INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS IRREGULAR SEMINAR

GABRIEL C. DRUMMOND-COLE

1. May 26: Ziming Nikolas Ma: Witten–Morse theory and mirror symmetry

I'm happy to be here and give a talk. Let me begin on part one, which is Witten–Morse theory. I'm going to start from very fundamental stuff. What we are considering is an oriented (compact) Riemanian manifold M, we're considering a Morse function on M, $f : M \to \mathbb{R}$. The critical points $\{p_i\}$ are isolated and at each critical point the Hessian is nondegenerate. The set of critical points is $Crit(f)$.

The Morse Lemma basically tells you that locally near p_i I can write this in a standard form

$$f(x) - f(p_i) = \frac{x_1^2 + \cdots + x_{n-k}^2 - (x_{n-k+1}^2 + \cdots + x_n^2)}{2}.$$

The degree of p_i is defined to be k, the number of negative directions. This is the notation. The set of critical points can also be given an index to distinguish critical points of a fixed degree.

There is a well-known Morse complex that one can construct to detect the topology of the manifold. The complex looks like

$$\cdots \to CM_f^{k-1} \to CM_f^k \to CM_f^{k+1} \to \cdots$$

where CM_f^k is the sum of $\mathbb{C}p$ over critical points in $Crit^k(f)$. The differential, I'll specify by

$$\langle \delta p, q\rangle$$

is the count of gradient flow lines from p to q. This is a signed count, so some of them count as $+1$ and some as -1.

It is a known fact that $H^*(CM_f^*, \delta) \cong H^*_{dR}(M; \mathbb{C})$. This is the Morse complex and what Witten suggests is giving me a Morse function, a natural object to look at is the de Rham complex of the manifold

$$\cdots \to \Omega^{k-1}(M) \to \Omega^k(M) \to \Omega^{k+1}(M) \to \cdots$$

which is God-given. Now we have $f : M \to \mathbb{R}$ our Morse function and with a small constant $\hbar$, Witten suggests doing a twist, twisting the de Rham differential by the Morse function. So

$$d_f = e^{-\frac{f}{\hbar}} d e^{\frac{f}{\hbar}}$$

and this differential is $d + \hbar{-1} df\wedge$. There is also an adjoint operator $d_f^* = e^{\frac{f}{\hbar}} d^* e^{-\frac{f}{\hbar}} \iota_{\nabla f}$. Then we can define the Witten Laplacian $\Delta_f = d_f d_f^* + d_f^* d_f$.

What Witten did is to look at the eigenvalues of the twisted Laplacians and look at how they move as $\hbar$ moves. We want to see how the eigenvalues move. At the beginning, the eigenvalues are spread out and the eigenvalues, some of them go to zero exponentially fast (finitely many). Some will go to positive ∞. We are interested the eigenvalues that eventually are less than one in the whole de Rham complex. We are interested in the eigensubcomplex with eigenvalues less than 1. It's kind of obvious that this subcomplex has the same homology as the de Rham complex, and he identified it with the Morse complex.

There is a map from the Morse complex $\phi_\hbar : CM_f^* \to \Omega^*_{<1}(M)$ and this is an isomorphism of vector spaces. It will send the critical point p, and the image is a function concentrated around p.

To be more precise let me give an example. Let me locally take my Morse function to be $f = \frac{x_1^2 - x_2^2}{2}$ on $\mathbb{R}^2$, with critical point the origin. The eigenform

$$\phi_h(0) = \frac{1}{(\pi\hbar)^{\frac{1}{2}}} e^{-\frac{x_1^2+x_2^2}{2\hbar}} dx_2$$

which is

$$e^{\frac{f(x)-f(0)}{\hbar}} \Big(\frac{1}{(\pi\hbar)^{-\frac{1}{2}}} e^{-\frac{x^2}{\hbar}} dx_2 \Big).$$

Some pictures of this.

After this identification, there is a next level of identification, you can identify the differentials:

Theorem 1.1. *(Witten, Helffer–Sjöstrand)*
The limit as $\hbar \to 0$ of d_f is δ.

More precisely, if p is of index k and q of index $k+1$, and there are the two eigenforms associated to them, I'll look at the operation

$$\langle d_f(\phi_h(p)), \frac{\phi_\hbar(q)}{||\phi_\hbar(q)||} \rangle = e^{-\frac{f(q)-f(p)}{\hbar}} \langle \delta p, q \rangle (1 + O(\hbar))$$

The left hand side is an integration

$$\int_M d_f \phi_\hbar(p) \wedge \frac{*\phi_\hbar(q)}{||\phi_\hbar(q)||}.$$

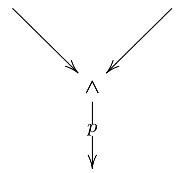
2. Witten deformations of product structures

This is a conjecture of Kenji Fukaya. Originally we just considered the de Rham complex, which is a differential graded algebra. As a differential graded algebra it satisfies the following property, the defining property of a differential graded algebra:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta.$$

After the twist, there is a reasonable operation, to look at $\Omega^*_{<1}(M) \otimes \Omega^*_{<1}(M) \to \Omega^*_{<1}(M)$, and what you do is to wedge together and then project to the subspace of small eigenvalue, orthogonal projection using the Riemannian metric.

To satisfy the correct properties, then you should take three different Morse functions, so that in the codomain the Morse function is the sum of the other two. Graphically, we have m_2 represented by a trivalent directed tree with two inputs and one output. In the internal vertex you put the wedge product and on the outgoing edge the projection.



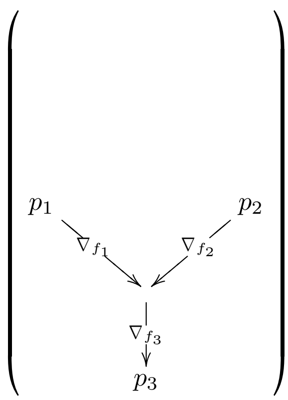
Theorem 2.1. *(Case of m_2)(Chen, Leung, [M]) The limit $\lim_{\hbar\to 0} m_2$ is the counting of some number of trees, satisfying some genericity conditions.*

Let me spell it out. I have three critical points, p_1, p_2, p_3, and I associate the eigenform to them

$$\langle m_2(\phi_\hbar(p_1), \phi_\hbar(p_2)), \frac{\phi_\hbar(p_3)}{\|\phi_\hbar(p_3)\|^2}\rangle$$

is the same as

$$e^{-\frac{1}{\hbar}(f_3(p_3)-f_1(p_1)-f_2(p_2))}\#$$



$$(1+O(\hbar))$$

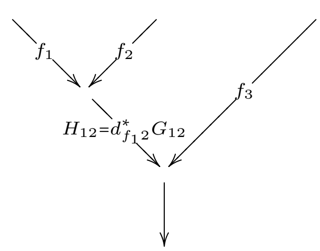
Now m_2 is not associative so there are higher structures, let me start with m_3

$$m_3 : \Omega^*_{f_1,<1} \otimes \Omega^*_{f_2,<1} \otimes \Omega^*_{f_3,<1} \to \Omega^*_{f_4=f_1+f_2+f_3,<1}$$

which satisfies

$$[m_2, m_2] = \pm[m_1 = d, m_3].$$

So how to write down m_3? basically there are two trees:



and the other tree with the same inputs.

