CGP WORKSHOP ON HOMOTOPICAL METHODS IN QUANTUM FIELD THEORY

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1. January 11: Alberto S. Cattaneo: Perturbative BV–BFV theories on manifolds with boundary

[I do not take notes at slide talks]

2. Pavel Mnev: Cellular BV–BFV–BF theory

[Missed the beginning]

$$S = \int_M \langle B\wedge, d_E A\rangle$$

for $A \in \Omega^1(M,E)$ and $B \in \Omega^{n-2}(M,E^*)$

So I want to talk about a cellular model. We heard in the last talk, we can put ghosts and antifields and so on, and then I should just move to

$$A \in \Omega^*(M,E)[1], B \in \Omega^*(M,E^*)[n-2]$$

Now I will work on a cobordism M with a cellular decomposition X.

The space of fields is $C^*(X,E)[1] \oplus C^*(X^\vee, E^*)[n-2]$, so E is an $O(m)$-local system, and the structure group can be relaxed to $SL_\pm(m)$, this is the most relaxed version of coefficients for this model. So cochains, essentially, the data of the local system is encoded in the differential on the space.

So fields are elements of this space. So

$$A = \sum_{e\in X} A_e e^*$$

where e^* is the index cochain, and $A_e \in \mathbb{R}^m$, which has an internal degree (ghost number) and

$$B = \sum_{e^\vee \in X^\vee} B_{e^\vee} (e^\vee)^*$$

So the ghost number of A_e is $|A_e| = 1 - \dim e$ and the ghost number of $B_{e^\vee}$ is $|B_{e^\vee}| = n - 2 - \dim e^\vee$.

The cell decomposition defines a ball complex, so the dual cell decomposition, for every, this top picture corresponds to a closed circle. For every cell I have a dual cell in complementary dimension. They have intersection number +1 if you orient correctly.

For a manifold with boundary, there are some choices to make, for example [pictures], so this decomposition $X^{\vee +}$ has too many cells, and this one $X^{\vee -}$ has too few. In $X^{\vee +}$ a cell e in the boundary has a dual cell $\kappa(e)$ as before and alos one of degree one lower $\kappa_\partial(e)$. To define the dual for a cobordism we combine the two constructions. We use the + construction at the in boundary and the − construction at the out boundary.

One can do a pedantic definition but I just want to do some pictures. [pictures]

Then the pairing of chains $C_*(X, X_{\mathrm{out}}) \otimes C_{n-*}(X^\vee, (X^\vee_{\mathrm{in}})) \to \mathbb{Z}$ is non-degenerate. Then at this level you have nondegenerate Poincaré–Lefschetz duality.

I spent some minutes talking about this because the correctness depends on the definitions of the duals.

Then we have $\omega = \langle \delta B, \delta A \rangle$. In terms of components, this is

$$\sum_{e \in X - X_{\mathrm{out}}} \langle \delta B_{\kappa(e)}, \delta A_e \rangle$$

So this will be degenerate because some of the boundary terms won't participate. This has ghost number -1 and looks like a BV two-form but is degenerate.

The action

$$S = \langle B, dA \rangle_X - \langle B, A \rangle_{X_{\mathrm{in}}}$$

where I'll comment on the second term later.

Then the other part of the structure is the cohomlogical vector field (I'm suppressing e throughout)

$$Q = \langle dA, \frac{\delta}{\delta A} \rangle + \langle dB, \frac{\delta}{\delta B} \rangle$$

lifting $d_X + d_{X^\vee}$ to $\mathcal{F}$.

Then

$$\mathcal{F}_\partial = C^*(X_\partial)[1] \oplus C^*(X^\vee_\partial)[n-2]$$

and this carries the one-form

$$\alpha_\partial = \langle B, \delta A \rangle_{\mathrm{out}} - \langle \delta B, A \rangle_{\mathrm{in}}.$$

There is some choice here, related to adding the boundary term before to the action, outherwise I'd have to write $\langle B, \delta A \rangle_{\mathrm{in}}$. Then this satisfies

$$\omega_\partial = \delta \alpha_\partial = \langle \delta B, \delta A \rangle_\partial$$

and this α_∂ is compatible with the polarization $\mathcal{P} = \mathrm{Span}(\frac{\partial}{\partial B_{\mathrm{out}}}, \frac{\partial}{\partial A_{\mathrm{in}}})$, so this is the vertical distribution of $\mathcal{F}_\partial \xrightarrow{p} \mathcal{B} = C^*(X_{\mathrm{out}})[1] \oplus C^*(X_{\mathrm{in}})[n-2]$, and I'd call an element here $(A_{\mathrm{out}}, B_{\mathrm{in}})$ or $(\mathbb{A}, \mathbb{B})$ to keep with the notation of the previous talk. This is the base of the Lagrangian fibration on our phase space. This $\mathbb{F}_\partial$ takes the restriction map π to the boundary from $\mathbb{F}$.

If we're given an element b in $\mathcal{B}$ it defines a Lagrangian subspace $p^{-1}(\{b\})$ and we have in $\mathcal{F}$ the fields with a boundary condition $\mathcal{F}_b = \pi^{-1} p^{-1}(\{b\})$. Then restricting here we get a nondegenerate 2-form which is what we need.

All of these for different b are affine translates of each other, so calling their isomorphism class Y, we get $\mathcal{F} = B \oplus Y$ and ω lives on Y via restriction to b.

Now we have the main equation

$$\iota_Q \omega = \delta S + \pi^* \alpha_\partial$$

and this is easy to check in the cellular case and follows from the "cellular Stokes theorem"

which says that for cochains a and b on X and its dual, that

$$\langle db, a \rangle_X \pm \langle b, da \rangle_X = \langle b, a \rangle_{X_{\mathrm{out}}} - \langle b, a \rangle_{X_{\mathrm{in}}}$$

One can calculate the Poisson bracket of S with itself. I can invert ω in the fiber fixing a boundary condition and get

$$\frac{1}{2}(S, S)_{\omega_b} = \pi^* S_\partial$$

where S_∂ is the action on the boundary phase space $\langle B, dA\rangle_\partial$ (there's also the definition for Q on the boundary which is exactly the same formula as in the bulk but for cochains of the boundary).

Now let us go on to the quantization.

For M *closed* and X we have $\Delta(E^{\frac{i}{\hbar}S}\mu^{\frac{1}{2}}) = 0$ on $\mathcal{F}$. I have $\mu^{\frac{1}{2}} \in \mathrm{Dens}^{\frac{1}{2}\mathrm{const}}(\mathcal{F})$ defined as

$$\prod_{e\in X}(DA_e)^{\frac{1}{2}}(DB_{e^\vee})^{\frac{1}{2}}$$

Half densities can be identified with

$$\mathrm{Det}\, C^*(X)/\pm 1.$$

I'd like to evaluate, what kind of integral I'll calculate, pushforwards to zero modes, the zero modes or residual fields are, in this case

$$\mathcal{F}_{\mathrm{res}} = \frac{\text{zero } Q}{Q}$$

and in this case that's easy, it's $H^*(X)[1] \oplus H(X^\vee)[n-2]$ and since homology is independent, this is $H^*(M)[1] \oplus H^*(M)[n-2]$.

I want to split this space $\mathcal{F}$ as $\mathcal{F}_{\mathrm{res}} \oplus Y''$ and choose a Lagrangian from a Hodge decomposition $\mathcal{L}$ in Y'' and do my integral here.

So we define

$$Z = \int_{\mathcal{L}} e^{\frac{i}{\hbar}S}\mu^{\frac{1}{2}}$$

and this is in $\mathbb{C}\otimes \mathrm{Dens}^{\frac{1}{2}}\,\mathcal{F}_{\mathrm{res}}$.

