2016 CGP SEMINAR

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1. February 16: Grigory Mikhalkin: Holomorphic disk potential for exotic monotone Lagrangian tori

Thank you for the invitation and the possibility to speak for two hours. Instead of technical details I'll try to give an introduction and in the second hour more into monotone Lagrangian tori. The larger story is very old. In the second part I'll be talking about work in progress with Galkin. Part of this subject is even 19th century mathematics.

Let's start from the following question about lens spaces. Look at lens spaces $L(p,q)$. I don't need to remind you what a lens space is. It's a quotient of S^3 by, think of it as the unit sphere in $\mathbb{C}^2$, and choose ξ a primitive pth root of unity and take a quotient of S^3 by the map that takes ξ to ξ^q. These are the simplest 3-manifolds after the 3-sphere. This is a topological description, well, topologically, glue two solid tori $D^2 \times S^1$ along their torus boundary. The result will always be a lens space, unless you get $S^2 \times S^1$.

The way it's represented analytically makes this a holomorphic manifold. We can take $\mathbb{C}^2$ quotient by the same relation, $(z,w) \mapsto (\xi z, \xi^q w)$. The quotient is an affine toric manifold (singular). As in 3-manifolds, we knew arithmetically it was given by two numbers. Any toric surface, here, well let's recall the description. In $\mathbb{Z}^2$ we have a convex cone generated by two integer vectors. The dual moment map for this, you'll have [pictures]. From the toric point of view it's better to think of this as two vectors up to automorphisms of the plane rather than two numbers p and q. This toric surface is a cone over $L(p,q)$. Now let's ask the next question, is this toric surface smoothable? If this was smooth you would find a four-manifold with this lens space as boundary. Topologically we can always find, any three-manifold is cobordant to zero. We want the same thing algebraically. So the next question is, how about holomorphically? It's not always possible and here arithmetic related to Markov triples appears. If the surface is X, we should have $X = X_0$ in a family X_t so that $\mathcal{X} \to \Delta$, this is a fibration over a disk with the fiber over t being X_t so that X_t is smooth at $t \neq 0$ and $X_0 = X$. We'll put a condition on the smoothing. Let's recall that the main tool for intersection theory in situations like this is the canonical class. We'll restrict to those for which the canonical class $-c_1(X)$ is $\mathbb{Q}$-Cartier, which means that the intersection with it is well-defined over the rationals.

Then immediately we'll see that there's a condition. Topologically if we look at a three-manifold, the lens space, and ask if we can, well, if $\partial Y = L(p,q)$, $b_2(Y)$ the second Betti number, is not determined. But holomorphically it is. Let us recall the case of singularities, isolated singularities of hypersurfaces, due to Milnor. Suppose X is in $\mathbb{C}^3$. If we do the smoothing X_t of this, then the smoothing is a smooth manifold and in such a case we can say that $b_2(X_t) - b_2(X_0) = \mu$, this is the "Milnor number" of the singularity, the number of vanishing cycles at our singular point.

Another thing we know which is familiar from the regulary case of hypersurface singularities, is that the Milnor number is positive, so for example $L(2,1) = \mathbb{RP}^3$ is an A_1 singularity, Morse singularity, [unintelligible]. Similarly we have A_n singularities, and you have n-spherical cycles centracted

For quotient singularities, we can also say what will be the number μ, the number of cycles that will get contracted. This is determined by p and q. It's easier to give the answer in [unintelligible]. [Missed some]

It turns out that the Milnor number is $\frac{\text{width}}{\text{height}} - 1$, and this implies that not for any p and q this quotient singularity is smoothable at all.

If μ is an integer then this can be smoothed, and this is the so-called T-singularity (Kollár, [unintelligible], 80s). Now we can ask a topological question, which is when $L(p,q)$ bounds a rational homology disk. We know this implies that, by secondary Poincaré duality, that $b_1(Y)^2 = b_1(L(p,q)) = p$ so p must be a square.

