $m_2$ is associative up to a homotopy $m_3$. The rest of them say that this is associative up to higher homotopies.

One example is differential graded algebras. These are $A_\infty$ algebras with $m_k = 0$ for $k \geq 3$. So $m_1$ is a differential which squares to zero and is a derivation of $m_2$ and $m_2$ is actually associative.

Okay, so now I want to define maps between $A_\infty$ algebras.

I'll define an $A_\infty$ homomorphism $F : A \to B$ to be a sequence of maps $F_k : A^{\otimes k} \to B$ of degree $1-k$, you can think of this in terms of the suspension as being a map of degree zero. These are required to satisfy an equation

$$\sum m_\ell^B(F_{i_1}(\ldots) \ldots, F_{i_\ell}(\ldots)) = \sum F_{n-j+1}(\ldots m_j^A(a_1, \ldots, a_i + j - 1), \ldots)$$

A naive $A_\infty$ homomorphism has $F_k = 0$ for $k \geq 2$. That's too much to ask in real life so that's why you have these more complicated things.

So $F_1$ is a chain map, that's the first thing this says, it induces a map on $m_1$ cohomologies. Then $F_2$ gives a homotopy between $m_2^B(F_1(\ ), F_1(\ ))$ and $F_1(m_2^A(\ ,\ ))$. In this world where everything is loose, these two guys are not equal but homotopic with respect to this specified homotopy $F_2$.

One more thing, I need to tell you that you can actually compose $A_\infty$ homomorphisms. If you have $F$ and $G$ $A_\infty$ homomorphisms you define the composition

$$(F \circ G)_K = \sum F_\ell(G_{i_1}(\ldots), \ldots, G_{i_\ell}(\ldots))$$

and this is actually associative.

Maybe a few more general things about $A_\infty$ algebras. If $F_1$ is an isomorphism of vector spaces, then $F$ is invertible. The identity in this setting has $F_1$ the identity and all other maps zero. You can solve this iteratively in the number of inputs to invert.

You're interested in when $F_1$ is just an isomorphism on cohomology. Then we call $F_1$ a quasiisomorphism.

Now comes kind of the main advantage of working in the $A_\infty$ world. You can actually invert these guys. You define a dg algebra map as just having $f_1$. To say that two dg algebras are quasiisomorphic, you usually say there is a zigzag of algebra maps to make this an equivalence relation. But this is not necessary in the $A_\infty$ world. This is sometimes known as the Whitehead theorem in the $A_\infty$ world. This is an advantage:

**Theorem 1.1.**

- *(Whitehead theorem) If $F : A \to B$ is a quasiisomorphism then there exists an $A_\infty$ quasiisomorphism $G$ such that $F \circ G \cong id$ and $G \circ F \cong id$. I haven't defined homotopy but you can think that it induces the identity on cohomology, it's much more than that.*
- *(Homological perturbation lemma) This is a homotopy notion and you can transfer these by homotopy retracts. Say you have $(A, m_k)$, an $A_\infty$ algebra, and you start with a chain complex $(V, d)$. Let's say I have these maps $i : V \to A$ and $p : A \to V$, chain maps with respect to $d$ and $m_1$. I have $h$ of degree $-1$ such that $m_1h + hm_1 = i \circ p - id_A$. So $h$ gives a homotopy between $i \circ p$ and the identity. Then there exists an $A_\infty$ structure on $V$, $\mu_k$, such that $\mu_1 = d$, and there exists $F : (V, \mu_k) \to (A, m_k)$ an $A_\infty$ homomorphism with $F_1 = i$. There's actually more, a homomorphism the other way beginning with $p$ and an $A_\infty$ homotopy beginning with $h$ but let's ignore it.*

If now we require that the other composition is the identity on cohomology, this would be a quasiisomorphism. If $A$ is very big and $V$ is some sub-chain complex with the same homology, you can push the $A_\infty$ structure to the small vector space. The main application (classically) is as follows. Say you start with $(A, d, \cdot)$, a dg algebra such as the de Rham complex of a manifold, and inside take a vector space inside with the same cohomology, such as the cohomology itself. So split $A$ into the cohomology, the exact things, and the things that are not closed. In the case of the de Rham complex of a manifold, you can take harmonic forms and $H$ is the Green operator, say. So you choose this so that $p \circ i = id$ and the output will be that the cohomology of $A$ has the structure of an $A_\infty$ algebra quasiisomorphic to the original.