Note that $\mathrm{Dens}^{\frac{1}{2}}\,\mathcal{F} = \mathrm{Dens}^{\frac{1}{2}}\,\mathcal{F}_{\mathrm{res}} \otimes \mathrm{Dens}^{\frac{1}{2}}\,Y''$, and you can restrict from half densities on Y'' to full densities on Lagrangians, that's how it works in this kind of geometry, so you restrict to $\mathrm{Dens}^{\frac{1}{2}}\,\mathcal{F}_{\mathrm{res}} \otimes \mathrm{Dens}^1\,\mathcal{L}$ and then integrate out to $\mathrm{Dens}^{\frac{1}{2}}\,\mathcal{F}_{\mathrm{res}}$.

Now we need to fix homological perturbation data, so a choice of representatives $i : H^*(X) \to C^*(X)$, a projection $C^*(X) \to H^*(X)$, and a homotopy K on $C^*(X)$ between the identity on $C^*(X)$ and ip, and so you hae $C^*(X) = i(H^*) \oplus \ker p$ and inside that last factor is $\mathrm{im} K = \mathcal{L}$. This is an analogue of Lorentz gauge in physics. When K is $d^*\Delta^{-1}$ this is actually Lorentz fixing.

Then $Z = \tau(X,E)\zeta$, where $\tau(X,E)$ is the Reidemeister torsion, an element in $\mathrm{Det}\, H^*/\pm 1$, and then we have a transport $\mathrm{Det}\, C^*(X) \to \mathrm{Det}\, H^*(X)$, transport this volume element (up to sign to avoid orientations), and this is a well-known invariant of a space and a local system. But we have this annoying complex number ζ from i and $\hbar$ and stuff like that. The number ζ depends on X, while τ only depends on the manifold. You can write $\zeta = \frac{\xi_H}{\xi_C}$, where ξ_H depends only on Betti numbers of the manifold, while ξ_C is *extensive*, has to do with the cells of X. I want to do a very baby version of renormalization, I'm not using the naive cellular density for my partition function, rescaling $\mu^{\frac{1}{2}}$ locally by a number depending only on the dimension of the cell, and then the rescaled one will be topological.

This topological part is curious, it contains a phase. Let me call it Z_{res}, then

$$Z_{\mathrm{res}} = \tau(X,E)\xi_H$$

and the phase is of the form $e^{\frac{2\pi i}{16}q}$ and q is a linear combination of Betti numbers that is not the Euler characteristic. So how do you get a phase modulo 16 where you expect only integer products, I don't know, square roots of i out of such an integral.

It comes from the splitting into the topological and extensive part. The topological part has a strange phase after you do that. The phase looks like the phase of the Chern–Simons partition function, a little, and the mechanism is similar, it's also an eta invariant (of a much simpler Dirac operator). This never appeared in old [unintelligible]theory because people never considered this kind of local system.

One point I should make is that if I have two cell decompositions of M such that X is a subdivision of X', then I get an aggregation, and I can do a BV pushforward and

$$P_*(e^{\frac{i}{\hbar}S_X}\mu_X^{\frac{1}{2}}) = e^{\frac{i}{\hbar}S_{X'}}\mu_{X'}^{\frac{1}{2}}$$

So this is like a baby version of renormalization, jumping to subdivisions with sparser cells.

Let me move now to the case with boundary, then my states $\mathcal{H}_\partial$ is $\mathrm{Dens}^{\frac{1}{2}}\mathcal{B}$, and an element here is like $\psi(\mathbb{A},\mathbb{B})$, where recall these are the restrictions of A and B to out and in boundary, respectively. So then this is like

$$\mathcal{H}_\partial = \mathcal{H}_{X_{\mathrm{in}}}^{(B)} \otimes \mathcal{H}_{X_{\mathrm{out}}}^{(A)}$$

and then Ω is, let me remind you of the phase space $\mathcal{F}_\partial$ with its Lagrangian fibration over $\mathcal{B}$, and this is projectible (true only for very simple theories) so $Q_B = p^*Q_\partial$, and then

$$\Omega = i\hbar Q_B = -i\hbar\langle d\mathbb{A}, \frac{\partial}{\partial\mathbb{A}}\rangle_{\mathrm{out}} + \langle d\mathbb{B}, \frac{\partial}{\partial\mathbb{B}}\rangle_{\mathrm{in}}$$

and $\mathcal{F}_{\mathrm{res}}$ is

$$H^*(X, X_{\mathrm{out}})[1] \oplus H^*(X^\vee, X_{\mathrm{in}}^\vee)[n-2]$$

and I choose homological perturbation data for $C^*(X, X_{\mathrm{out}})$ and $H^*(X, X_{\mathrm{out}})$. Now I get a well-defined extension (by zero) of $\mathbb{A}$ to the bulk and likewise for $\mathbb{B}$. Then $A = \tilde{\mathbb{A}} + a + \alpha$ and $B = \tilde{\mathbb{B}} + b + \beta$, introducing fluctuations, with the (residual) zero modes a and b, and we do some integrals and let me tell you the result, repeated in some parts from the last talk.

Then

$$Z(\mathbb{A},\mathbb{B},a,b) = \int_{\mathcal{L}\subset\mathbb{F}_{\mathrm{fluct}}} e^{\frac{i}{\hbar}S}\mu^{\frac{1}{2}} \in \mathbb{C}\otimes\mathrm{Dens}^{\frac{1}{2}B\otimes\mathrm{Dens}^{\frac{1}{2}}\mathcal{F}_{\mathrm{res}}}$$

so the first two terms are the space of fields, and the answer will be

$$\xi^{H(M,M_{\mathrm{out}})} \quad \underbrace{\tau(M, M_{\mathrm{out}})}_{\in\mathrm{Det}\,H^*(M,M_{\mathrm{out}})/\pm1} \quad \underbrace{e^{\frac{i}{\hbar}\left(\langle\mathbb{B},a|_{\mathrm{in}}\rangle_{X_\epsilon}+\langle b|_{\mathrm{out}},\mathbb{A}\rangle_{X_{\mathrm{out}}}-\langle\varphi^\vee\mathbb{B},K\varphi A\rangle_X\right)}\mu^{\frac{1}{2}\mathcal{B}}}_{\text{[unintelligible]part of the effective action}}$$

So I have three terms, the parts where a talks to the in boundary, where b talks to the out boundary, and where the boundaries talk to each other.

The properties of this answer are the following. First of all, it satisfies this quantum master equation, and this is very easy to derive, deriving this is an immediate consequence of the fact that the exponential of the action times the half densitysatisfies

$$(\Delta + \Omega)e^{\frac{i}{\hbar}S}\mu^{\frac{1}{2}} = 0$$

(I've omitted properties of $\hbar$) but this implies that

$$\Delta_{\mathrm{res}} + \Omega)Z = 0$$

and if I change gauge fixing (i,p,K), then this changes by an exact term, Z goes to $Z + (\Delta_{\mathrm{res}} + \Omega)(\text{something})$.

I can only make the statement if the decomposition doesn't change at the boundary. So Z is invariant with respect to aggregation of cells relative to the boundary. If I just change the cell decomposition relative to the boundary, I go to the cohomology of Ω, the smallest model of this space of states, going from $(\mathcal{H}, \Omega)$ to H^*_Ω, and this is $\mathcal{H}^{\min}$ and then there's a reformulation completely independent of the cell decomposition.