However, the special case $L(n^2, n-1)$ or more generally $L(n^2, dn-1)$ if d and n are relatively prime. In four dimensional topology, this was the source of rational blowdown technique of Fintushel and Stern.

Let's see how this is possible. The simplest example is when $n = 2$, we'll look at $L(4,1)$. Topologically this is almost the same as $L(4,-1)$ (which is an A_3 singularity) but holomorphically this is quite different, this is $\mu = 0$. Let's see how it is possible, but the part of the audience familiar with Fintushel–Stern knows how this goes.

What happens is that the singular point is replaced with, the vanishing cycle is Lagrangian, but if it's non-orientable it won't contribute to integer homology. The simplest rational blowdown is on a real projective plane. Removing $\mathbb{RP}^2$ from $\mathbb{CP}^2$ leaves a neighborhood of an imaginary conic. This is a resolution of [unintelligible], the minus version of this and $\mathbb{RP}^2$ have the same [unintelligible]. So $\mathbb{RP}^2$ is $|x|^2 + |y|^2 + |z|^2 = |x^2 + y^2 + z^2|$, and the quotient here is a Morse function with the fiber over 1 $\mathbb{RP}^2$ and the fiber over zero an imaginary conic $x^2 + y^2 + z^2 = 0$.

This is transparent from the point of view of 2-dimensional homology. What was noticed in particular by, there were two papers exploring this trick [unintelligible]. We have a complete description of the quotient singularities which are smoothable and transparent, and if our singularity, if we projectivize, pass to a compact manifold, compact surface only with T-singularities, then you can hope to get simply connected manifold with the same second homology from singular T-surfaces. This was explored in two papers, first Hacking–Prokhorov in the case $K < 0$ and then Lee–Park in the positive case. For us it will be more interesting to consider $K < 0$ because this is the case for $\mathbb{CP}^2$. Lee and Park, let me say though, constructed simply connected general type surfaces with $K^2 = 2$, so exotic $\mathbb{CP}^2 \# 7\overline{\mathbb{CP}}^2$.

So Hacking and Prokhorov did classify completely, gave a classification of toric Del Pezzo surfaces with T-singularities. Let us recall first the non-singular, the 19th century classification, the Del Pezzo classification. If we have a surface which is a smooth surface, if it is smooth and the canonical class is negative, then this is a Del Pezzo surface and the classification is $\mathbb{CP}^2$, $\mathbb{CP}^2$ blown up in up to eight generic points, or $\mathbb{CP}^1 \times \mathbb{CP}^1$.

If we go back to the smoothing, if we smooth a Del Pezzo, it'll also be a Del Pezzo. If we know where we are before and we know the Milnor fibers we know where we are afterwards.

Let me describe the classification, which is separate in each case. It's essentially arithmetic. In particular for $\mathbb{CP}^2$, the classification is given by so-called *Markov triples*, the equation $a^2 + b^2 + c^2 = 3abc$, this is the Markov equation.

Then in $\mathbb{P}(a^2, b^2, c^2)$, the weighted projective space, then all three singularities are T-singularities with $\mu = 0$.

[pictures]

This answers the problem of finding triples of vectors such that if I look at the triangle, each pair is the pair that givse this kind of T-singularity with $T = 0$. So the height should be equal to the weight in every case.

So far we saw the appearance of the triples is toric geometry, toric degenerations of $\mathbb{CP}^2$. Markov triples have a rich history. Markov himself noticed that the solutions are all obtained from the obvious solution $(1,1,1)$, because this equation is quadratic in each variable, by elementary moves (which also reappeared as cluster mutations) $(a,b,c) \mapsto (a',b,c)$ with $a + a' = 3bc$. So we can mutate $(1,1,1)$ to $(1,1,2)$, and mutate that as well to $(1,5,2)$, and from there to $(1,5,13)$; if we preserve 1 then we get every other Fibonacci number; we get $(29,5,2)$

[missed a lot thinking about Markov triples, verifying that $(1,1,1)$ and $(1,1,2)$ are the only Markov triples with repeated index]

As a corollary, there are infinitely many monotone Lagrangian tori inside $\mathbb{CP}^2$. The list misses one case when $k^2 = 7$. Each has its own version of a Markov equation. The example is the Markov triple corresponding to $(2,1,1)$, the Chekanov torus.

