INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS IRREGULAR SEMINAR

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1. October 30: Seonhwa Kim, An introduction to Volume conjecture, its generalizations and related topics

It's my great pleasure to talk here. This is the first time to give a formal talk in English so I'm really nervous. I'll try to give a tiny brief crash course in the volume conjecture and describe the importance of this problem. I'll try to explain in a graduate student level because I was a graduate student two months ago. Ask questions but please consider "three S" which are "slow speaking" and "short sentences" and "simple English."

Before the start, I'd like to mention several references. Recently there were lots of papers related to the volume conjecture. If you are really unfamiliar, I'll suggest several names of mathematicians or physicists as keywords for introductory articles. Hitoshi Murakami and Sergei Gukov and Hiroyuki Fuji. They are a Japanese knot theorist, a Russian physicist, and a Japanese physicist. You can google the volume conjecture and easily get a pdf of a paper or slides. Many things in this talk come from various articles of these guys.

The statement of the volume conjecture is where I'll start. In 1994 Russian physicists and mathematicians R. Kashaev and L. Faddeev discovered the quantum dilogarithm. After a year Kashaev suggested a certain knot invariant, denoted $\langle K\rangle_N$, where N is an integer greater than two and K is a link or knot. In this talk I don't talk about the quantum dilogarithm or the Kashaev invariant. I'll just need two facts.

(1) $\langle K\rangle_N$ is a sequence of complex numbers for $N \geq 2$
(2) the absolute values of the sequence has exponential growth

A natural question is what is the exponential growth rate of this invariant. He tested it numerically and suggested an answer based on physical reasoning.

$$\lim_{N\to\infty} 2\pi \frac{\log |\langle K\rangle_N|}{N} = vol(S^3 \backslash K)$$

This is the Kashaev volume conjecture.

He gave a rigorous proof for the figure eight knot. There is a critical assumption which is that K is hyperbolic. I will explain what is the meaning of this, that the knot is hyperbolic.

After about five years, in 2001, H. Murakami and Jun Murakami proved the following theorem (the two Murakami theorem):

Theorem 1.1.

$$\langle K\rangle_N = J_N(K, q = e^{\frac{2\pi\sqrt{-1}}{N}})$$

Since the two Murakami theorem, this conjecture has been very attractive to mathematicians (knot theorists) and physicists since the Jones polynomial is popular since Ed Witten's Chern Simons theorem.

I anticipated the questions at this point:

(1) What is the volume?
(2) What is the *colored* Jones polynomial?
(3) What does this have to do with physics?

I'll try to answer this. In 1984, V. Jones announced his famous Jones polynomial, which came from the study of von Neumann algebras. This is difficult to me. This is a very important result. Many interpretations came up. The most important is from Kauffman and the second from Ed Witten. My plan had that everyone knew Kauffman's definition, but I'll say what that is because some don't.

He started by describing a bracket polynomial for a knot or link, embedded copies of S^1 in S^3. The bracket polynomial of $\langle K\rangle$ (this is similar to the Kashaev notation but I won't say that again) is defined in the following way:

- $\langle \text{unknot}\rangle = 1$
- $\langle \text{unknot} \sqcup K\rangle = (-A^2 - A^{-2})\langle K\rangle$. This uses the quantum number $[2] = A^2 + A^{-2}$ and can be written $-[2]\langle K\rangle$.
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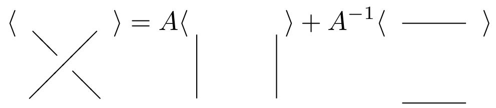

We can mark the region where an undercrossing rotates clockwise to an overcrossing as $+$ and this is A times the positive smoothing (joining two positive regions) plus A^{-1} times the negative smoothing.

[Example of the Hopf link.]

Theorem 1.2. *The bracket polynomial of K is a regular isotopy invariant.*

Theorem 1.3. *The bracket polynomial times $(-A^{-3})^{w(k)}$ is equal to the Jones polynomial with the change of variable $A^{-4} = q$*

The writhe is the sum of the the crossings, signed, which depends on an orientation. For a knot reversing the orientation preserves the invariant but for a link it is not an invariant, so we make it an oriented link invariant.

Remark 1.1. *We can modify the definition of the Jones polynomial to get an unoriented link invariant.*

Recall the Reidemeister theorem. There are three Reidemeister moves [pictures] and two knot diagrams are connected by a finite number of these moves if and only if they represent the same knot. If you design an invariant, you need to check that it is invariant under these three moves. If you have a smooth isotopy between two knots, not ambient isotopy, we can extend to get an ambient isotopy. Smoothness is crucial. Without smooth this is false.

A regular isotopy invariant only preserves Reidemeister moves two and three and a regular knot invariant preserves all three. One more thing, a framed knot invariant also preserves two and three as well as the modified Reidemeister one where you add two opposing Reidemeister one moves.

Regular isotopy invariants are the same as framed knot invariants in S^3 but not in $\mathbb{R}^3$ because you can slide around the whole sphere and come back in S^3. We can see this by looking at the winding number at infinity.