So I should say that the gluing, [pictures], if I have two cobordisms which I glue, M_I to M_{II}, with interface Σ_2 and other boundaries Σ_1 and Σ_3. I have Z_I and Z_{II}, and Z_I depends on $\mathbb{A}_2$ and $\mathbb{B}_1$ and M_{II} depends on $\mathbb{B}_2$ and $\mathbb{A}_3$. Also some zero modes. The first step is to compute $Z_I * Z_{II}$, by which I mean integrating over $\mathbb{A}_2$ and $\mathbb{B}_2$, inserting something to change polarizations. But then I don't obtain the final result, I depend on zero modes, on too big a space of zero modes, zero modes for two different choices of component for a and b. So I need a smaller space, and that's also done by a BV pushforward P_*. So for instance I have $H(M_I, \Sigma_2) \oplus H_{M_{II}, \Sigma_3}$, which is bigger than or equal to $H(M, \Sigma_3)$, but you can push forward here. You can also do the same thing for the b fields.

Okay, so our gluing procedure is going in two steps, pairing the states in the gluing interface and then choosing the residual model we prefer.

I chose the simplest possible model, and there's a non-Abelian version that is more interesting but that I couldn't start until I got here. I can only define this for prismatic complexes, where cells are products of simplices.

You can construct such a cellular theory, which will also satisfy the quantum master equation, you can construct such an action and it will satisfy the quantum master equation, and you can play the game of pushforwards with more interesting Feynman diagrams, tree and one-loop Feynman diagrams, and your ultimate partition function has information on the torsion, but explores the formal neighborhood of a flat connection, and looks at how singular the point is in the moduli space and what happens to the torsion, this is the tree part. This is somehow the rational homotopy type of the manifold, this is the squeezed version of what this gives you.

3. Andrey Losev: Feynman geometry

Before I start to talk, I'd like to tell, I prefer private talk from person to person, so for general talks I give general ideas, this is an appetizer for what people can discuss with me in private. I'd like to talk not only about Feynman geometry but also about how and why A_∞ algebras are the proper geometry in twenty-first century physics. The reason for this will come from two places. The first place is of course Feynman geometry. The second reason is a kind of unification of space-time and fields.

The second actually comes from, there is an idea that space-time is doomed, and if so then we need to do something else, and that's the second part.

First I need to explain what Feynman geometry is and why I call something Feynman geometry. Actually I'd like to give a brief review of a situation in quantum field theory. There are actually two approaches. I call the first the Dirac–Segal approach, to consider cobordisms colored by geometrical data (possibly empty) and you associate to this thing by some operation I some element $V_1 \otimes \cdots \otimes V_{\#\partial X}$ of linear algebra and the main axiom is of course that $I(X_1 \cup X_2) = I(X_1) \circ_V I(X_2)$ where the contraction is along the cutting boundary.

This is simple notation but the people who know it understand what I mean and the people who don't can study it if they are interested.

The geometric data is an interesting part, $I(X,g) = I(X_1,g_1) \circ I(X_2,g_2)$. The simplest example was found before 1930 by Dirac. For him X was an interval, g was the length of the interval, and it was basically the condition $I(t_1 + t_2) = I(t_1) + I(t_2)$ of a semigroup, and the universal solution is $I(t) = e^{tH}$ and this is quantum mechanics. Let me stress, there is no $\hbar$, this is *internally* quantum theory.

My point of view is that $\hbar$ is a distance to classical theory. So I prefer to think not about quantization of classical theory but classicization of quantum theory.

This is not a piece, a main thing, of my talk. I'll return to this a bit later.

I should put another approach to QFT, which is Feynman. This approach has $\hbar$. We have classical physics on this side.

So $\int \mathcal{D}\varphi e^{\frac{1}{\hbar}S(\varphi)}$ is the symbolic form in which we want to write a theory. This is not a glorious way to write down a theory. This is not a mathematical construction. This is an "idea," not a definition. To make a definition, you need to explain how to deal with infinite dimensional integrals.

I'm from Russia and in Russia we have two things, a constitution of the country, like the "idea" and then we have the actual practice of what is going on, called [unintelligible]. You need additional tools, and a tool we have here is called renomalization. Like in Russia, this is very geometric, you can see symmetries and everything here, and like in Russia, the [unintelligible]and the renormalization are quite ugly.

I'm a mathematician and I cannot tolerate this situation. Let's try to describe this situation in some kind of geometry, I'd like to put this under the same mathematical law without tricks, guesses, and some other things.

The hint, in 2002, Kontsevich visited Russia, and he told people they were doing old outdated algebraic geometry. He explained that geometry has to go this way. First you have commutative geometry then differential operators, then noncommutative geometry, then physical [unintelligible]. People went this way for target space geometry.

But quantum field theory is about maps from source to target, so let me ask a question: could it be that the world sheet geometry is not the outdated geometry of the nineteenth century but the modern geometry of the twenty-first.

Suppose I'm a religious person and I think God created the laws of the universe. Maybe we can guess them, maybe not, but probably he wouldn't create them in terms of outdated geometry. So the theory of the universe should not be based on the geometry of the 19th century.

So how could the idea be implemented? Let me make a first observation, that all action of classical field theory could be written in terms of the de Rham differential graded algebra.

The language of classical physics is the de Rham differential graded algebra. Let me give you several examples to convince you.

0 Example 0 is Chern–Simons theory, $\int \mathrm{Tr}(AdA + \frac{2}{3}A^3)$, you may say this is non-physical.

1 Gravity. We'll use Palitini, Einstein's theory is unphysical because it ignores fermions that are spinors. Then it's impossible to write things down in terms of Einstein gravity. I'm also a physicist. So Einstein's theory is a nice mathematical model, and this is equivalent to the Palitini action, something

like $\int \langle e \wedge \cdots \wedge (d\omega + \omega \wedge \omega)\rangle_\epsilon$. In terms of geometry, actual data is Spin(n)-bundles on X, V-associated vector bundles associated to an n-dimensional representation, a connection $\nabla = d + \omega$ on this bundle, a morphism e so that the metric g is the pullback of a metric on the bundle under e [missed some]. The action is again written in terms of the differential graded algebra.

2 Let's look at Yang–Mills, the action is $\int F * F$, and the star prevents me from writing this down, but to first order it's $\int pF + p * p$, and $*$ belongs to Riemann, Hodge, but not to differential forms.

If I understand $p = e \wedge \cdots \wedge e\tilde{p}$, then this action could be replaced by $e \wedge \cdots \wedge e\tilde{p}(dA + A^2) + \mathrm{Tr}(\tilde{p}\tilde{p}) \wedge e \wedge \cdots \wedge e$

So in every case I get a differential graded algebra.

For all these things I could add ghosts and I'd get solutions to the BV master equation.

Now I should explain what Feynman geometry is. The idea is to replace the de Rham differential graded algebra by an A_∞ algebra with operations belonging to trace class.

The operations, pointwise multiplication, don't belong to the trace class, and this is the origin of the ∞s. Propagators are not strong enough to make things trace class.

Yesterday I was on the excursion, and we went to Buddhist churches. Buddha is not giving you a solution, just a way. I'm inspired by the excursion, so I will not give you the solution, I'll give you the way. Take the replacement and then study how all these actions could go over to the A_∞ trace class *Feynman geometry* (so-called because you can do these integrals in this geometry).

So Chern–Simons theory could clearly be put in such geometry. It's interesting to study other theories. The picture I have in mind is the following. I have a space of Feynman geometries We want to go some particular place, to our classical geometry. We need to unfortunately quantize our geometry to see if there are theories over Feynman geometries that exist and consider the limit when Feynman geometry goes to classical geometry.