Theorem 2.2. *(Case of m_3) (Chen, Leung, [M])*
Suppose I have four critical points, do the same thing,

$$\langle m_3^{T_1}(\phi_\hbar(p_1), \phi_\hbar(p_2), \phi_\hbar(p_3)), \frac{\phi_\hbar(p_4)}{\|\phi_\hbar(p_4)\|^2}\rangle$$

is the same as

$$e^{-A/\hbar}\#\,(\textit{count of these trees})\,(1+O(\hbar^{\frac{1}{2}})).$$

I have ten minutes, let me just sketch the proof. So the difficulty comes from doing the approximation for the Green's kernel. Look at the diagonal in $M\times M$. Originally I have a kernel function defined, a kernel $\Delta_{f,y}H_f(x,y) = d^*_{f,y}(I-P_y)\left(e^{-\frac{f(y)-f(x)}{\hbar}}\delta_\Delta(x,y)\right)$ and this is too singular, so I replace $\delta_\Delta(x,y)$ with a smoothing $\delta_{\Delta\hbar}(x,y)$. [Explicit formula]. Life is better but we still need to, from the a priori estimate, one can localize to a sum over gradient trees and each gradient tree localizes the integral over the two points x and y.

Then for each gradient tree one needs to comput this thing, that's where the WKV approximation comes in. So because I have cut off from the gradient tree, these two points are away from the critical points, so now the equation becomes something like

$$\Delta_{f,y}\widehat{H}_f(x,y) = d^*_{f,y}e^{-\frac{f(y)-f(x)}{\hbar}}e^{\frac{-\|x-y\|^2]}{\hbar}}(\quad - \quad - \quad).$$

So we should rewrite this without x, so look at the equation

$$\Delta_f\xi = d^*_f e^{\frac{-f}{\hbar}}(e^{\frac{-g}{\hbar}}\nu)$$

where hat means smoothed versions. So $\xi = e^{-\psi/\hbar}(w_0 + w_1\hbar^{\frac{1}{2}} + \cdots)$ and we need a guess for ψ.

A first guess is

$$\psi_0(x) = \min_{z\in M}(f(z)+g(z)-\|z-x\|_f)$$

which is the distance associated to $|df|^2 - g_0$. If I write this, well

$$f(x) + \min(g(z) + \|z-x\|_f - (f(x)-f(z))).$$

So the two pieces are positive and the absolute minimum happens along the a gradient flow line.

Now this is still too singular. Now we have a hypersurface passing through x_0, which I call U. Then one can parameterize the neighborhood of the flow line using the geodesic lines associated to this metric. I think I will stop here for the first half.

3. Scattering diagrams

I want to talk about SYZ mirror symmetry (Fukaya). I want to consider a Calabi–Yau manifold, compact for now, with ω and J. As an idea, we can assume it to be compact, it's Calabi–Yau with Lagrangian fibration B. The base is real dimension two and X is complex dimension two. For technical reasons we assume there is a Lagrangian section. There may be some singular points. The base is a smooth part B_0 union with B_{sing} which is codimension two. The generic fiber is a smooth torus and a singular fiber is a degenerate torus.

Basically we are looking at the A-model. I mean I am counting holomorphic disks which have boundary inside a smooth torus and hit the singular fibers.

To take the dual torus fibration, we need to remove the singular fibers and then do fiberwise duality. So we get (X_0,ω,J) over B_0 (which has an affine structure). This is T^*B_0/Λ^* with the standard symplectic form. Then we should add back in the holomorphic disks. So there is some quantum data to get to (X,ω) which is precisely those holomorphic disks.

What exactly is that holomorphic disk data? To be more precise, on B_0 there is an exact sequence of lattices so that at each point on B I can look at $\pi_2(x, p^{-1}(u))$ where u is a generic point, and we get

$$0 \to \pi_2(x) \to \pi_2(x, p^{-1}(u)) \to \pi_1(p^{-1}(u)) \to 0$$

So locally the fiber we get $\pi_1(T_u^N) = \mathbb{Z}^N$ and $\pi_2(x)$ gives me $\mathbb{Z}^{M-N}$. This is in N complex dimensions.

What I'm trying to look at is $\alpha : D \to X$ such that the class of α is inside $\pi_2(X, p^{-1}(u))$ with Maslov index 0. You quotient by a relation and I won't be precise. The moduli will help me to recover my manifold. There is a map from this to $\mathcal{M}$, the lattice bundle $\pi_2(x, p^{-1}(u))$ which projects to B_0. This map is "take homotopy class" and this map is codimension one from dimension counting.

This structure tells me how to go from X_0 to X. So to be more precise, the image on the base should be some lines, the codimension one submanifolds. Those are called walls. To make life easier, we look at a contractible neighborhood on the base. We look at what happens inside a contractible neighborhood. First of all we can trivialize, take the product with our contractible neighborhood U. Then we can look at the strcuture of the walls.

A *scattering diagram* is a union of walls $\bigcup(\ell_i, f_i)$. What does each one look like? ℓ_i is a ray or a line. This means there is a holomorphic disk which propagates in the direction of the ray. A wall has a holomorphic disk on top of it. Then f_i is basically a generating series which counts the number of holomorphic disks are on top of it. So $f_i = \sum a_m^{(i)} w^m$ where $a_m^{(i)}$ is the count of holomorphic disks. The lattice m has some restrictions. I'll make it clear.

Anyway I have a structure like this that will help me to capture the holomorphic disk data. What is scattering? It's a procedure that governs this package of data. It basically comes from the gluing of holomorphic disks. The picture is, say you have a part of the base and one ball propagating in one direction and another one hitting it in a transversal direction (pictures). So basically these two disks can glue by a pair of pants to give a new disk. The there is a new wall where they intersect. This is the scattering effect.

In general the structure of walls can be very complicated and the generating series can have infinite sums at a point.

Suppose I have $\ell_i = \mathbb{R}_{\geq 0} r_i + b_i$ or $\mathbb{R} r_i + b_i$, then $f_i = \sum a_m^{(i)} w^m$. So $r(m)$ gives me a function and then I need to fix a metric, flat, Riemannian, on U. Then this $r(m)$ is a one-form on U. I will need Novikov theory to lift to $\mathcal{M}$. Anyway, there is a gradient vector field V_m associated to this metric. I need to look at only the m such that V_m is parallel to r_i.

Instead of thinking about this as a function, we'll think about it as a vector field. Where does the vector field come from? Formally I attach a vector field and I get a Lie algebra structure $\mathfrak{h}$, and I need to specify

$$[w^{m_1} \otimes \partial_{n_1}, w^{m_2} \otimes \partial_{n_2}] = w^{m_1+m_2} \partial_{\langle r(m_2) n_1 \langle n_2 - \langle r(m_1), n_2 \rangle m_1}.$$

Now imagine I have two walls. [picture] Exponentiating the Lie algebra elements I get an automorphism inside the vertex group. At first if I look at a loop around the origin in the counterclockwise direction, I get $\Theta_0^{-1}\Theta_1\Theta_0\Theta_1^{-1}$. There is a unique way to add walls in the first quadrant and what they show is that this is unique such that the product is equal to the identity:

$$\Theta_0^{-1}\Theta_1(\prod \Theta_{q_i})\Theta_0\Theta_1^{-1} = \mathrm{id}.$$

We want to say that this scattering procedure is equivalent to solving the Maurer–Cartan equation.