In general if you just have a dg algebra and you take its cohomology, then the cohomology is a dg algebra with zero differential. But it might not have anything to do with the original, it might be completely different. But you can find a structure on the cohomology with no differential equivalent to the original one. For example, the $\mu_3$ is essentially the Massey products. These are essentially $\mu_3$, you need some condition on the elements and then that agrees with $\mu_3$.

This is the background, generalities on $A_\infty$ algebras. Now let's look at some disadvantages. If you have $A$ and $B$ dg algebras, you can take $d^\otimes$ on $A \otimes B$ which is $d^A \otimes id_B + id_A \otimes d^B$ and define a componentwise product $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (-1)^* a_1a_2 \otimes b_1b_2$. What do you do for $m_3$? You could take a guess and do it componentwise but that's not even the right degree.

Let me give an idea motivated by operads. Let me talk about Stasheff polytopes. For $n \geq 2$, it'll be a manifold of dimension $n - 2$ with corners. As a symplectic geometer, let me tell you, $K_n$ is the moduli space of stable disks with $n+1$ boundary marked points, cyclicly ordered. So $K_2$, there is only one disk with three marked points. Mobius transformations act transitively. Okay, so $K_3$, you have four marked points, I can fix three in standard position and the fourth one can move around. The space between is actually an interval. The endpoints correspond to the singular things. When the marked points collide we interpret that as a bubble.

So I'm going to, a way of describing this moduli space is by trees. It has a stratification by how many disks you have and how they're attached. I take the $\infty$ as the root, put a vertex on each disk with an edge for each intersection point and a leaf for every marked point other than $\infty$. So just for fun, $K_4$ is a pentagon. The interior is a tree with a single vertex. The codimension one faces have two interior vertices. The vertices of the polytope correspond to binary trees.

Now there's a way of gluing these disks to go from two $K_n$ to a higher $K_n$, A map I'll call

$$\circ_i : K_{n_1} \times K_{n_2} \to \partial K_{n_1+n_2-1}.$$

If I have two trees I glue the root of the first tree to the $i$th leaf of the second one. So now what I say is, maybe I'll state it as a theorem:

**Theorem 1.2.** *The Stasheff polytope $K_n$ is a manifold with corners of dimension $n-2$ and is contractible, has the cohomology of a point.*

Another important thing is the boundary. I told you that the gluing operation goes into the boundary, but in fact, it's better than that. $\partial K_n = \amalg K_j \circ_i K_n - j + 1$.

So now for us what's usually called the $A_\infty$ operad is precisely this, it's the cellular chains (using the stratification from the manifold with corners structure) $A_\infty(n) = (C_*(K_n), \delta)$, and $\circ_i$ are cellular so they induce maps on cellular chains $C_*(K_{n_1}) \otimes C_*(K_{n_2}) \to C_*(K_{n_1+n_2-1})$. The operad is this entire thing.

Another operad, the $End$-operad, if you have a chain complex $A$, I'll call it $End(A)_n$ is $Hom(A^{\otimes n}, A)$. This is also a chain complex, you get a natural differential from $d_A$. Here you can really compose and you get $\circ_i$ like before.

In this language, what's an $A_\infty$ algebra? In the redefinition, an $A_\infty$ algebra structure on $(A, d)$ is a map of operads $\rho : A_\infty \to End(A)$. That means I have for each $n$ a map $\rho_n : A_\infty(n) := C_*(K_n) \to A^{\otimes n}$ which respects the compositions and which is a chain map.