Talking about physics, by the way, saying the fact that our spacetime is continuous, exactly, is not something we could ever prove. We could only show it up to some length scale. Since we could not imagine anything else before, we said "it should not be something we cannot imagine, so it should be continuous." But this is nineteenth century logic. Thanks to mathematicians, we can still imagine more things. But even if this is still true to all scales, the place where this stuff lives is not over Feynman geometry but over classical geometry.

So there's a dichotomy between the Dirac–Segal quantum physics/classical geometry/no $\hbar$ picture, and the Feynman $\hbar$ and Feynman geometry.

Then when I explain this idea, let me give you some examples of Feynman geometries that we already know and study. We studied it, not calling them Feynman geometries, but as constructions to do some computations. That's what I'm going to review.

In the construction of Feynman geometry, there is an *anomaly* phenomenon, this means there are two things that you cannot preserve simultaneously. Here the two notions are commutativity and associativity. This means that when you try to go to finite dimensions, you cannot preserve both. This phenomenon was observed around the 60s in one of the first conferences where Russian and Western people got together. This was called the Kolmogorov problem. They tried to replace Ω^* by cochains. How to construct a supercommutative associative multiplication on cochains. People tried and found that it is impossible. So when you are passing the exam in topology, you are using the cup non-commutative associative comultiplication. How does it go? On a simplification, you have a set $(1,\dots k)$, and you comultiply $\sum(1,\dots k_1)\otimes(k_1,\dots,k)$. This was considered to be a technical tool to show that you have a comultiplication that leads to the multiplication on homology. This is associative but it's not commutative. Kolmogorov tried to conserve commutativity but not associativity. You can't have both, the way out, as we saw with Pasha, is to give up and replace associativity with an A_∞ structure.

Let me give other examples. Consider something like fuzzy space, this is a very clear thing, you consider dimension N representations of $SU(2)$, let me write it in terms of differential operators $\sum_{i,j=0}^{s} z^i \frac{\partial}{\partial z^j}$ acting on $P_{N-1}(z_0,z_1)$. I basically want to call this sum (renormalized by $\frac{1}{N}$) by the name Z^{ij}. Then $[Z,Z]=\frac{1}{N}Z$. So when $N\to\infty$, you get a commutative thing, and it's functions on the sphere.

This is finite-dimensional, round (equivariant with respect to $SU(2)$) commutative, but not associative.

Let me give one more example, which is published, more or less, by Costello. Consider the standard multiplication, it's multiplication on de Rham forms. Put on the output $e^{-\beta\Delta}$, the product we use in index theory. The product is nice. It's homotopically the product we wanted, and it's [unintelligible]. But it's not associative.

However, since this box is exact, it's not a problem to deform it to a homotopically associative product, which is the following product. Let me just write down m_3. This is [pictures].

It's possible to get a similar product as follows. Consider induction of A_∞ on the space of differential forms such that the eigenvalues of the Laplcian or less that E. This will have no associativity, A_∞ structure, but the space is finite dimensional, so it's another example of Feynman geometry.

Let me give you one more example. The last example that keeps me from being destroyed in the mathematical community for not knowing the main developments of the last fifty years, is that string theory in the Zweibach formulation is Feynman geometry. Basically you know this, this is Feynman geometry that goes to loops. As far as I understand, Pasha knows that A_∞ geometry on simplices go to loops. So it's an interesting question to study. But this formulation is exactly the Feynman geometry. It's the geometry on the target space. Moreover, the construction of Costello may be considered as the tropicalization of the Zweibach construction. It's some part of the space of graphs with lengths on the edges. For Zweibach it's surfaces.

I want you to have a new look at what we're doing in quantum field theory. The thing I've written down is the proper way to do cutoffs.. Simplicies is how to do lattices. We want a quantum field theory over Feynman geometry. The study of quantum field theory is like, you go to Feynman geometry and explain what you're doing in definite examples, and study universal relations, which actions could be put to Feynman geometry, and what are the conditions that obstruct this or that action.

That's what I'd like to tell you about Feynman geometry.

[some discussion]

Feynman geometry is dual to A_∞ algebra under the algebra-geometric correspondence in the same way that standard geometry is related to C^* associative and commutative algebra. What we are doing on the algebraic side is replacing this algebra with A_∞-algebra. Chern–Simons already can be done this way.

[some discussion]

4. Brian Williams: Chiral differential operators from curved beta-gamma

I'd like to thank the organizers for inviting me to speak. I'm going to start with some motivation. The motivation comes from a model we've talked a lot about last week, that's σ-models, concerned with studying maps $\phi : S \to Y$, from a source manifold S to a target manifold Y. The formalism I like to think about is the idea to describe Y the target by some Lie algebraic data and then form a field theory on the spacetime source S that looks formally similar to a gauge theory. This is all joint work with Owen Gwilliam and Vassily Gorbanov.

We'll talk more about that later, let me, there are a plentitude of examples that exploit this prescription in the literature, let me review some.

Some examples, Grady–Gwilliam and Grady–Si Li–Qin Li study the example where S is the circle and Y is a symplectic manifold. We heard about this in Qin's talk and this is called "topological quantum mechanics." The next example, studied by Li–Li and by Rozynblyum, S is a Riemann surface of genus g and Y is the cotangent bundle of a complex manifold, and they formulate this as a model of the "topological B-model." The last example, studied by Kevin Costello, S is taken to be an elliptic curve E and Y is the cotangent bundle of a complex manifold S, and we study holomorphic maps form E to the cotangent bundle.

For this talk, study a variant of Costello where $S = \mathbb{C}$, so local in spacetime, and look at maps $\mathbb{C} \to T^*X$ that are near the constant (zero) section. Another way to say this is that we're studying a cotangent theory for holomorphic maps $\mathbb{C} \to X$, perturbing around degree 0 holomorphic maps.

The output of BV quantization for our setting is a factorization algebra, which I'll try to say a little bit about later, and our theory that we're going to consider, and this is called *curved* $\beta\gamma$ in the physics literature (called *bc* in the supersymmetric version), and this is what is called a *holomorphic* theory, and rather than definining this, the output is a *holomorphic factorization algebra on* $\mathbb{C}$ and this is basically a vertex algebra. We'll identify this vertex algebra as a familiar object in algebraic geometry.

The main result is that quantization of curve $\beta\gamma$ is the sheaf of chiral differential operators, some sheaf CDO_X of vertex algebras on X.

Okay great, so let me start witha brief review of quantization the way that I'm thinking about it and the way that we'll use in this talk. Everything we do is in the Lagrangian formalization of classical field theory, so we start with a space of fields, an action functional, this describes a classical moduli problem. We could ask to quantize this data to give (perturbative formal) quantum field theories, and by the way everything we do is in the formalism of an effective approach in terms of heat kernel methods developed by Costello.

The outputs of these quantum field theories, there are two main things you can do, there's a global story (global on the source) and a local story. The global story you can study things like the partition function, related to the path integral, and the local story produces factorization algebras on spacetime (which is $\mathbb{C}$). We'd like to compare these to well-known objects from physics and algebraic geometry as a secondary step.

For curved $\beta\gamma$, a result of Costello relates the partition function for curved $\beta\gamma$ to something called the Witten genus of the target manifold. I think Dan will talk a bit more about the Witten genus in his talk. We're concerned with the local story anyway.

Okay, so let me make some remarks, so quantization may not always exist. This is exactly the failure to satisfy the quantum master equation. Another remark is that there may be many quantizations, we'd like to consider deformations of our theory. Luckily for us, there's a complex that I'll call the deformation complex and write Def, that contains all of this information. So $H^1(\mathrm{Def})$ is where these obstructions (anomalies) live, and $H^0(\mathrm{Def})$ is where the deformations live, and $H^{-1}(\mathrm{Def})$ is where automorphisms live.