Another example is $L_{(1,1,1)}$, the standard torus. For the second example, for the Chekanov torus, this is dual to the triangle [picture]. In this case, the statement about potential for the standard torus, the projections of holomorphic disks are the three intervals and nothing les. For the $(1,1,2)$ triangle, you have the three projections and in addition, two more inside of the cone, so in the dual language, the disk will go to the singular point. Then we'll have to make sense of this holomorphic disk from the smoothing of singularities. Here we'll have to look how it behaves under smoothing.

[pictures]

Now let's go to our work with Sergei. Let's see how to get the mutations of the potential defined with [unintelligible]. For this, to get the potential, we pass to the tropical limit for the smoothing. When we pass to the tropical limit for the smoothing, let's recall our situation. We have a projective, we have a family over the disk, and X_0 is a projective surface and X_t is a projective surface, these are toric, but $\mathcal{X}$ is not toric. Then if we pass to the tropical limit, rescaling of the norm part, a certain rescaling and combining, we use the log, use the log map which is in some sense a moment map from $\mathbb{CP}^N$ to $\mathbb{R}^N$, and this goes to ∞ as the parameter in the base disk Δ goes to 0. Then the logarithm at t takes $(z_0 : \cdots : z_N)$ goes to $[unintelligible]$. Then images of holomorphic disks are trees here, and the surface X_t becomes a two dimensional polyhedral complex. Let me first consider the example which corresponds to $(2,1,1)$. In the case here you have a mutation to $(1,1,1)$ and this mutation can be made inside, is given by a pencil in weighted $\mathbb{P}^3$, which will always be the case. Then $\mathbb{P}^3$ in the weighted tropical limit is $\mathbb{R}^3$ completed at ∞ according to the fan.

This means we start from the standard triangle, the $(1,1,1)$ triangle, and pass to another triangle, and in the tropical limit of a general surface in the pencil, this is a surface such that inside the triangle we do a tropical [unintelligible]-ification. The modification is that I get another branch [picture].

Let me draw the same picture in logarithmic coordinates [picture]. I get a union of three half-planes, connected at the boundary in two different ways. We have a book with three pages. Two of them correspond to bigons and the other to a corner, and has two ways to let the disks go to ∞ and end. The whole pencil is obtained like this [picture]. This allows us to compute tropical disks.

In the remaining five minutes, let me say the rule for mutation of the potential and this rule will be consistent with [unintelligible]. We'll independently prove a relative correspondence theorem for tropical disks. This is joint work in progress with [unintelligible].

Okay, so let me say how to get a mutation. Let's make the first step. How do we do the mutation? What was the conjecture? To do the mutation, the easiest thing, the rule for the mutation starts from a Markov triangle with three parts. The first thing is that we sholud decide which one we want to mutate. [picture] Then we make that one horizontal. Then we have the cluster mutation, divide by $1+x$, essentially. [pictures]

2. Ping Li: May 31: On the uniqueness of the complex projective spaces

My talk will be divided into two parts. In the first part I'll discuss the uniqueness of $\mathbb{CP}^n$ in terms of topology. This is the main part. If time permits I'll also discuss uniqueness in terms of spectra of the Laplace operator.