So $\mathfrak{h}$ has elements $\sum a_m^{(i)} w^m \otimes \partial_n$ and we enhance it to $\mathfrak{g} = \bigoplus \Omega^*(B_0) w^m \otimes \partial_n)$. This is a dg Lie algebra with the standard generalizations $\bar\partial$ and $\{,\}$. So for example $\bar\partial(\varphi w^m \otimes \partial_n) = d\varphi w^m \otimes \partial_n$ and some generaliziton for $\{,\}$.

[missed some] I need to do some gauge fixing $H : \mathfrak{g} \to \mathfrak{g}[-1]$.

So $H(\alpha w^m \otimes \partial_n)$, I can pull back and get $\left(\int_{-\infty}^0 \tau_m^*(\alpha)\right) w^m \otimes \partial_n$ where $\tau_m : \mathbb{R} \times B_0 \to B_0$.

Now there is a sum over trees formula. Basically to solve the equation, we look at all the sums over trees, we put ψ_1 at the inputs. The internal vertices get $-\frac{1}{2}$ the Lie bracket and on the internal edges the H operator. So each tree gives ℓ_T, and I can define ψ to be the sum over all trees of all these operations. This is a solution to the Maurer–Cartan equation.

If we produce a solution in this way, well, let me write down the theorem

Theorem 3.1. *(Chen (a different Chen), [M]) The limit to ψ as $\hbar$ goes to 0 (I've hidden $\hbar$ everywhere) gives me back the scattering diagram.*

The solution looks like it's on each ray, but I need to remove a small disk around the origin. Then I get $\psi \to \varphi_0 + \varphi_1 + \sum \delta_{a_i,\hbar} f_{a_i} + O(\hbar^{\frac{1}{2}})$. That's basically what I wanted to say.

4. November 23: Carlos Shahbazi Alonso: On the mathematical formulation of four-dimensional supergravity and its supersymmetric solutions I

I'll be talking about supergravities. In contrast to quantum field theory, they admit a rigorous mathematical formulation. They can be explained in terms of differential, algebraic, and spin geometry.

Why are we interested in supergravity? They include general relativity and low energy string theory.

I'll focus on four-dimensional supergravities. Time permitting I'll talk about ten dimensions as well.

Let me explain the basics a little bit, of 4-dimensional supergravity.

For me it's formulated in a 4-dimensional Lorentzian manifold (M_4, g). Physically it is supposed to describe gravity. I need to be able to define spinors on the manifold. I will assume, although it is not strictly speaking necessary, that M_4 is oriented and spin. Let us denote by $Cl(M,g)$ the Clifford bundle over (M,g). For those of you who don't know it, the typical fiber is the Clifford algebra at that point, $Cl(M,g)|_p$ is the Clifford algebra of the vector space (T_pM, g_p). Given a finite dimensional vector space with a quadratic form (V,q), the Clifford algebra $Cl(V,q)$ is the unital associative algebra generated by V subject to the condition that $v^2 = q(v)$ for all v in V. So $T(V)/I(V)$ where $I(V)$ is generated by $v \cdot v - q(v)$.

Once we've constructed this bundle, we can try to associate to it a bundle S of Clifford modules over $Cl(M,g)$. There is an obstruction to constructing S.

Riemannian and Lorentzian signatures are different. The obstruction for a compact manifold to admit a Lorentzian metric is Euler characteristic, it has to be

0. If you're interested in Ricci flat or Einstein metrics, you have an elliptic set of differential equations, whereas in Lorentzian metrics they are hyperbolic. Moduli spaces are not usually well-defined in Lorentzian situations. Often situations in supergravity, trying to find solutions is related to a question in Riemannian geometry about an associated manifold.

Now the bundle of Clifford algebras, we want a bundle of irreducible representations of our Clifford algebras. This depends on the dimension and signature. The standard way to proceed is to assume the manifold is oriented and spin, so that the first two Stiefel–Whitney classes w_1 and w_2 are zero. Then we can make Clifford modules. This is sufficient but not necessary.

By spin I mean you can lift your structure group from $SO(1,3)$ or $SO(4)$ to $Spin(4)$.

Now I'll focus on four dimensions, So $Cl(1,3)$ is the Clifford algebra of $\mathbb{R}^4$ with the Minkowski metric η. This guy here is is isomorphic as a unital associative algebra to $M_4(\mathbb{R})$, a matrix algebra. This is not signature-independent. We can define a subalgebra of our Clifford algebra, the even part of the Clifford algebra Cl^0, which is generated by an even number of products. It's clear that it's a subalgebra of this guy. What do they have to do with the spinors? Well, the following holds. The spin group $Spin(d) \subset Cl^0(V,q) \subset Cl(V,q)$, so by restricting to the even part we get representations of the spin group. Irreducible representations of the even part stay irreducible when restricted to the spin group.

In our simplified setup, we'll want representations of the spin group, which are Fermions, so relevant for describing fundamental particles. In our case, this $M_4(\mathbb{R})$ has a unique irreducible representation on $\mathbb{R}^4$, which stays irreducible as a representation of the even part, and then remains irreducible on $Spin(1,3)$. This is what physicists call Majorana spin. This is important in supergravity. You have all these fields, tensor products of spinors and tensors, which are invariant under the infinitesimal transformations generated by a spinor. So the Lie derivative of a vector field acts on tensors. So we'll have something similar, generated by a spinor. Depending on the situation, how many spinors we have, we'll have different supergravities, you can have $1,\dots,8$ (but not 7) spinors. The $N = 1$ is invariant under 1 supersymmetries and for $N = 8$ you get something invariant under 8 different spinors. We'll be concerned mainly with $N = 1$ and $N = 2$. The more supersymmetries you have, the more constrained the theory is. For $N > 4$, the theory is unique, there is only one supergravity. For $N = 3$ and $N = 4$ there is a discrete (infinite) set, say $\mathbb{Z}$, of supergravity. For $N = 1$ and $N = 2$, the moduli space of supergravities is not known but is an interesting problem. We will see it can be phrased very rigorously.

Once we know this, physicists don't always like to work with Majorana spinors, sometimes they prefer complex spinors. We can complexify the Clifford algebra we just used, and then we get a Clifford algebra over the complexes $\mathbb{C}\ell_4$, which splits, and you can define, if you complexify, there are two representations of the even part $\mathbb{C}\ell_4^0$, call them the $+$ and $-$ chirality representations of the even part of the Clifford algebra, which induce complex representations of the spin group, so-called Weyl spinors.

These generate the transformations that leave invariant the theory we want to consider.

Let me give you an example, a Lorentzian manifold with a bundle of Clifford modules, and the sections of the corresponding bundle are what physicists call spinors, which generate transformations, and our theory must be invariant under these transformations.

The actual calculations were done in the 70s and 80s and are very messy. I won't review them, they're not very conceptually interesting, but we'll pick up the results.

The supergravity action, someone did the calculation, writing the supergravity action is difficult because it's very long. People usually truncate to the Bosonic part of the action, which doesn't depend on the spinors. Looking for macroscopic solutions people only care about the Bosons.

$$S = \int_M \omega\{R + G_{ij}\partial_r\phi^i\partial^\mu\phi^\gamma + 2\Im N_{\Lambda\Sigma}\langle F^\Lambda, F^\Sigma\rangle - 2\Re N_{\Lambda\Sigma}\langle F^\Lambda, *F^\Sigma\rangle + V\}$$

The R is the curvature of the metric. The $G_{ij}\partial_r\phi^i\partial^\mu\phi^\gamma$ is a nonlinear sigma model. Let me explain a little. We assume we have a map $M_4 \xrightarrow{\phi} (\mathcal{M}, G)$, to a Riemannian manifold, and the ϕ are expressions of this map. So $||d\phi||^2$ is the global form of this guy. Let me tell you what $\mathcal{M}$ has to be. It follows from supersymmetry. For $N = 1$, $(\mathcal{M}, G)$ must be a Kähler–Hodge manifold. For $N = 2$ this is a product, a Riemannian product, of a special Kähler manifold and a quanternionic Kähler manifold. For $N > 2$, $(\mathcal{M}, G)$ must be a symmetric space. For $N = 8$, this is interesting at a quantum level, this $(\mathcal{M}, G)$ is $E_{7(7)}$ over $SU(8)$, the maximally [unintelligible]form of the exceptional Lie group $E(7)$. So supersymmetry highly restricts the theory already. In the $N = 8$ case it's unique.