So formal moduli problems are completely described by Lie algebras, Koszul duality if you want to say it in a fancy way. There's a totally Lie algebraic version putting all of this together.

Classical field theories are equivalent to the data of a local L_∞ algebra with the extra data of a type of pairing. Local means it's on spacetime S.

So if $\mathcal{L}$ is a classical field theory, there's a nice description of this deformation complex as the Chevalley–Eilenberg cohains on $\mathcal{L}$, but a special type called local cochains, Qin called these local functionals.

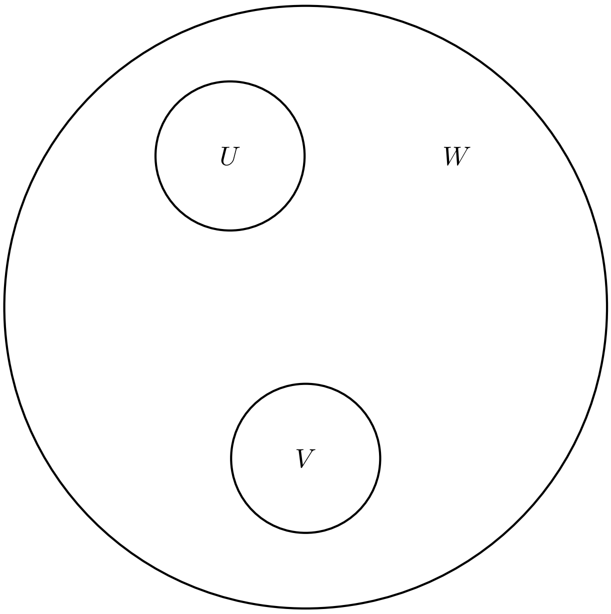
The condition is twofold. Local functionals means Lagrangian densities, so built from jets of fields, so partial derivatives of fields, and you can thus think of these as sitting inside all functionals.

Let me say something about observables. Morally speaking, they are taking measurements of our theory. We do this as functionals on our fields, which in this Lie algebra setting is cochains $C^*(\mathcal{L})$ on $\mathcal{L}$. Of course this is a familiar object, a complex computing this looks like

$$(\mathrm{Sym}(\mathcal{L}^\vee[-1]), d_{\mathrm{CE}})$$

where $d_{\mathrm{CE}} = \{S^{\mathrm{Cl}},\quad\}$.

Maybe I don't have time to say too much but this exhibits a factorization structure but it means that when I have two opens U and V in a bigger open W all in S, disjoint:



then you get a map $\mathrm{Obs}^{\mathrm{Cl}}(U) \otimes \mathrm{Obs}^{\mathrm{Cl}}(V) \to \mathrm{Obs}^{\mathrm{Cl}}(W)$, and quantization gives quantum observables.

Let me give an example. Let $\mathcal{L}$ be $\Omega^{0,*}(\mathbb{C})[-1] \oplus \Omega^{1,*}(\mathbb{C})[-1]$. Now don't worry too much about the cohomological shift. this is describing holomorphic maps $\mathbb{C} \to \mathbb{C}$.

Calling pairs in this (γ, β), the action functional has this form

$$S(\gamma, \beta) = \int_{\mathbb{C}} \langle \gamma, \bar{\partial} \beta$$

where the pairing is wedge and integrate.

There's an n-dimensional version where my target is $\mathbb{C}^n$, and then I get n copies under direct sum of my previously stated $\mathcal{L}$, and then I can say $S(\gamma_i, \beta_j)$ and use the same formulas.

This is really *flat* $\beta\gamma$. What is Costello's model. As I alluded to, you start with a complex manifold X, and we want to describe it by some Lie algebraic data g_X, which is actually a sheaf of L_∞-algebras on X. I won't provide the construction, but a defining piece of information is that $C^*(g_X) \cong \mathcal{O}_X$.

Okay, so we use this description to write down curved $\beta\gamma$ on X,

$$\mathcal{L}_X = \Omega^{0,*}(\mathbb{C}, g_X) \oplus \Omega^{1,*}(\mathbb{C}, g_X^\vee[-2])$$

This means that if I apply this to an open set U, it looks like this:

$$\mathcal{L}_U = \Omega^{0,*}(U, g_X) \oplus \Omega^{1,*}(U, g_X^\vee[-2]).$$

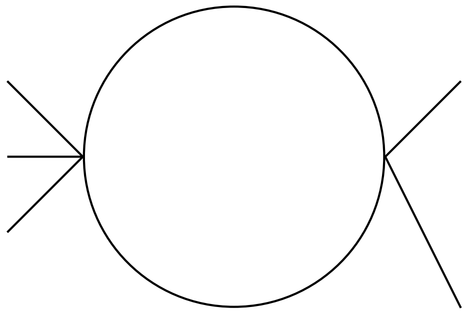
The main theorem that I'll reference from Costello's work is the following

Theorem 4.1.

- $$\mathrm{Def} \cong \Omega^2_{\mathrm{Cl}(Bg_X)}[1]$$
  which means that $H^1(\mathrm{Def})$, where the anomalies live, can be identified with
  $$H^2(X, \Omega^2_{\mathrm{Cl}} X)$$

- *moreover, the obstruction to quantization is identified with an object in* $H^1(\mathrm{Def})$*, which is the second Chern character of the holomorphic tangent bundle on* X*,* $\mathrm{ch}_2(T_X)$

Remark 4.1. Let me write down the Feynman diagram that gives this obstruction. Consider two vertex diagrams like this, and sum over the number of inputs of things that look like this



and this is ch_2

Theorem 4.2. *Suppose that* $\mathrm{ch}_2(T_X) = 0$*, so that there is a quantization. Then* $\mathrm{Obs}^q \cong CDO_X$

What does $CDO_X(U)$ *do for* $U \cong D^n$*, this is* V_n*, the vertex algebr, generated by* $2n$ *fields* $\beta_i(z)$ *and* $\gamma_i(z)$ *with an operator product expansion of the form*

$$\beta_i(z)\gamma_j(w) \sim \frac{\delta_{ij}}{z-w}.$$

A theorem of [unintelligible]says the following. Chiral differntial operators may not always exist, there's some kind of problem.

Theorem 4.3. *(GMS) If* $\mathrm{ch}_2(T_X) = 0$ *then there is a sheaf* CDO_X *which is locally* V_n*, this* $\beta\gamma$ *vertex algebra.*

4.1. **Localization.** So in the next section I'd like to talk about our approach to proving our theorem. This passages through a procedure called *Harish–Chandra localization*. What is the picture? We start with objects defined on the formal n-disk, $\hat{D}^n$. I mean modules over the formal n-disk. You can call this formal geometry if you like, There's some construction Loc_X that produces from these objects (in arbitrary categories, though for us it's vertex algebras, factorization algebras), gives sheaves on X.

The main object, we have coordinatizations X^{coor} sitting over X, where over p we have jets of formal coordinatizations near p. You have $\mathrm{Aut}(\hat{D}^n)$ acting here, and given M an $\mathrm{Aut}\,\hat{D}^n$-representation, you get $\mathcal{M}$, which performs the Borel construction, $X^{\mathrm{coor}} \times_{\mathrm{Aut}} M$, but this is huge, always infinite dimensional in examples we care about.

An example to keep in mind, the infinite jet bundle on X, J_X, how is it related to a more familiar object? There's a canonical flat connection ∇^{jet} on J_X, and flat sections are precisely functions on X. So we want to try something similar in this setting.