First of all, let's recall some basic facts of $\mathbb{CP}^n$. On this, we have a canonical metric, the Fubini–Study metric g_{FS}, and it's a complex manifold with the standard structure J_{stand}. This is Fano (meaning $c_1 > 0$); the metric is Kähler and Einstein. The integral homology ring is $H^*(\mathbb{CP}^n, \mathbb{Z}) = \mathbb{Z}[t]/\langle t^{n+1}\rangle$. Because $c_1 > 0$ we can choose t in H^2 to be positive. Then the total Chern class is $c(\mathbb{CP}^n) = (1+t)^{n+1}$ and the total Pontrjagin class is $p(\mathbb{CP}^n) = (1+t^2)^{n+1}$.

Conversely, you can ask, among all this data, how we can pick up as little information as possible to characterize $\mathbb{CP}^n$.

Theorem 2.1. *(Hirzebruch–Kodaira 1957) If M is Kähler and is diffeomorphic to $\mathbb{CP}^n$ with the standard smooth structure, then*

1. *M is biholomorphic to $\mathbb{CP}^n$ if n is odd.*
2. *M is biholomorphic to $\mathbb{CP}^n$ if n is even and c_1 is not of the form $-(n+1)t$.*

Here t is still the positive generator of $H^2(M, \mathbb{Z})$.

The idea of the proof, note the year of publication. That year Kodaira has just found his fundamental results, Kodaira embedding and vanishing, and Hirzebruch has established higher dimension Riemann–Roch. So this is a combination of these two remarkable theorems.

So first show that the Todd genus of M is $\mathrm{td}(M) = 1$. Then apply Kodaira embedding. Now we know that Hirzebruch–Riemann–Roch is valid for all complex manifolds; at the time it was only valid for projective space. So they needed to embed holomorphically into some projective space.

Then use Herzebruch–Riemann–Roch to show that $c_1(\mathbb{CP}^n) = (n+1)t$ for n odd or $\pm(n+1)t$ for n even. Suppose $c_1(L) = t$, since t is positive, we can choose a holomorphic line bundle L in the Picard group of M so that the class of L is t. Then $H^q(M, \mathcal{O}(L))$ is 0 if $q > 0$ by Kodaira vanishing. This implies that only $H^0(M, \mathcal{O}(L))$ matters, which is $\chi(M, L)$, and then by Hirzebruch–Riemann Roch you can show that this is $n+1$. Then we can establish a biregular equivalence to the projectivization of $H^0(M, \mathcal{O}(L)) \cong \mathbb{CP}^n$.

The second part of the argument can be refined, actually. This was done by Kobayashi and Ochiai as follows. Let M be Fano, so $c_1 > 0$. We define $I(M)$ to be the Fano index of M, defined as th the largest possitve integer which divides the first Chern class. So for example, $I(\mathbb{CP}^n) = n+1$ and I of the hyperquadric in $\mathbb{CP}^{n+1} = n$.

Then Kobayashi and Ochiai showed the following result, inspired by this. Suppose M is Fano. If the Fano index is at least $n+1$ then M is biholomorphic to $\mathbb{CP}^n$. I didn't need to assume Fano, to define index. If the index $I(M) = n$, then M is biholomorphic to a hyperquadric in $\mathbb{CP}^{n+1}$.

The additional requirement is because $c_1(M)$ might be negative in the even case.

In 1977, Yau noticed that, as a corollary to the Calabi conjecture, there is a Chern number inequality involving c_1 and c_2.

(1) The condition "diffeormorphism" can be relaxed to "homeomorphism" due to Novikov's result from the 1960s on the homeomorphism invariants of rational Pontrjagin class.
(2) The even dimensional requirement can be removed by the Chern number inequality (negative case), which states that if M is Kähler and $c_1(M) < 0$, then
$$c_2(-c_1)^{n-2} \geq \frac{n}{2(n+1)}(-c_1)^n$$
with equality if and only if M is holomorphically covered by the unit ball. If the first Chern class is $-(n+1)t$, then we find that we attain equality, and then this is holomorphically covered by the unit ball, which it is not.