Now you can see why the moduli space is not solved, it would involve classifying all Kähler–Hodge manifolds. The goal of this lecture is to do the $N = 2$ case in [unintelligible]full generality.

So $F^\Lambda : \Lambda : 0, \dots, n$ and [unintelligible]are curvatures of $U(1)$ bundles. N is a matrix whose entries depend on ϕ. We call ϕ the scalars. You pull back the connection, compute the curvature, and pull the curvature back to the base to get a 2-form on the base. You take the pullback $\phi^*(F^\Lambda)$ and $\phi^*(F^\Sigma)$, so

$$\langle F^\Lambda, F^\Sigma\rangle = \int_M \epsilon\langle\phi^*(F^\Lambda), \phi^*(F^\Sigma)\rangle_M.$$

Now you need the equations of motion coming from the action. This is very different from having a Ricci flat metric. Usually you are not interested in the [unintelligible]here. Using supersymmetry, the problem of finding solutions to this system of equations can be addressed.

Let me give you an example of supersymmetry calculations. As Calin said, there is an object called the gravitino, a section, a 1-form tensor a spinor, ψ. As I said we will be interested only in the bosonic sector, all the things that contain the spinors we'll truncate to 0, then the supersymmetric transformation of ψ, I'll write δ_ϵ, in the $N = 1$ case, the simplest one, is just $D\epsilon$. Now D is a connection on the spinor bundle, $\Gamma(S) \to \Omega^1(M) \otimes \Gamma(S)$. This is an infinitesimal transformation of gravity. We have to work more to see D. In the $N = 1$ case, as we said, the manifold was Kähler–Hodge, so there' is a holomorphic line bundle over it. We have this map $M_4 \xrightarrow{\phi} \mathcal{M}$, a differentiable map.

So we can pull back this bundle $\mathcal{L}$ to $\phi^*\mathcal{L}$. Our spinor will be a section of the spinor bundle times this bundle, it's not exactly this line bundle but so far, for our purposes it's enough to say this. I can give you the details if you want. I want to focus on a trivial case. To this holomorphic line bundle you can associate a $U(1)$, the pullback of the $U(1)$ bundle is what you need here. Let's take the case where $\mathcal{M}$ is a point, then $\mathcal{L}$ is trivial and we can use the trivial connection. If we use the trivial connection and the bundle is trivial, then the spinor (people say in this case there are no scalars) is a section of the spinor bundle $\epsilon \in \Gamma(S)$, and this becomes the standard spin connection $\delta_\epsilon \Psi = \nabla^\epsilon$, the standard lift of the Levi–Civita connection.

Supersymmetric solutions have to be invariant so they satisfy $\nabla^S \epsilon = 0$. We're in a well-defined mathematical setup. We need to worry about 4-dimensional manifolds equipped with parallel transport. There's a theorem about this. In particular, the holonomy of the connection is in the stabilizer $Stab(\epsilon)$. As I was pointing out at the beginning, Lorentzian and Riemannian geometry is different. Classification of things like this was only started in the 1990s for the Lorentzian case. If your holonomy acts reducibly in the Riemannian case, then your manifold is a product. That's not true, it's much more complicated.

Anyway, there's a theorem and then we can finish here.

Theorem 4.1. *Let (M, g) be a complete simply connected Lorentzian manifold of dimension $d = n + 2$ admitting a parallel spinor $\nabla^S \epsilon = 0$. Then the following holds. Either*

(1) *(M, g) is isometric to $(\mathbb{R} - dt^2) \times (M_1, g_1)$, where M_1 is a Riemannian manifold with special holonomy, admitting a parallel spin itself. But then a lot is known. It should be Ricci flat of special holonomy and then you have the Berger list. This is three dimensional for us. There is no irreducible three-manifold with special holonomy, not generic. So in the four dimensional case this is flat space. This is a pretty strong result.*
(2) *the holonomy group is contained in $SO(n) \ltimes \mathbb{R}^n$, and the projection to $SO(n)$ is one of the groups allowed in special holonomy, in Berger's list, G_2, $Spin_7$, so on.*

Let me summarize. I'll tell you who is the author of the theorem tomorrow. We started with a complicated scalar maniflod, we can truncate and do a simple case. In this simple case the gravitino superconnection reduced to $\nabla^S \epsilon$. This should be invariant which makes it 0. Then the geometry of the manifold is highly constrained as shown in the theorem. I didn't say how this was related to solving the equation, but I'll talk about this tomorrow. But you see there is something that you can say. The goal is to solve the differential equations that define the theory.

5. November 24: Carlos Shahbazi Alonso: On the mathematical formulation of four-dimensional supergravity and its supersymmetric solutions II

So I prepared for today, I hope it's something a little clearer than yesterday. Today we will focus on a particular supergravity. Let me make some general remarks.

Let me start with some references for what I said yesterday. The last theorem, the classification of spin or $Spin_c$ Lorentmanifolds, I'll give one name, Aziz Ikemakhen. For general supergravity, you can check Gravity and Strings by Tomás Ortin and Supergravity, by Freedman and van Proeyen. These are books for physicists, you won't have rigor. There may be errors but not in the physics. You can see here the formalism to build the Lagrangians. They developed in the 80s a formalism to find these Lagrangians. Some chapters are not so easy, they're quite technical. The first book is more conceptual, I'd say.

So in this one hour lecture, we will focus on $N = 1$ supergravity. I've been talking about "theories" but maybe I should say what a theory is. For me here, this misses many physical details, but a theory is a set of PDEs on an open set in $\mathbb{R}^n$ involving a metric, curvature of a bundle, and other similar things. This is very restrictive. In Riemannian geometry one is used to working with a given manifold (M, g), if it is complex, you can study the moduli space of complex structures or something, but the manifold is given, an underlying fixed differentiable manifold. In Lorentzian geometry, more importantly for physics, this is not the case, you don't know your underlying manifold. Consider the case where the unknown is the metric. This is general relativity. The vacuum equations, when there is no matter, the set of PDEs says that $Ric(\nabla^{LC}) = 0$. Now you could say, fix a manifold and study the Ricci flat metrics. The way to proceed is to solve these equations on an open set in $\mathbb{R}^n$. Then when you know the local form of the metric, you need to find what is the manifold, globally defined, that has this metric. This is called obtaining the analytic extension.