So we'll look for the data of a flat connection on this bundle here. One can identify the tangent space of this coordinate bundle TX^{coor} at the coordinatization φ with $\mathrm{Vect}(\hat{D}^n)$. This tells you that we need to consider actions of this Lie algebra of vector fields on our modules M. I should be pedantic, this is more data that just a representation of Aut, it contains [unintelligible]. From now on I'll only talk about actions of Vect due to the following remark (why we're working in the formal setting)

Remark 4.2. There exists a semidirect product splitting of the following form

$$\mathrm{Aut} = \underbrace{\mathrm{Aut}_+}_{\text{pronilpotent}} \ltimes GL_n$$

We can integrate our Aut_+ so that we only have to keep track of GL_n, and that will be pretty obvious. I won't talk about it in this talk but it's easy.

The idea is [missed a little]

(1) σ-model $\mathbb{C} \to \hat{D}^n$
(2) $(\mathrm{Vect}, \mathrm{Aut})$-action
(1) So we choose

$$\mathcal{L}_{\hat{D}^n} = \Omega^{0,*}(\mathbb{C}, g_{\hat{D}^n} \oplus \Omega^{1,*}(\mathbb{C}, g^\vee_{\hat{D}^n}[-2]))$$

Remark 4.3. This theory is free so automatically admits a quantization $\mathrm{Obs}^q_{\hat{D}^n}$ on the formal n-disk.

The action looks like

$$S(\beta, \gamma) = \int_{\mathbb{C}} \beta \wedge (\bar{\partial}\gamma + \ell_0 + \ell_1(\gamma)).$$

Okay, so that's great, the next thing, I'd like to address:

Lemma 4.1. *We can identify* $\mathrm{Obs}^q_{\hat{D}^n}$ *with* V_n

so we want to bootstrap this to get our global object.

(2) We need a Vect action. Well, Vect acts on g so they also act on $\mathcal{L}_{\hat{D}^n}$ at the classical level. The question we'd like to ask is whether we can quantize equivariantly with respect to this structure.

The method of proof is something called a *background field* approach. The data of the action is equivalent to an L_∞ structure that is the structure of a classical field theory on $\mathcal{L}_{\hat{D}^n} \oplus \mathrm{Vect}(\hat{D}^n)$. My new action has the following form,

$$S(\beta, \gamma) + S_{D^n}(\beta, \gamma) + \underbrace{\int \langle \beta, X \cdot \gamma \rangle}_{\text{interaction term}}$$

The reason I'm saying background term is that the only kinetic term is in the middle term, with no interaction with vector fields, so they don't propagate.

So what is the deformation complex, it looks like

$$\mathrm{Def} \cong C^*(\mathrm{Vect}, \mathrm{Def}_{\hat{D}^n})$$

where $\mathrm{Def}_{\hat{D}^n}$ is just the formal deformations associated with the n-disk.

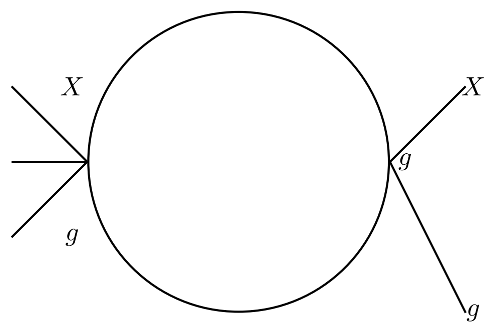
A quick calculation (bootstrapping Kevin's result) says that $\mathrm{Def}_{\hat{D}^n} \cong \Omega^2_{\mathrm{Cl}}(\hat{D}^n)[1]$.

Then the last calculation is

Lemma 4.2. *The obstruction to quantization is equivalent to an extension of* Vect *by this deformation module*

$$\mathrm{Def}_{\hat{D}^n} \to \widetilde{\mathrm{Vect}} \to \mathrm{Vect}$$

So in terms of Feynman diagrams, we again have two-vertex diagrams, but now with one input on each side decorated by X and the others by g. Sum over all



numbers of inputs.

We identify this deformation complex with $C^*(\mathrm{Vect}, \Omega^2_{\mathrm{Cl}})$, and identify it with something we call the universal second Chern character ch_2^u.

What's going on here? There's a cohomology theory for Harish–Chandra modules that I won't describe, so I can consider

$$H^*_{\mathrm{HC}}(\langle \mathrm{Vect}, \mathrm{Aut}\rangle, M)$$

which localizes over X to $H^*(X, \mathcal{M})$. Now, there's a map from this Gelfand–Fuchs cohomology $H^*(\mathrm{Vect}, \Omega^2_{\mathrm{Cl}})$ into this Harish–Chandra thing,

$$H^*(\mathrm{Vect}, \Omega^2_{\mathrm{Cl}}) \to H^*_{\mathrm{HC}}(\langle \mathrm{Vect}, \mathrm{Aut}\rangle, \Omega^2_{\mathrm{Cl}}) \to H^*(X, \Omega^2_{\mathrm{Cl}})$$

and it's basically defining that ch_2^u maps under this to [unintelligible].

A corollary is that

Corollary 4.1. $\mathrm{Obs}^q_{\hat{D}^n} \cong V_n$ *is an equivalence of Harish–Chandra modules.*

The important part is that they're not modules for the original Harish–Chandra pair, but actually for the central extension.

The last fact is that X^{coor}, this kind of Aut-bundle, has a lift $\widetilde{X^{\mathrm{coor}}}$ to an $\widetilde{\mathrm{Aut}}$ if and only if $\mathrm{ch}_2 T_X$ is zero.

A corollary of GMS identifies in this situation CDO_X with $\mathrm{Loc}_{\widetilde{X^{\mathrm{coor}}}}(V_n)$, which implies that the localization of CDO_X is isomorphic to $\mathrm{Loc}(\mathrm{Obs}^q_{\hat{D}^n})$.

So this felt very different from Costello's construction. We started with an equivariant quantization, which gave us a factorization algebra on $\mathbb{C}$ with a $(\mathrm{Vect}, \mathrm{Aut})$-action, and what did we say, we said that we could localize this guy along $\widetilde{X}$ to a sheaf of factorization algebras, this was the output of Costello's quantization. We could also identify this factorization algebra by the vertex algebra relationship to with a vertex algebra witha an $\langle \mathrm{Aut}, \mathrm{Vect}\rangle$-action, and CDO_X was constructed by localizing this guy, and there's a nice functor from the sheaf of factorization algebras that feels like a fiberwise thing.

The last thing to do is to say that Costello's quantization agrees with our equivariant quantization. I didn't talk about this today, but it can be done with similar methods.

$$\begin{tikzcd}
\{\text{Equiv. quant.}\} \arrow[d] & \{\text{Costello quant.}\} \arrow[d] \\
\{\mathbb{C}-\text{fact. alg.}, \langle \mathrm{Vect}, \mathrm{Aut}\rangle - \text{action}\} \arrow[r, "\mathrm{Loc}_{X^\vee}"] \arrow[d, "\mathbb{V}ect"'] & \{\text{sheaf of fact. alg.}\} \arrow[d, "\mathbb{V}ect"] \\
\{\text{vert. alg.}, \langle \mathrm{Vect}, \mathrm{Aut}\rangle - \text{action}\} \arrow[r, "\mathrm{Loc}_{X^\vee}"'] & \{\text{sheaf of vert. alg.}\}
\end{tikzcd}$$

5. January 12: Damien Calaque: Deformation quantization of shifted symplectic and Poisson structures

The purpose of my talk today is to introduce shifted symplectic and Poisson structures so other people can use them.

This is after and joint with Pantev–Toën–Vaquié–Vezzosi.

For an introduction, let me look at AKSZ versus PTVV. In the first, we look at maps from ΠTX to Πg, and you pull back the Killing form, basically a symplectic form, and then integrate over ΠTX and then you get back some kind of two-form in the mapping space.