Then the theorem, due to Hirzebruch–Kodaira, Yau, is that M is Kähler and homeomorphic to $\mathbb{CP}^n$ then it is biholomorphic to $\mathbb{CP}^n$.

Then we ask whether the conditions homeomorphism and or Kähler be further relaxed?

For general n, we don't have any essential relaxation but when n is small enough we have the following, some results.

Theorem 2.2. *(many)*

(1) *(Yau, 1977) For $n = 2$, a compact complex surface, we have a very strong characterization. We don't need Kähler, and $H^*(M^2, \mathbb{Z}) = H^*(\mathbb{CP}^2, \mathbb{Z})$ (as a ring) implies that M^2 is biholomorphic to $\mathbb{CP}^2$. This solved the [unintelligible]conjecture.*
(2) *(Lanteri–Strappa, 1980) for $n = 3$, for M compact Kähler and $H^*(M^2, \mathbb{Z}) \cong H^*(\mathbb{CP}^3, \mathbb{Z})$ then M is biholomorphic to $\mathbb{CP}^3$.*
(3) *(Libgober–Wood, 1999; Debarre 2015) If $n = 4$ or $n = 6$, M is compact–Kähler and has the same cohomology ring, then M is biholomorphic to $\mathbb{CP}^4$ or $\mathbb{CP}^6$ or is covered by the ball. If $n = 5$ then M is biholomorphic to $\mathbb{CP}^5$.*

Let me give a short proof of Yau's result. So $b_1(M) = 0$ by this assumption. Then by classification of compact complex surfaces, it is Kähler if and only if b_1 is even.

So Kählerness can be deduced. We can use Riemann–Roch to show that $c_1^2(M) = 0$ and $c_2(M) = 3$. Then if $c_1 > 0$, then $c_1 = 3t$, and so by Kobayashi–Ochiai, this is $\mathbb{CP}^2$. If $c_1 < 0$ then by Yau, this is covered by B^2, so this is a fake projective plane. A very recent result of Prasad–Yeung implies that $H_1(M, \mathbb{Z})$ is nonzero torsion for a fake projective plane, so $H^2(M, \mathbb{Z})$ has torsion, a contradiction.

You need more and more complicated things to get to the expected case as the dimension rises.

Conjecture 2.1. (Libgober–Wood and Debarre) If M is compact Kähler and homotopy equivalent to M, or more weakly, M has isomorphic integral cohomology, then M is biholomorphic to $\mathbb{CP}^n$.

If you read Kobayashi–Ochiai and Yau and Hirzebruch–Kodaira carefully,

Theorem 2.3 (L.). *if M is compact Kähler and has the same cohomology ring as $H^*(\mathbb{CP}^n, \mathbb{Z})$, so that the total Pontrjagin class $p(M) = (1+t^2)^{n+1}$, this technical assumption, this is strictly weaker than homeomerphism. This has no torsion, so rational and integral Pontrjagin classes coincide. Then*

1. *M is biholomorphic to $\mathbb{CP}^n$ if n is odd*
2. *M is biholomorphic to $\mathbb{CP}^n$ if n is even or covered holomorphically by the unit ball.*

Remark 1.
1. The assumptions are weaker than homeomorphism, due to Novikov's result.
2. If n is even, we still have two possibilities. But if we further assume that $\pi_1(M)$ is finite when n is even, you can drop this possibility of being holomorphically covered.

Why should we have this? It's a biproduct of the reading of the original papers. It's also due to another result in transformation groups. We have the assumption that the total Pontrjagin class must be the standard form.

A sketch of the proof. We show that $h^{p,q}(M)$ should be 0 for $p \neq q$ and 1 for $p = q$. Then the Todd genus of M, $\operatorname{td}(M) = \sum_{g=0}^n (-1)^q h^{0,q}(M) = 1$.