Why is this the same as before? The $A_\infty$ operad is in a way free, or quasifree, for each number of inputs, forgetting the differential, you can build any tree as coming from gluing trees with one vertex. There are no relations. So the map $\rho$ is determined by the image of the trees with single interior vertex. Call this $c_n$, the $n$th corolla. This gives me $m_n$ in the previous notation. Because any tree can be glued, the $\rho$ is completely determined by $\rho_n(c_n)$ which can be arbitrary except that $\rho_n(\partial c_n)$ should be $\delta m_n$. So we saw $\rho_n(\partial c_n)$ is the sum of trees with two vertices. That must be $\delta m_n$, which by definition is $\sum m_n(\ldots d \ldots) \pm d m_n(\ldots)$. Remember that $\rho$ respects composition, so this left hand side is

$$\sum m_j \circ_i m_{n-j+1} = \sum m_n(\ldots d \ldots) \pm d m_n(\ldots)$$

This is the same as the $A_\infty$ relation I gave before.

Okay, why did we do this? An $A_\infty$ algebra is a map from this operad to the $End$ operad. If I want a tensor product, I want a map $A_\infty \to End(A \otimes B)$. So maybe you can guess what I do. I have a map $End(A) \otimes End(B)$ to $End(A \otimes B)$. By definition I have maps $\rho_A$ and $\rho_B$ so I can take $A_\infty \otimes A_\infty to End(A) \otimes End(B)$.

So we essentially need a diagonal $A_\infty \to A_\infty \otimes A_\infty$ to get

$$\begin{tikzcd}
A_\infty \arrow[d, dotted] & & End(A \otimes B) \\
A_\infty \otimes A_\infty \arrow[rr, "\rho_A \otimes \rho_B"] & & End(A) \otimes End(B) \arrow[u]
\end{tikzcd}$$

We have a diagonal $K_n \to K_n \times K_n$ but that's not cellular. $K_2 \to K_2 \times K_2$ is easy because it's a point. The higher ones you get into trouble. $K_3$ is an interval, and $K_3 \times K_3$ is a square but the diagonal is not a cell. I deform the diagonal to the sum of two sides.

Then what is tricky? When we first have a course in topology, you have a diagonal map on singular chains which gives you cup product, and you can use Alexander-Whitney which is canonical. Here with cellular chains there is no canonical diagonal.

**Theorem 1.3.** *There exists a diagonal. You can get this from the Stasheff polytopes being contractible.*

Compatibility with composition takes care of the rest. Then you can define up to level $C_{n-1}$ and then write down an equation and solve it, but it's in an acyclic complex.

That won't give you explicit expressions, but it shows it exists. Can you find an explicit construction? You can, this was done first by Saneblidze-Humble, by brute force. There was later work by Markl-Schnider, which says it's a reinterpretation. They take the cellular complex and subdivide it into a cubical complex. These are like simplicial complexes, and there's a canonical diagonal. Then you have to go back, that's the tricky part. That involves choices and everything gets messy. Loday did a similar construction. He subdivided into a simplicial complex instead of a cubical complex. Miraculously, they all get the exact same diagonal.

Now you can ask for additional properties of this diagonal.

You could ask for it to be coassociative. That means $\Delta(1 \otimes \Delta) = \Delta(\Delta \otimes 1)$, which would make our tensor product strictly associative. You could ask it to be cocommutative $\Delta\tau = \Delta$. You could ask for $\Delta$ to be cyclic. What does this mean? For the $A_\infty(n)$ there is a $\mathbb{Z}/n+1$ action, the last leaf becomes the first, the first becomes the second, and so on. You could ask that the $\Delta$ respects this action for each $n$. What would this imply? On the other side you take the diagonal action. What would this mean about our tensor product? This would mean that if $A$ and $B$ are cyclic then $A \otimes B$ is cyclic. This means, well, $(A, m_k)$ is cyclic if you have an inner product, nondegenerate, and then $\langle m_k(a_1, \dots a_\alpha), a_0\rangle$ is invariant up to a sign of cyclic permutations of the inputs.

It is not ever possible to get a coassociative $\Delta$, this was proved by Markl-Shnider. The answers to the other two are yes, and even better, yes simultaneously.