Everything is an actual form, this is all strict, and when you try to write down degeneracy you have an infinite dimensional space, so you say it's injective on tangent and that's enough. But we want to do everything in a homotopical way. You can't talk about being injective. Here's what people usually do. Replace Πg by BG, this captures more global information, if you look at the moduli space of G-bundles with flat connection. You get more symmetries, not just the trivial G-bundle. There is a stack X_{dR}, and we look at stack maps (X_{dR}, BG) and here on BG we'll get a 2-shifted symplectic form, that's basically the Killing form still. Your Berezin measure is a fundamental class $[X_{dR}]$ and this gives some version of Poincaré duality, let's say. Then we integrate over the fundamental class $\int_{[X_{dR}]} \mathrm{ev}^* \omega_{BG}$. The main two differences are that we do everything up to homotopy, so we can't, say, that things are closed, but only closed "up to homotopy" (I'll give more detail later) and secondly, this is not an infinite dimensional object, how do you compute a tangent? Say f is a map, then $T_f\,\mathrm{MAP}$ is global sections of $f^*\mathbb{T}_{\Pi g}$. This is some complex, but it's a perfect complex, which comes from the fact that since X is compact, its homology is finite dimensional. Here nondegeneracy is that the map between tangent and cotangent is a quasi-isomorphism. We're now in algebraic geometry instead of differential geometry, but that's not a big deal here, everything can be done in differential geometry.

So let me start by introducing shifted differential forms. The basic objects I will work with are X a derived geometric stack. Fortunately or unfortunately I'm not going to define what this is. One way to do differential geometry is to say that spaces are maps from test spaces to your space. So X is some functor from differential graded commutative algebras (test spaces) with nice properties.

So one can construct something $DR^{\#}(X)$, that's a graded complex, a complex with an auxilliary grading. Take $\Gamma(X, S^*(\mathbb{L}_X[-1]))$, this is wedge power of the cotangent bundle, there are two degrees, one from cohomological degree and one from the symmetric power. The second degree I'll always call weight.

So that's the graded complex of forms on X. In concrete terms, if X is $\operatorname{Spec} A$, then $DR(X)^{\#}$ is just $S^*_A(\mathbb{L}_A[-1])$ where $\mathbb{L}_A$, you resolve A, it will be $\Omega^1_{\tilde{A}} \otimes_{\tilde{A}} A$. It's the derived module of one-forms, if you wish.

Notice that even if A is not derived, a perfect commutative ring, we might have nontrivial cohomological degree on $\mathbb{L}_A$ because of the resolution.

If X is BG, then it's a known calculation that $\mathbb{L}_X \cong g^\vee[-1]$, and it's also known that $\mathrm{QCoh}(X) \cong G-\mathrm{mod}$. Then the de Rham complex of X is $C^*(G, S^*(g^\vee[-2]))$, just using the fact that $\Gamma(X, \quad) \cong C^*(G, \quad)$.

I'm ready to tell you what a form is. a p-form of degree n on X is an element in π_0 of $\mathrm{Map}_{\mathrm{gr-cpx}}(\mathbf{k}(p)[-n-p], DR(X)^{\#})$. So taking the weight p part is a pth symmetric power, so we'll get a component of $\mathbf{k}[-n-p], \Gamma(X, S^p(\mathbb{L}_X[-1]))$.

So this is an n-cocycle in $\Gamma(X, S^p(\mathbb{L}_X[-1])[p])$ up to coboundary.

If X is BG, then these are n-cocycles in $C^*(G, S^2(g^\vee([-2])[2]))$, so 2-forms of degree n, so $(n-2)$-cocycles in $C^*(G, S^2(g^\vee))$. So if $n=2$, then 2-forms of degree 2 are elements in $S^2(g^\vee)^G$, a 2-form on BG is symmetric invariant pairings on the Lie algebra.

Now I want to talk about nondegeneracy. Assume $\mathbb{L}_X$ is perfect, so it has a dual which we'll define to be $\mathbb{T}_X$. Then a 2-form $\omega_0 \in \mathcal{A}^2(X,n)$, a 2-form of degree n, is said to be non-degenerate if the induced map $\mathbb{T}_X \to \mathbb{L}_X[n]$ is a weak equivalence.

In the case of BG, non-degenerate 2-forms of degree 2 are $(S(g^\vee)^G)^{ND}$, actually non-degenerate pairings (since $\mathbb{T}_X$ and $\mathbb{L}_X$ are in one degree)

If $X = \operatorname{Spec} A$, for A a nonpositively graded commutative differential graded algebra, then there are no nondegenerate 2-forms of positive degree. The fact of getting an isomorphism between tangent and cotangent, this puts bound on the amplitude of your cohomology. You can't get positively shifted nondegenerate forms.

You can also check that if you have a perfect tangent complex, you have a nondegenerate thing of one degree, then you won't be able to pair nondegenerately with any other degree.

Now I want to talk about closed forms. There exists some graded muxed complex $DR(X)$ such that $DR(X)^{\#}$ is $DR(X)^{\#}$. This sounds silly. This has a different differential of weight one which commutes with the other. So it's $V^{\cdot}$, where the dot is for the auxilliary grading, with maps $\epsilon_p : V^p \to V^{p+1}[1]$, and $\epsilon_{p+1}\epsilon_p = 0$. Then $\#$ means the underlying graded complex.

We care mainly how this looks on test objects. $DR(X) = S^*_{\tilde{A}}(\Omega^1_{\tilde{A}}[-1])$ with $epsilon = d_{dR}$, it has degree 1, commutes with the internal differential, and weight 1. If you take the underlying graded complex, it's the same guy without the de Rham differential, and since A and $\tilde{A}$ are quasi-isomorphic, this guy is equivalent to $S^*_A(\mathbb{L}_A[-1])$, what we were calling $DR(X)^{\#}$

So if $X = BG$, with G reductive, then $DR(X) = S^*(g^\vee[-2])^G$, and because of degrees, we see that $\epsilon = 0$.

Definition 5.1. A *closed p-form of degree n* is an element in $\pi_0 \mathrm{Map}_{\mathrm{gr-cpx}}(\mathbf{k}(p)[-n-p], DR(X))$.

You want something that's ϵ-closed up to a boundary.

So anyway, that's the same thing as an element of

$$\pi_0(\mathrm{Map}(\mathbf{k}[-n-p], \prod_{q\geq p} DR(X)(q), d+\epsilon))$$

and what is that? A closed p-form of degree n is a series $(\omega_0, \omega_1, \ldots)$ where ω_0 is a p-form of degree n, something of weight p and degree $n+p$, closed under d, and ω_1 has weight $p+1$ and degree $n+p-1$, and $\epsilon(\omega_0) = d(\omega_1)$, and so on and so forth, so $\epsilon(\omega_i) = d(\omega_{i+1})$.

As I said, on BG for G reductive, all forms are closed on the nose without adding additional homotopies.

Definition 5.2. An n-shifted symplectic structure on X is a closed 2-form of degree n on X, called ω such that ω_0 is nondegenerate.

Examples of this are, well, BG, when G is reductive, together with t a nondegenerate pairing in $(S^2(g^\vee)^G)^{\mathrm{ND}}$, this is 2-shifted. Perf is another example, inside this you have the stack of vector bundles, basically $\cup BGL_n$. Whenever you have $\mathbb{T}^*[n]X$, that's n-shifted symplectic. For instance, if you take $\mathbb{T}^*[1]BG$, that happens to be $[g^*/G]$, that's 1-shifted symplectic.

If you have a symplectic groupoid, then its quotient stack $[G_0/G_1]$ will be 1-shifted symplectic as well.