This *genus* is a complex cobordism invariant, in the sense of Hirzebruch. Every complex genus corresponds to a power series with constant term 1, whose characteristic power series is $\frac{x}{1-e^{-x}}$, which we can slightly rewrite as $e^{\frac{x}{2}} \frac{x}{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}$, you can recognize that this is the $\hat{A}$-genus, which can be defined for any oriented smooth manifold, because it is an even power series.

So if we formall write $c(M) = \prod_{i=1}^m (1+x_i)$ for $m \geq n$, so $c_1(M) = \sum x_i$ then

$$\begin{aligned} 1 = \operatorname{td}(M) &= \int_M \prod_{i=1}^m \frac{x_i}{1-e^{-x}} \\ &= \int_M e^{\frac{\sum x_i}{2}} \prod \frac{x_i}{e^{\frac{x_i}{2}} - e^{-\frac{x_i}{2}}} \\ &= \int_M e^{\frac{kt}{2}} (\frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}})^{n+1} \end{aligned}$$

because for oriented smooth manifolds you can use Pontrjagin classes to define your genus.

Using the resudie formula you get the coefficient of y^n in $(1+y)^{\frac{k+n-1}{2}}$, which, well, it depends on whether this is an integer or half-integer. So something elementary, you still have to say something, this is $n+1$ if n is odd; if n is even, this is $\pm(n+1)$, there is a sign problem. We conclude the theorem.

Let's complete the technical result. This has the assumption about the Pontrjagin class, being in the standard form. This result is related to a famous conjecture in transformation group theory.

Conjecture 2.2. (Petrie, 1970s) Let M^{2n} be a smooth oriented manifold homotopy equivalent to $\mathbb{CP}^n$ and M admits a smooth and effective circle action, then $p(n)$ is the standard form, $(1+t^2)^{n+1}$, where t is any generator of $H^2(M,\mathbb{Z})$.

Up to now the conjecture is still open. But there are partial results toward this conjecture, which is why this technical result makes sens.

Theorem 2.4. *([unintelligible]–Witking, 2004) Let M^{2n} be a smooth manifold homotopy equivalent to $\mathbb{CP}^n$ and T^k acts on M^{2n} with $k > \frac{n+1}{4}$, smoothly and effectively, then the conjecture holds; $p(M)$ is the standard form.*

Our second result is just a combination of my result and this result. We wanted compact Kähler and homotopy equivalent (or weaker, the same cohomology ring). Now if the underlying manifold has this underlying symmetry, we get this.

Theorem 2.5. *(L.) If M is compact Kähler and homotopy equivalent to $\mathbb{CP}^n$ and M admits a torus action, smooth and effective, for $k > \frac{n+1}{4}$, then M is biholomorphic to $\mathbb{CP}^n$.*

I want to briefly outline the second part in terms of the spectrum of the Laplacian.

Let's begin for some general observations. Suppose we have a closed oriented Riemannian manifold. Then we have $d : \Omega^p M \to \Omega^{p+1} M$, and the metric induces a formal adjoint of this operator, $d^* : \Omega^p(M) \to \Omega^{p-1} M$, and then we get the Laplacian $\Delta := dd^* + d^*d : \Omega^p(M) \to \Omega^p(M)$. The eigenvalues are discrete and have a sequence $\mathrm{Spec}^p(M,g)$; the operator is nonnegative, so these eigenvalues are non-negative and tend to ∞. The eigenvalues may have multiplicity. We denote it, they may repeat. By Hodge theory, the multiplicity of 0 is the pth Betti number of the manifold. So we have this question. How does the spectrum $\{\lambda_{k,p}\}$ reflect the geometry of (M,g)? Or you may use a more fantastic sentence.

Can one hear the shape of the drum by its frequency?

In general, due to examples (this is an observation of Milnor), there exist non-isomorphic pairs (M_1,g_1) and (M_2,g_2) with the same spectra. In general the spectrum cannot determine the manifold. We can continue, a well-known result of Patodi says whether or not the metric g is flat, of constant (sectional) curvature, or Einstein, is completely determined by the spectrum of the Laplacian.