[discussion of analytic extension vis à vis Riemannian completion]

I thought, I hope I'm not wrong, I thought I'd copy the equations explicitly so you can see how easy to understand or not they are. It's easier to do this in local coordinates. Let (M, g) be our Lorentzian manifold. Let $U \subset M$ be an open set with coordinates x^μ for $\mu = 0, \ldots, d-1$. Given a tensor, for example a 1-form, I'll write $\xi(\frac{\partial}{\partial x^\mu})$ as $\xi(\partial_\mu) = \xi_\mu$. Then we can write the PDEs locally on U in a nice way.

$$Q_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + 2G_{i\bar{j}}(z)\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} - g_{\mu\nu} G_{ij} \partial_\rho z^i \partial^\rho \bar{z}^{\bar{j}} - 4\Im N_{\Lambda\Sigma} F_\mu^{\Lambda+} \rho F_{ij}^{\Sigma-}$$

The nonlinear sigma model $\int_M G_{i\bar{j}} \partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}}$ and then the effective potential $\int \omega V(z)$, and then the curvature, let's forget that. Let's explore the nonlinear σ model. If you remember there was a map $M \xrightarrow{\phi} (\mathcal{M}, G)$, where $\mathcal{M}$ was a Riemannian manifold. Given this map to a Riemannian manifold, you can take $U \subset M$ and $\mu \subset M$ such that $\phi(U) \subset \mu$, and the first thing that supersymmetry tells us, take this for granted, is that inside μ you have local complex coordinates z^i. Then the local form of ϕ evaluated in u is $z^i(x^\mu)$, this is a map, locally, I compose with the coordinates. This is to explain what was upstairs.

Now another thing that supersymmetry tells us is that the metric G locally on μ is given $\partial\bar{\partial}K$ where K is in $C^\infty(U)$. Then it tells me that the effective potential V, a function on the manifold, is locally writen as $e^K(DW\overline{DW} - W\bar{W})$ where W is a holomorphic function in μ where $DW = \partial W + \partial K W$. In order to have a globally consistent picture, we need to see what happens to these guys in an overlap between the two steps.

Supersymmetry tells us that on the overlap, we have $K_a = K_b + f_{ab} + \bar{f}_{ab}$ where f_{ab} is holomorphic in U_{ab}. We have $W_a = e^{-f_{ab}} W_b$.

Then one can say that $DW_a = e^{-f_{ab}} DW_b$. Pretty standard, right? Given this data, one can construct a holomorphic line bundle on $\mathcal{M}$, $\mathcal{L}$, as $\{\mathcal{M}, \{\mu_a\}, e^{-f_{ab}}, \mathbb{C}\}$.

Now we define on this holomorphic line bundle a holomorphic metric, $H|_{U_a} = e^{K_a}$, using the holomorphic local section W_a. With this definition one can see that the associated Chern connection acting on W is locally in U_a given by $\partial W_a + \partial K_a W$. This is clear, right?

I'll show you that $c_1(\mathcal{L}) = [\omega]$, the Chern class is the Kahler form.

If you want W_a to be invariant, you have to use this transformation rule, I don't know how you do this.

If you define the metric H, then V globally is $H(DW, DW) - H(W, W)$.

After identifying this, one can easily see that $iF_H = \omega$. Rescaling this one deduces that $c_1(\mathcal{L}) = [\omega]$. This implies that $\mathcal{M}$ is Kähler–Hodge.

The standard choice is the upper-half plane. If you specify $(\mathcal{M}, G)$ and W, along with the matrix N, (and gauge) then the theory is completely fixed. You don't need to explicitly write the equations of the Lagrangian.

[degenerated discussion]

So you can reduce the structure group to $U(1)$, doing this you can get a bundle $\tilde{\mathcal{L}}$ over $\mathcal{M}$ and pull it back to $\phi^*\tilde{\mathcal{L}}$ over M, the same as a choice of Hermitian metric. You need a family of these, choosing transition functions g_{ab}, and take $e^{qi\Im f_{ab}}$ and get a family of $U(1)$ bundles. q is an integer. In principle you need something more general, $e^{-q_1 f_{ab} - q_2 \bar{f}_{ab}}$.

Now I want to write the supersymmetric transformations which I wrote yesterday but didn't explain. There is a spinor ϵ in $\Gamma(S^+_{\mathbb{C}})$, this is a generator of supersymmetry in general, but we need to tensor with $\Gamma(\phi^*\tilde{\mathcal{L}}^{\frac{1}{2}, -\frac{1}{2}})$. This is the supersymmetry spinor. Now the supersymmetry transformation, the thing that should be zero to have a solution, is $D\epsilon$, where D is the spin connection on the spinor complex bundle and the pullback of the connection on the second factor. The $\tilde{\mathcal{L}}$ have a connection from the Chern connection and then you can pull it back. This $D\epsilon = 0$ is the Killing spinor equation of the gravitino, which means that $\delta_\epsilon \psi = D\epsilon$. This is what we said yesterday that one could use it to classify at least part of the supersymmetric solutions of the theory.

6. December 10: Matt Young: A mathematical approach to BPS state counting in orientifold string theory

Thank you for the chance to speak here. I want to talk about ongoing work to understand counting BPS states in the context of orientifolds.

For the first part of the talk I'll talk about why mathematicians might care about orientifolding. I'll also try to put it into a precise mathematical framework.

Some motivation. Let's start with a well-known caricature. Given an oriented string theory, of type IIA or IIB are the most familiar examples, this is a theory of maps ϕ from oriented surfaces with boundary into some space X which I'm happy to think of as a Calabi–Yau three-fold for this talk. This is a physics theory that we can't define, but it's a physics theory which predicts we should look at B, the category of D-branes. Some familiar examples are the Fukaya category of X, the derived category of coherent sheaves on X, or the derived category of representations of a quiver, matrix factorizations, et cetera. Once we have passed to the categorical setting, we can ask the following question. From the physics point of view you're interested in counting BPS D-branes. From the mathematical point of view we want to count stable objects of B with respect to some Bridgeland stability condition.

This is the kind of familiar story you get from oriented string theory. I want to talk about something slightly different. There's a procedure called orientifolding, a physical procedure that starts from an oriented string theory to produce a theory of maps of unoriented surfaces. This is a theory of maps of non necessarily orientable surfaces with boundary into X. In order to define this theory, we need some extra data σ on X. So now we'll start from the oriented string theory and try to replace things in our sketchy diagram. We can still get a category B of D-branes, the same D-brane category, and the extra data we get, I'll call orientifold data. It's a lowbrow piece of data. We get a contravariant functor $S : B^{op} \to B$ and secondly a way to identify $\theta : 1_B \to S \circ S$. This stability problem becomes the problem of counting stable self-dual objects of (B, S, θ), which I have to define.

Definition 6.1. A *self-dual object* is a pair $N \in B$ and $\psi_N : N \xrightarrow{\sim} S(N)$ such that $S(\psi_N) \circ \theta_N = \psi_N$. So this is some kind of transpose.

In some of these examples, what is this extra data? Suppose we start with B as the Fukaya category. We need to fix $\sigma : X \to X$, which is an antisymplectic involution, a smooth diffeomorphism which squares to the identity so that $\sigma^*\omega = -\omega$. What are objects of the Fukaya category? They are pairs consisting of a unitary local system E on a Lagrangian L of X, and the functor S takes the pair to the pullback $\sigma^* E^\vee \to \sigma(L)$, where $\sigma(L)$ is again a Lagrangian. The morphisms are roughly Floer cohomology groups, and because this is an anti-involution, it changes the direction of the cohomology, so it's contravariant.

The main type of examples will be the derived category of coherent sheaves or the local version.

What about when B is $D^b(X)$? Then we can take $S = RHom(\ \ , \mathcal{O}_X)$, the derived dual. Then $\theta = sev$ where ev is the isomorphism of a finite dimensional vector space with its double dual and s is ± 1, which is σ in this case.