One last comment, a framed knot invariant is closely related to 3-manifolds. Dehn surgery is an integral surgery on a link complement, and the integer is the framing.

The second topic is the Witten invariant in Chern-Simons theory. In 1989, he discovered this. I know very little about the physical approach. Let me say a few words. Say you have L embedded in a compact M^3 with a compact gauge group G (let's say a principal G-bundle). Then

$$Z(M,L) = \int \mathcal{D}A e^{\frac{i\hbar}{k}CS(M)} W_R(K)$$

over all connections quotiented by gauge equivalence where $CS(M) = \frac{1}{4\pi}\int tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$ We can approach this formally. The quantity k is an integer, the coupling constant, and the remaining thing is the Wilson loop. I don't know the physical meaning but mathematically this is the trace of the holonomy of K_i, the product of these over the components of the link. R is a representation from G to $GL_n(\mathbb{C})$. If you fix R and take the trace of these matrices for the holonomy, that gives us the Wilson loop. This expression is defined by construction to be a knot invariant because the holonomy is invariant under isotopy and the trace is invariance up to conjugacy. Integration over connections, well, we can do that formally, anyway, Witten defined this integration which he called Chern-Simons theory, and the surprising result is that when $M = S^3$ and $G = SU(2)$ and coupling constant 1, with R the fundamental representation from $SU(2)$ on $\mathbb{C}^2$, in this case $\frac{Z(S^3,L)}{Z(S^3,\text{unknot})}$ is the classical Jones polynomial with variable $e^{\frac{i\hbar}{k+2}}$ or something close to that, I might have the denominator off. This is a physical interpretation. In this case, you can take any manifold, any link, any gauge group.

The Witten invariant can be defined for this pair. This is difficult to formulate in a rigorous way because this integration is difficult. Mathematicians formulated this rigorously using TQFT functors. This is a functor from the category of surfaces to vector spaces. Your objects are disjoint unions of circles. You can design such a functor. We can make a 2+1 dimensional TQFT, this is designed by Reshetikin Turaev, so we call this an RT-functor. The detail is complicated, but the idea is the same, it's a functor from a topological category to the vector space category. Here, we have embedded dots in the plane which go to that many copies of that vector space tensored together.

The morphisms of the 2D TQFT case are cobordisms, and in the RT case the morphisms are tangles. Fix the input and output endpoints. If you design this functor, you need elementary moves for the morphisms. You need certain relations for the topological object. In the RT-case, we want a knot invariant, so we used some kind of Reidemeister invariance. We have generators which are the two knot crossings, the cup, the cap, and the identity. We can concatenate these horizontally and vertically. Then there are relations, the elementary sliced tangle moves. The algebraic structure is called a quasi-triangular braided ribbon hopf algebra. They get these from quantum groups.

Let me explain briefly what is a quantum group representation. So let me review $sl_2\mathbb{C}$, which is $\langle X, Y, H | [H,X] = 2X, [H,Y] = 2Y, [X,Y] = H\rangle$. We could consider these in representation as

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Let $U(sl_2\mathbb{C})$ be the universal enveloping algebra, generated again by X, Y, and H subject to the relations $HX - XH = 2X$, $HY - YH = 2Y$, and $XY - YX = H$.

Every direct sum is a direct sum of irreducible representations. Let me give a fact: there exists a unique irreducible representation on $\mathbb{C}^n$ for each n.

Why is this fact important? Basically colored means this dimension. So each dimension N we get only one quantum invariant. That's the colored Jones polynomial. So we act on V^n by $\langle X^{n-1}, X^{n-2}Y, \ldots, Y^{n-1}\rangle$.

For example, the Clebsch-Gordon equation decomoses $V^n \otimes V^m$ into the direct sum $V^{n+m-1} \oplus \cdots \oplus V^{n-m+1}$.

A fundamental lemma is Schur's lemma: if V and W are irreducible modules and $f : V \to W$ then f is an isomorphism (multiple of the identity) or 0.

An example: If you consider $f : V \otimes V \to V \otimes V$, where each factor is V^2, this is $V^3 \oplus \mathbb{C} \to V^3 \oplus \mathbb{C}$. If you assume f is a module map, then we know that f must be of the form $pId + q$ times the inner product and co-inner product.

For example you can consider the map $V^{\otimes 2}$ to $V^{\otimes 2}$ by flipping, $a \otimes b \mapsto b \otimes a$, then p and q are 1.

Three years ago I went to a big conference on the volume conjecture and in the opening talk the speaker said "quantum" meant unknown. So I don't know what it means, but I'll discuss the "quantum" deformation of $V(sl_2\mathbb{C})$ to $V_q(sl_2\mathbb{C})$. Here $V_q(sl_2\mathbb{C}) = \langle X, Y, K | KX = A^2XK, KY = A^{-2}YK, XY - YX = \frac{K^2 - K^{-2}}{A^2 - A^{-2}}$. Here this is the same A as in the bracket. Also $K = A^H$.