This means that given two Riemannian manifolds, if they have the same spectra, if one is flat, the other is. If one is constant sectional curvature, the other is. If one is Einstein, the other is.

Let me go to $\mathbb{CP}^n$. For general manifolds, you cannot hear the shape. But if the drum is special enough, you should have some positive result.

Another result, maybe Gilkey, you can check, for a compact Kähler manifold (M,g,J), and the spectrum is the same as $(\mathbb{CP}^n, g_{FS}, J)$ then the two are holomorphically isometric.

Another question, then, is whether, for (M,g,J) compact Kähler, if Spec^p agrees for some fixed p, can we conclude the same?

Some partial results. When $p = 0$ or 1, for $n \le N$ (some concrete integer), this is true. For $p = 0$ this is maybe $n \le 6$ and for $p = 1$ something like $n \le 45$. I forgot who did this.

I'm concerned with the next case, when $p = 2$. If $n \ne 2, 8$, this holds. This is due to B. Y. Chen–Vanhecke. For $p = 2, 8$, Goldberg claimed a result that was incorrect. Farther together with [unintelligible]obtained some other results for other p. This is in Tohoku.

What I want to say, is $p = 2, n = 2$, for this case, this is a compact complex surface, this is true, you use some argument to see that. For $n = 8$, the proof of Goldberg doesn't work, the mistake was not easy to find, he used a result of Kobayashi, an early result. I noticed this only very recently, but later, Perrane published a result reproving $p = 2$ case by Kobayashi–Ochiai but this is wrong too.

Let me briefly explain why their proofs are false. The false idea was carried over to the second paper.

So he wanted to use Kobayashi's earlier results. The claim is that for (M, g) Kähler–Einstein with positive scalar curvature (so Fano with a compatible Einstein metric) and so that the scalar curvature is $S_{g_{FS}}$ on $(\mathbb{CP}^n, g_{FS})$, then $\mathrm{Vol}(M, g) \le \mathrm{Vol}(\mathbb{CP}^n, g_{FS})$, with equality if and only if M is biholomorphically isometric to $\mathbb{CP}^n$.

I thought this must be wrong, because this is a very famous conjecture. This is the same as the following case.

For M with $c_1 > 0$, and a Kähler–Einstein metric, then $c_1^n(M) \le (n+1)^n$

with equality if and only if M is biholomorphic to $\mathbb{CP}^n$. The reason is easy. It's Kähler–Einstein, so the Ricci form $\mathrm{Ric}(\omega_g) = \omega_g$ so $\mathrm{Ric}^n(\omega_g) = \omega_g^n$. This was recently resolved by Kento Fujita.

What Goldberg outlined is the result of the following weaker form. If M has $c_1 > 0$ and there exists a Kähler–Einstein metric then $c_1^n(M) \le \frac{n+1}{I(M)}(n+1)^n$ with equality if and only if M is biholomorphic to $\mathbb{CP}^n$. This should be well-known for several decades in complex algebraic geometry.

This was one thing we needed to correct. The second thing was some differential geometry, where he calculated some pointwise, this is not easy; some details must be, he used some early results but his idea is very good. He misused some key point of Kobayashi. The Kobayashi results are now so comprehensive, so many details should be checked to make sure that the otuline of Goldberg's proof is this result. When I carefully check, there is even no detailed proof of this Chern class inequality because it's been well-known to experts for decades. [some discussion of the difference between the two]

He also made some mistakes in calculating some square norms of tensors. So for other p the statement must be reproved. The Perrone proof is also false. He missed, it's my duty to rewrite this topic again to make everything clear and clean and correct. He said he rescaled the metric if necessary, but this is not valid, because at the beginning you fix the spectrum. If you rescale the metric, the eigenvalues are rescaled as well. You cannot rescale the metric later. This is a key mistake. My duty is to rewrite or reprove all of these results. This is ongoing. I hope I can complete it this summer.