What are the self dual objects? If N is a vector bundle, then $\psi : N \to N^\vee$ should be an isomorphism so that $\psi_N^\vee = s\psi_N$. We have tensor products here, this is the same as a nondegenerate bilinear form on N which is symmetric or nonsymmetric (depending on s). So this is the same as an orthongonal ($s = 1$) or symplectic ($s = -1$) vector bundle.

The usual problem of counting stable objects is counting vector bundles, but now you're counting slightly different objects that wouldn't fit in that basic framework.

One subcase of this problem is, can we get some handle on the moduli of whatever these bundles are.

This is the motivation. Today I'm going to focus, maybe I should write here, what is, what are the counting theories? For the Fukaya case, this is work of Joyce, counting special Lagrangians. The second to cases this is work of Kontsevich–Soibelman and Joyce–Song. The matrix factorization case is work of Toda, relatively basic compared to the others. The counting theories on the orientifold side, there's no work in any of the cases.

I want to focus on the middle example, which is Donaldson–Thomas theory, so $D^b(X)$ or $D^b(Rep\ Q)$, I want to explain the standard story first.

It's useful to recall the original example. This is Thomas in 1998. Suppose you are given X a smooth projective Calabi–Yau three-fold. You want to study the moduli space of sheaves on X. You can define $\mathcal{M}_d^{st}$, which is the moduli of stable (with respect to some polarization) sheaves on X with fixed characteristic numbers (which I'll just call d). This is a perfectly good moduli space. To do counting theory you need control on the geometry of this space. If there are no strictly semistable sheaves, you can think roughly that strictly semistable sheaves come from points that have too many automorphisms, so if there are none, then $\mathcal{M}_d^{st}$ is a (singular) projective scheme with (due to the Calabi–Yau condition on X) a symmetric perfect obstruction theory, and so we can define an integer DT_d to be the degree of the class $[\mathcal{M}_d^{st}]^{vir} \in \mathbb{Z}$. So we can define this virtual fundamental class, and this sits in degree zero, we can just count this and get the stable sheaves.

The main problem with this is that usually there are many strictly semistable sheaves. We can look at the moduli space of semistable sheaves, and this is a stack, not projective, with no virtual fundamental class. The second, well, not problem, but we may want more. This defined an integer that knows about the space but maybe we want more, a Poincaré polynomial or something like this, more refined information about this moduli space.

This is not so much a problem but we're greedy so we want more. So Kontsevich–Soibelman and Joyce–Song have different programs to address these issues. These programs are most well-founded in the case of quiver representations for $D^b(Rep\, Q)$. To do it for coherent sheaves, there are conjectures to relate them, but in the quiver case everything is well-established.

So I want to talk about the Kontsevich–Soibelman approach. Let's let Q be a quiver, just an oriented graph, maybe multigraph, so you can have multiple arrows between each node, and a representation of Q is a complex vector space for each node and a map for each arrow. So for this quiver [picture] you get U_i and U_j and then maps $u_\alpha : U_i \to U_j$ and $u_\beta : U_j \to U_j$, where U_i and U_j are $\mathbb{C}^{d_i}$ and $\mathbb{C}^{d_j}$ respectively. I'll stick to the node with $m \geq 0$ loops. So this will be a vector space with m endomorphisms. Let's call R_d the space of all tuples of all $d \times d$ matrices, and you have an action of the group GL_d which acts by automorphisms on this space. So this is the same as n-tuples of matrices up to simultaneous conjugation.

So we have an Abelian category of representations $Rep\ Q$. This is the main category we'll be interested in. We want to form a moduli problem, so what are the moduli spaces we're interested in? There are two. I'm interested in M_d, a stack, which is $[R_d/GL_d]$. We have a map from the stack to its coarse moduli space, the GIT quotient $\mathcal{M}_d$, which is $\operatorname{Spec} \mathbb{C}[\mathbb{R}_d]^{GL_d}$, this is a varienty but horribly singular, and sitting inside, open, we have a smooth but non-projective variety $\mathcal{M}_d^{st}$. These are the three moduli spaces I'm interested in.

Much of what I'll say will work for moduli problems in Abelian categories, and this should be basically the same as studying Bridgeland stability.

What is the basic geometry of these moduli? The first is that this space $\mathcal{M}_d$ is not really a moduli space. So $\mathcal{M}_d$ parameterizes closed orbits of GL_d on R_d, we'd really like to parameterize actual isomorphism classes but there's a lot of collapsing that happens here. More precisely, U has a filtration $0 = U_0 \subset \cdots \subset U_\ell$ such that U_i/U_{i-1} is stable (no nontrivial subrepresentations). This means the n matrices have no common eigenvector. This is the Jordan Holder filtration, any object has a stable filtration like this. Then U and $\bigoplus U_i/U_{i-1}$ are identified in $\mathcal{M}_d$. So we lose all the information about the extensions.

On the other hand, $\mathcal{M}_d^{st}$ really parameterizes isomorphism classes of stable representations. This is a very good moduli space. This filtration gives us a very useful heuristic. This lets us build the singular moduli space from the stable one. I wnat $\mathcal{M} = \bigsqcup \mathcal{M}_d$ and $\mathcal{M}^{st} = \bigsqcup \mathcal{M}_d^{st}$ and we have an identification $\mathcal{M} = Sym(M^st)$. Then the whole moduli space is built from the stable locus. This is not even close to true as varieties. You can get closer from a constructible stratification but that's not close enough. We'd like to use this set identification and lift it to the level of varieties, and this is what Kontsevich–Soibelman give a recipe to do.

We can make this precise via the Hall algebra. We have an Abelian category, we can talk about short exact sequences, and we should think if we have $0 \to U \to V \to W \to 0$, then V is built out of U and W. If I form V as an extension it will never be stable. Anything that we multiply and get as a product will never be stable so we won't be interested in it.

So how do you make this statement more precise? You consider the correspondence $M_{d,d'}$ which maps to $M_d \times M_{d'}$ and $M_{d+d'}$, where $M_{d,d'}$ is the stack of short exact sequences where U has dimension d and W has dimension d'. Then the projections are the maps to (U, W) and to V. Then let $\mathcal{H}_Q$ be $\bigoplus H^*(M_d)$ which is $\bigoplus H^*_{GL_d}(R_d)$. Then we can define an algebra on $\mathcal{H}_Q$ using the correspondence diagram. The map $\mathcal{H}_d \otimes \mathcal{H}_{d'} \to \mathcal{H}_{d+d'}$ is then $(p_2)_! \circ p_1^*$ and it makes $\mathcal{H}_Q$ into an associative $\Lambda_Q^+ \times \mathbb{Z}$-graded algebra. Here Λ_Q are the dimension vectors and $\mathbb{Z}$ is the cohomological grading. This gives a precise way of smashing together and getting a bigger representation.

This is the Hall algebra. If Q is symmetric (it has the same number of arrows going in each direction) then the algebra $\mathcal{H}_Q$ is supercommutative with $\mathbb{Z}_2$-grading coming from the $\mathbb{Z}$-grading. So odd elements are those with odd cohomological degree. This is a supercommutative algebra graded by the big lattice.

Now we can make geometric the bijection.