**Theorem 1.4.** *(A,Tu) It is possible to get both cocommutativity and cyclicity simultaneously.*

Shall we continue? So maybe I'll comment, why you can make 2 and 3 work is that they're invariant under a $\mathbb{Z}/n+1$ action. You can take the usual solution and average at each step. We can show that we can solve this inductively. The Markl Schnider have explicit formulas for all $m_k$. We can write down just $m_3$ and $m_4$.

So $m_3$ being cyclic determines the formula. Then $m_4$ being cyclic isn't enough to determine it but being commutative is then enough. Above that you have a lot of freedom.

Let me give one application. Let $A$ be a cyclic $A_\infty$ algebra, finite dimensional (at least at the cohomological level). Given this data, you produce $c_A$, a cohomology class on the moduli space of Riemann surfaces, $H^*(\amalg_{g,n} M_{g,n}, \mathbb{Q})$. This is the uncompactified moduli space. The way it goes, you have a combinatorial model using ribbon graphs for this guy. These are graphs, a vector space generated by graphs with a cyclic order at the edges incident on each vertex. Then the differential is given similar to the Stasheff polytope, expand the vertex in all possible ways. That's the differential. It's very nice combinatorially. Cyclicity says that the formula doesn't depend on some sign. Then you take the product over vertices and edges, pairing along edges using the inner product. At the end that gives you a number. You have to check that it is a cocycle.

As it's written here, I should have an "even" $A_\infty$ algebra, so that the pairing is even degree. You get a cohomology class on a twisted version of this space for an odd algebra.

The theorem we proved, and that's the reason we studied this thing, is that the Kontsevich class of the tensor product $C_{A\otimes_{cyc}B} = C_A \cup C_B$. The idea of this is that the cyclic diagonal not only lets you define tensor products of cyclic $A_\infty$ algebras, it also gives a diagonal on $M_{g,n}$ which you use to compute the cup product.

There is a paper on the arxiv, the theorem is correct but the proof has a mistake. We use the diagonal on the Stasheff polytope to give a diagonal on the ribbon graph complex. It should be a cellular approximation of the ribbon graph complex diagonal, and the way we argue that is wrong.

We're all happy. Why isn't this the end of the story? Symplectic geometers work with curved $A_\infty$ algebras. It's very easy to define these.

Curved means that you have an $m_0$ term. Now you start at zero. It's an element of degree 2 in $A$. Why is it curved, what's the main example? Take a vector bundle over a manifold with some connection $(E, \nabla)$. Then de Rham forms valued in $End(E)$, $\Omega^*(End(E))$, well, here you have the curvature $F_\nabla$. Then $m_1 = d_\nabla$ and $m_2$ is $\wedge \otimes \circ$. The higher ones are zero. The first equation is $m_1(m_0) = 0$. This is the Bianchi identity. The rest are more or less formal. Then $m_1^2$ is the commutator with $m_0$, that's more or less the definition of curvature.

Curved $A_\infty$ algebras are very weird. You can show the following theorem.

**Theorem 1.5.** *(Lazarev-Schedler) If you have a curved $A_\infty$ algebra with $m_0 \neq 0$ then this is isomorphic to $(A, m_0, 0, \ldots, 0, \ldots)$.*

We've seen this theorem but not in this language, This is the box flow theorem. When you have a vector field which is nonzero, you can change coordinates to make it constant. The bar construction of the dual of $A$, the $m_k$ is a vector field there. Then the $m_0$ is nonzero at the origin, you can renormalize to make it constant.

We don't like these curved ones, so we'll talk about a special kind that shows up in symplectic geometry, called filtered $A_\infty$ algebras.

You have a vector space $A$ and $G$ a submonoid of $\mathbb{R}_{\geq}0 \times 2\mathbb{Z}$, say it's discrete so that if the projections $E$ and $U$, then $E^{-1}|_G[0, c]$ is finite for each $c$. This is given by Gromov compactness. So take $\beta \in G$ and you get $m_{k,\beta} : A^{\otimes k} \to A$ of degree $2 - K - \mu(\beta)$. We enforce that $m_{0,0}$ is 0. The curvature lives in this positive part.

These will satisfy the same equations as before but now you split by $\beta$. You fix $\beta$ and split it in all possible ways.