Theorem 6.1. *(Efimov, 2012ish) There exists a $\Lambda_Q^+ \times \mathbb{Z}$-graded vector space $V^{prim} = \bigoplus V^{prim}_{d,k}$ with the properties*

1. *Integrality: V_d^{prim} is finite dimensional, where this is the sum over all k.*
2. *The Hall algebra is the free supercommutative algebra on $(V^{prim} \otimes \mathbb{Q}[u])$ where u is in degree $(0,2)$.*

What does this theorem mean? Once we fix d, it's got the cohomology of a finite dimensional variety. We could hope that V_d^{prim} looks like $H^*(\mathcal{M}_d^{st}$. The second point makes precise the statement $\mathcal{M} = Sym(\mathcal{M}^{st})$ at the level of stacks, $M = \mathbb{L}_d M_d \sim Sym(\mathcal{M}^{st}/\mathbb{C}^\times)$, this is actually correct once we take cohomology. The cohomology on the left is the Hall algebra. On the right this is $V^{prim} \otimes \mathbb{Q}[u]$.

This is how we think of the theorem in the first heuristic.

Definition 6.2.
1. V^{prim} is the cohomological Donaldson–Thomas invariant
2. $\Omega_{d,k} = \dim V^{prim}_{d,k}$ is the motivic Donaldson–Thomas invariant.
3. $\chi(V_d^{prim})$ is the usual Donaldson–Thomas invariant

Unfortunately the first thing $V_d^{Prim} \not\sim H^*(\mathcal{M}_d^{st})$ is not quite true.

Theorem 6.2. *(Chen) $V_d^{prim} \cong PH^*(\mathcal{M}_d^{st})$, the pure parts in the mixed Hodge structure.*

I want to give you two examples that show that things are computable. You have no chance for computing the Hodge structure directly, but using the Hall algebra you can. So for my loop quiver, $\mathcal{H}_Q = \bigoplus H^*_{GL_d}(gl_d^{evm} = \bigoplus H^*(BGL_d) = \bigoplus \mathbb{Q}[x_1, \ldots x_d]^{S_d}$. Then linearly this doesn't know about the number of loops but the product does. $f_1 f_2 = \sum_{\pi \in Sh_{d,d'}} \pi(f_1 f_2 \prod_{i=1}^{d} \prod_{j=1}^{d'} (x_i' x_j'')^{m-1})$.

Let's start with $m = 0$. Then you can calculate directly that the Hall algebra is $\bigwedge^*[x^0, x^1, \ldots]$, which come from the $d = 1$ term. These are the vector space generators, and x^i are all independent algebraic generators. This si the same as $\mathbb{Q}_{odd} \otimes \mathbb{Q}[u]$. If we have a one-dimensional vector space, it's stable, so $\mathcal{M}_d^{st}$ is a point for $d = 1$, and empty for $d > 1$. So this has cohomology $\mathbb{Q}$ in degree 1, and that's the same $\mathbb{Q}$. The algebra is still interesting, it's really a point mod $\mathbb{C}^*$.

If I take $m = 1$ I get $\mathbb{Q}[x^0, x^1, \ldots]$, which is $\mathbb{Q}_{even} \otimes \mathbb{Q}[u]$. What's the geometric point of view? $\mathcal{M}_d^{st}$, a one-dimensional vector space is a vector space with an endomorphism, so I have $\mathbb{C}$ in degree 1 and again it's empty in $d \geq 2$.

For $m = 2$ you have infinitely many d for which you have nonempty stable moduli spaces.

After the break I'll explain how to modify this in the orientifold setting.

In the second half I'll explain my work. It will be shorter now that we have the right framework. I want to make one comment I forgot. The second part of Efimov's theorem, one corollary is the following, V^{prim} is determined by $\Omega_{d,k}$, let's go down one level and compute the Poincaré series, $A_Q = \sum_{d,k} \dim H_{d,k}(-q^{\frac{1}{2}})^k t^d$. We know as a vector space this is symmetric polynomials, we can get an expression in that way. We also know it's a supercommutative algebra, and it's also easy to compute the series. So A_Q is basically known, we can write it in terms of something so it only depends on $\Omega_{d,k}$. So $\Omega_d = \sum \Omega_{d,k} q^{\frac{k}{2}}$ a priori is rational in $q^{\frac{1}{2}}$ and this theorem shows it's actually polynomial in q.

Now I want to move to the orientifold setting. We'll put an involution on the category and identify the double. So I have (U, u) and the involution will go to $(U^v, \pm u^v)$. I take θ to be $\pm ev$. I can choose all the signs independently (even for individual matrices in u). So this was $gl_d^{\oplus m}/GL_d$ so we can get to $sp_d^{\oplus m}/SP_d$, if we do $-$ and $-$. If we do $+$ and $-$ then we get $(\bigwedge^2 \mathcal{C}^d)^{\oplus m}/Sp_d$, and if we change the second sign we get orthogonal rather than symplectic versions.

In the orientifold setting, only the Kontsevich–Soibelman approach will work. You can never do the naive thing with virtual fundamental classes because you always have strictly semistable things.

So e will always be a dimension vector for a self-dual guy, so you have $M_e^\sigma \to \mathcal{M}_e^\sigma \to \mathcal{M}_e^{\sigma,st}$ which is no longer smooth, but is now an orbifold.

Now let's try to play the same heuristic game, decompose the moduli space in the middle in terms of the stable guys again. Let's look at some basic facts about the GIT quotient. This will again be closed orbits of these groups acting on the space.

(1) A self-dual representation N is σ-stable if it has no nontrivial isotropic subrepresentations. Isotropic subrepresentations are subrepresentations whose underlying space is isotropic. The other moduli space $\mathcal{M}_e^{\sigma,st}$ is good, it parameterizes stable objects.
(2) Every self dual representation has an isotropic filtration $0 = U_0 \subset \cdots \subset U_\ell \subset N$ so that subquotients U_i/U_{i-1} is stable and the final guy $N//U_\ell = U_\ell^\perp/U_\ell$ is either 0 or σ-stable. This is the analogue of the Jordan–Hölder filtration for groups other than GL_n.

    Also, N and $\bigoplus H(U_i/U_{i-1}) \oplus N//U_\ell$ are identified in $\mathcal{M}_e^\sigma$, where $H(V) = V \oplus S(V)$. So again, there's a huge amount of collapsing going on here.

Then as sets, we can modify our prediction, $\mathcal{M}^\sigma$, let's look at its stratification, we have a doubled bunch of usual stable moduli spaces, and then get one extra factor which is σ-stable. So this will be $H(SymM^{st}/\mathbb{Z}_2) \times \mathcal{M}^{\sigma,st}$, where we factor out $\mathbb{Z}/2$ because the symmetric thing of V and SV are the same.

So a problem is that, we don't have an Abelian structure on self-dual objects, because they'd be automatically split. Then the key idea, we can look at the short exact sequences $0 \to U \xrightarrow{\text{isotropic}} N \to P \to 0$ where $P \cong N//U$. We'll never get an algebra, but we get an ordinary guy, a self-dual guy, smash them together and get a self-dual guy. So we look at stacks $M^\sigma_{d,e}$ where we get projections $M_d \times M^\sigma_e$ and to M^σ_{2d+e}.

Now the theorem is kind of obvious

Theorem 6.3. *(Y.) $\mathcal{M}_Q = \bigoplus H^*(M^\sigma_e)$ is an $\mathcal{H}_Q$-module. It's a $\Lambda^{\sigma+}_Q \times \mathbb{Z}$-graded supermodule*

We know from the Hall algebra setting, we wanted to study the generators for the algebra, now we want the generators for the module, which should tell us about the geometry of the stable moduli space. If we look at the short exact sequence, the action will never be stable, we always have an isotropic subobject. We only want things that can't be obtained from this.

Theorem 6.4. *(Y.)(Integrality for orientifolds) Let $W^{prim} = \oplus W^{prim}_{e,\ell}$ be a space of minimal generators as a module. Then* dim W^{prim}_e *is finite dimensional.*

Just proceeding by analogy with Efimov's theorem, the guess or hope is that this is the pure part of the cohomology of the stable space. Unfortunately I don't know how to prove this in general.

Theorem 6.5. *There is a canonical surjection $V^{prim}_d \to PH^*(\mathcal{M}^{\sigma,st}_e)$.*

If this were an isomorphism, there would be a strong similarity to the other case.

This is an orbifold, so there's more difficulty, but you can do surjectivity. Injectivity in the usual case uses Nakajima quiver varieties that just don't exist for general linear groups. The conjecture is that this guy is an isomorphism.

Let's look at an example. There's a combinatorial version of this module. Let's look at $m = 0$, and look at the point modulo the symplectic group. In this case, the module $\mathcal{M}_Q$ is generated by, well, first let me say, as a vector space $\mathcal{M}_Q = \bigoplus H_{Sp_e}() = \bigoplus H^*(BSp_e)$ which is symmetric polynomials in the squares of the variables $\mathbb{Q}[z_1^2, \ldots z_e^2]^{S_e}$. The invariants should only have squares because the Weyl group changes the sign. Then the module is generated by a single element, the constant function of zero variables, and is free over the subalgebra $\mathcal{H}^{odd}_Q$ (this was an infinitely generated Clifford algebra).

Again, what's the geometry? $\mathcal{M}^{\sigma st}_e$ is empty if the dimension is at least 2 and is a point if the dimension is 0. You need a point to generate anything. There's no higher stable moduli spaces, and that corresponds to the module being generated by this single guy in dimension 0.

Now what about the case when $m = 1$? Take the representations that are symmetric squares of $\mathbb{C}^d/O_d$. The module is generated by $1_0, 1_1, \ldots$, the constant function in different degrees. It's infinitely generated, and is free over the subalgebra $\mathcal{H}^{odd}_Q = \mathbb{Q}[x^1, x^3, \ldots]$. What's happening here? The stable moduli space $\mathcal{M}^{\sigma,st}_e$ is an orthogonal space with a symmetric map. In dimension 1, we get a copy of $\mathbb{C}$. In higher dimensions, the stable representations are diagonal matrices with no common eigenvalues, which is $Sym^e\mathbb{C}\backslash\Delta^{big}$. These are relatively complicated. Can we compute the pure part of the cohomology of this guy? It's $\mathbb{Q}$ in degree 0. There's only the one pure part. We get a one dimensional generator in each degree. So you'll never get the whole cohomology, but it sees all of, uh.

I have ten more minutes? These examples already show that the freeness part of Efimov's theorem is more complicated. Briefly, what's the analogue of freeness? This really differentiates the normal from the orientifold setting. It's not enough to go to the motivic one, we need the full cohomological one. The functor S defines an involution of our algebra $\mathcal{H}_Q$. At the level of stacks it's dualizing, and then you induce that map on cohomology. This descends to an involution of the primitive space V^{prim}. This is not entirely trivial, because this is not a canonically defined space.

Proposition 6.1. *If $f \in \mathcal{H}_d$ and $g \in \mathcal{M}_e$, then $f \star g = (-1)^{\epsilon(e,d)} S(f) \star g$. Here ϵ we have an explicit formula for.*

So $f - (-1)^{\epsilon(e,d) S(f)}$ annihilates g. This gives us something we need to kill. For example, if $f = S(f)$, then this gives us some sign condition about whether f can act nontrivially. Let $V(e)$ be the subspace of $V^{prim} \otimes Q[u]$ spanned by $f - (-1)^{\epsilon(e,d)} S(f)$ for $f \in V^{prim} \otimes Q[u]$. Now let $\mathcal{H}_Q(e)$ be the free supercommutative algebra on $V^{prim} \otimes \mathbb{Q}[u]/V(e)$.

Conjecture 6.1.
1. Let $g \in W_e^{prim}$. Then $H_q \star g$ is free of rank one over $\mathcal{H}_Q(e)$.
2. $\mathcal{M}_Q = \bigoplus_g \mathcal{H}_Q \star g$ where g ranges over a basis of W_e^{prim}.

This again is telling you, how did we define this module? We really needed to know V^{prim} with its module structure over $\mathbb{Z}_2$.

To finish,

Theorem 6.6. *The conjectures hold for the cases L_0, the zero loop quiver, L_1, the quiver with two arrows, and for Dynkin quivers. Beyond that, you can check for L_2 in low degrees.*

As a last comment, we ended the last lecture arguing that you could look at the Poincaré series, but the conjecture tells you that you can't hope to do this in the orientifold case. You really need to take the Hall algebra seriously, there seems to be no direct way around this problem.

7. Jongil Park: On symplectic fillings of quotient surface singularities

I'm not sure how far I can cover in my talk. Last time, as I told you, Lisca classified symplectic and Stein fillings up to diffeomorphisms (orientation preserving). As I told you, he conjectured a one-to-one correspondence between Milnor fibers and minimal symplectic fillings. In the first hour of this talk, I'll cover how he proved, classified, and review how [unintelligible]classified fillings for the non-[unintelligible]case. At the end I'll review how we can show how these are Milnor fibers. In the second half I'll review Milnor fibers and partial resolutions and so on.

For Lisca's result, he parameterized the fillings by, well, let $L(n,a)$ be cyclic quotient singularities of type $\frac{1}{n}(1,a)$. So n and a are relatively prime so $\frac{n}{a} = [b_1, \dots, b_r]$, and the counterpart is $\frac{n}{n-a} = [a_1, \dots a_e]$, the continued fraction expansion. The two numbers are closely related. For example, $b_1, \dots, b_r, 1, a_e, \dots a_1] = 0$.

[some discussion]

If we fix n and a then $K_e(\frac{n}{n-a})$ is the e-tuple of integers where $[n_1, \dots n_e]$ is zero and $0 < n_i \le a_i$ where a_i is the continued fraction of $\frac{n}{n-a}$. This parameterizes the minimal symplectic fillings. I review the construction because when you consider the non-[unintelligible]case, it's the same idea.

Then he constructed $W_{n,a}$ which is a symplectic filling of $L(n,a)$. For each $\mathbf{n} = (n_1, \ldots, n_e)$ in $K_e(\frac{n}{n-a})$, well, how can you construct this one. You blow up at p consecutively [pictures].

In summary we have a rational surface Z which is $\mathbb{CP}^2 \# N\bar{\mathbb{C}}\mathbb{P}^2$ and inside we have this configuration (picture).

Theorem 7.1. *(Lisca) Suppose* (W,ω) *is a minimal symplectic filling of* $(L(n,a),\xi_L)$*. Then there is an* $M > 0$ *such that* $W \cong \mathbb{CP}^2 \# M\bar{\mathbb{C}}\mathbb{P}^2 \backslash D\Gamma$ *where* $D\Gamma = C_1 \cup \cdots \cup C_e$ *of type* $(1, 1-a_1, -a_2, \ldots, -a_e)$.

That's the first part (it's not difficult). The key part is how to match the exact, what kind of $W_{n,a}$*?*

[too many pictures]