There is a Novikov ring $\Lambda_0 = \sum a_i T^{\lambda_i}$, with the $\lambda_i$ increasing and going to $+\infty$.

Now you take $A \otimes \Lambda_0$ and either complete with respect to the filtration or insist that $A$ is finite dimensional. Then $m_k = \sum_\beta m_{k,\beta} T^{E(\beta)}$. This is well-defined by the condition on discreteness.

Now there is a way to develop all the theory. You define the $F_k$ which are maps between $A_\infty$ algebras. You have $F_0$ but you also have $F_{0,0} = 0$.

Then when you define quasiisomorphism, you mean that $F_{1,0}$ is a quasiisomorphism. Then the Whitehead theorem and homological perturbation all go through. Now the question is how do you define the tensor product without thinking about operads?

You take the lazy approach and the lazy approach works. With dg algebras, you have, well, think of $A$ as being a right $A_\infty$-module over $A$. Now you take $Hom(A, A)$ as $mod - A$. Now a surprising thing happens. This is in fact a $dg$ algebra. The $m_2$ is composition. The $m_1$ is a commutator with the $m_k$. Now if you have units, $e$ which is unital for $m_2$ and vanishes for higher $m_k$. Then you have this map $A \to Hom(A, A)$. With associative algebras, you multiply on the left. Take $a$ to $a\cdot$. You can turn this into an $A_\infty$ homomorphism which is a quasiisomorphism if $A$ is unital. You go back by saying $P(\phi)$ is $\phi(1)$. So there's an issue here. We wanted to work with $A_\infty$ algebras with $m_0$. The ones with $m_0$ are strange. Then $A$ is not a right $A$-module over itself. But even though it's not a module, you can still try to write $Hom(A, A)$, and find that this is a curved dg algebra. It's filtered with all $m_k = 0$ for all $m_k \geq 3$.

So $A \otimes_\infty B = End_A \otimes_{dg} End_B$. So now maybe the only non-obvious thing, $\mu_0$ in the tensor product is $\mu_0^A \otimes e_B + E_A \otimes \mu_0^B$.

**Theorem 1.6.** *By the homological perturbation lemma, include this with $F_{1,0} \otimes F_{1,0}$, $A \otimes B \to A \otimes_\infty B$, and then you can project back by evaluating at the unit, $P \otimes P$ and there's a homotopy you can write, and you get a filtered $A_\infty$ algebra on $A \otimes B$.*

*In the classical case, with no filtrations, $n_k^\otimes$ agrees with the original tensor products defined for classical $A_\infty$ algebras.*

Maybe I'll just say one final theorem about this thing. I want to have a way to decide when an $A_\infty$ algebra is a tensor product of two subalgebras. In the associative case, if I have $A$ and $B$ in $C$, when do you have a map from $A \otimes B \to C$ induced by the product $a \otimes b \mapsto ab$. Well, $ab$ should be $ba$. That's a necessary condition.

The following is maybe not homotopic but here's a condition. So now $A$ is an $A_\infty$ subalgebra of a filtered $A_\infty$ algebra $C$. So it's a subalgebra if the higher than zero $m_k$ are respected. Now $A$ and $B$ should have units that are the same in $C$. Then $A$ and $B$ are commuting if, there are a bunch of conditions, if you take $\mu_k(\ldots, a, \ldots, b, \ldots) = 0$ except if there are only $a$s or there are only $b$s or $k = 2$, in which case $ab = ba$.

Another condition is that the same thing holds if one of these is whatever it wants to be, $(a \ldots a \ldots b \ldots c \ldots b)$, this is only nonzero if there are only $a$, only $b$, or [another condition that was too fast]

**Theorem 1.7.** *If you have $A$ and $B$ commuting subalgebras of $C$ and $A \otimes B \to C$ which sends $a \otimes b \to m_2(a, b)$ is injective and an isomorphism on $\mu_{1,0}$ cohomology. Then $C \cong A \otimes_\infty B$ as $A_\infty$ algebras.*

I don't have time, but this model is big, but it's good enough to prove the theorem.