Then there's this AKSZ/PTVV, say that $X = M_B$ (the stack classifying local systems on a compact oriented manifold of dimension d) or X is a projective smooth algebraic variety together with a trivialization of its canonical bundle, so a d-Calabi–Yau variety.

If X is one of these two, then the mapping stack $\mathrm{Map}(X,Y)$ will be $(n-d)$-shifted symplectic.

This will have a function on X tensor a 2-form on Map. Then you use the fundamental class to kill the function on X, basically Poincaré or Serre duality. So for $Y = BG$, the G-local systems on a closed oriented surface, this is 0-symplectic. For a 3-manifold it's -1-shifted symplectic. That's the starting point for classical BV. Then [unintelligible]on a K_3 is 0-shifted symplectic; for a 3-Calabi–Yau, it's again -1-shifted symplectic, this is the starting point for BV in holomorphic Chern–Simons.

Let me quickly define Lagrangian structures. Assume we have an n-symplectic guy Y and a map $f : X \to Y$. An isotropic structure on X is a homotopy between 0 and $f^*\omega_Y$ in the space $\mathcal{A}^{2,\mathrm{cl}}(X,n)$. It's important that it stays in the space of closed forms, a homotopy between the whole series.

A Lagrangian is isotropic and nondegenerate, and I have to explain what nondegenerate means in this context.

So we'll mimic for a sypmlpectic form. So nondegeneracy is a property of γ_0. We have $\mathbb{T}_X \to f^*\mathbb{T}_Y$, treat it as if X is a submanifold of Y. Then this is $f^*\mathbb{L}_Y[n] \to \mathbb{L}_X[n]$. The composed map is homotopic to zero so it has a lift to the homotopy fiber, and what is this fiber? They are the elements of $\mathbb{T}_Y$ killed with $\mathbb{T}_X$ under the symplectic form, so you can say it's $\mathbb{T}_X^o$, or $\mathbb{L}_f[n-1]$, these two are equivalent, it's saying that $\mathbb{T}_X$ sits inside its symplectic orthogonal, and you want this map to be surjective as well as injective. For us we require that this map is a weak equivalence.

Let's talk about examples. Take the point with the $n+1$-shifted symplectic structure 0, then look at Lagrangians in X mapping to the point, those are n-shifted symplectic structures on X.

Let $X \to g^*$ be a smooth symplectic G-scheme with a moment map, then $[X/G] \to [g^*/G]$ is Lagrangian.

There are a lot of constructions that have a nice interpretation in terms of these Lagrangian structures. Then the map $G_0 \to [G_0/G_1]$ is Lagrangian as well. This is a reinterpretation of work of Ping Xu.

I have ten minutes to talk about shifted Poisson structures. The reason I want to talk about these is that you want to [missed a little], we want to quantize stacks in a way that preserves field theories, so I want to get E_n structures, so first I should get Poisson structures. We all know that symplectic structures lead to Poisson structures. This is a place to do a lot of work to see that symplectic structures lead to Poisson structures.

You could say that an n-shifted Poisson structure on $\operatorname{Spec} A$ is a P_{n+1}-structure on A, that's by definition a P_{n+1} algebra B together with an equivalence between B and A as commutative algebras.

There's another way, there's an equivalence between Poisson structures and bivectors whose bracket with themselves is zero, so look at Maurer–Cartan elements in $\operatorname{Poly}(A, n+1)[n+1]$, and $\operatorname{Poly}(A, n+1)$ is $S_{\tilde{A}}(\operatorname{Der}(\tilde{A}, \tilde{A}))[-n-1]$, and here you have a Lie bracket of weight -1 and degree $-n-1$, it's basically the Lie bracket of derivations, and to make it an actual Lie algebra, you shift it. There's again an auxilliary grading, and the bracket again has degree [unintelligible]. It's not quite exactly Maurer–Cartan, because it's only up to homotopy for the auxiliary grading.

The two definitions do not exactly coincide, there's a map from P_{n+1}-structures on A to $\operatorname{Poiss}(A, n)$, you can turn a strict model into an actual Maurer–Cartan element with no homotopies. There is a theorem of Melani, which says if $\mathbb{L}_A$ is perfect then this is an equivalence. That's what we're interested in, and since we're starting with symplectic we expect this to be perfect.

Let me say, with a lot of work you can make sense of this for an arbitrary derived stack X of locally finite presentation. All this still works, for instance, if X is BG with G reductive, then shifted polyvectors on BG is the same as $S^*(g[-n])^G$ and $\operatorname{Poiss}(BG, n)$ will be in elements in $(\wedge^3 g)^G$ if $n=1$, elements $S^2(g)^G$ if $n=2$ and nothing otherwise.

The last thing is, there is a morphism, a map of spaces, which goes from Poisson structures of degree n on a stack to $\mathcal{A}^{2,\mathrm{cl}}(X, n)$ which sends nondegenerate Poisson structures to symplectic ones. The induced map $\operatorname{Poiss}(X, n)^{\mathrm{ND}} \to \operatorname{Symp}(X, n)$ to symplectic ones is an equivalence. This is not easy to prove.

Then you can use a similar kind of equivalence, get a P_{n+1} algebra and maybe quantize it to an E_{n+1}-algebra object, and as soon as $n \geq 1$, this works perfectly well. For $n=0$ you need some Kontsevich type deformation quantization, and for negatively shifted things, you need something completely different. [some description].

6. Pavel Safronov: Poisson geometry of groups and shifted Poisson structures

In this talk I'll continue Damien's talk, and give an application of the theory he described.

Let me start with some motivation for shifted Poisson structures. So you start with X a smooth manifold, and you know that X is Poisson, this means that if you look at $\mathrm{QCoh}(X)$, a symmetric monoidal category, you want to deform this, a Poisson structure gives a direction in the deformation space of this category, this gives a plain category $\mathrm{QCoh}_q(X)$. You can ask about symmetric monoidal deformations. If X is 1-shifted Poisson, you should try to deform QCoh to a monoidal category. If X is two-shifted Poisson, you should deform to a braided monoidal category, and so on.

My main example will be the stack $X = BG$ for the following reason. You can look at $\mathrm{QCoh}(BG)$, and this is Rep G, and your question for the deformations of this to a monoidal or braided monoidal category, this question of classification of deformations is related to classifications of shifted Poisson structures on BG. These categories of monoidal and braided monoidal categories are related to quantum representations of G, that's my main motivation.

Let me remind you how to define shifted Poisson structures. I'll also define coisotropic structures, so $L \to X$ is coisotropic, then you have an action $\mathrm{QCoh}(X)$ on $\mathrm{QCoh}(L)$. If this is 1-shifted, then this has a deformation to some monoidal category $\mathrm{QCoh}_q(Q)$ and a deformation to $\mathrm{QCoh}_q(L)$, an ordinary category. Coisotropic means that this action deforms as well. If things are two-shifted then this is a braided monoidal category acting on a monoidal category.

Let me remind you from Calaque–Pantev–Toën–Vaquié–Vezzosi, say X is a nice derived stack, then you can introduce $\mathrm{Poly}(X,n) = \Gamma(X, \mathrm{Sym}(\mathbb{T}_X[-n-1]))[n+1]$, this is a graded algebra, with grading from the symmetric algebra, this is a graded $P(n+2)$-algebra under the analogue of the Schouten bracket, and I'll say

Theorem 6.1. *(Calaque–Pantev–Toën–Vaquié–Vezzosi)* $\mathrm{Poly}(X,n)$ *is a graded differential graded Lie algebra.*

Definition 6.1. A *shifted Poisson structure* on X is

$$\pi \in \mathrm{Map}_{\mathrm{gr\,dgla}}(\mathbf{k}(2)[-1], \mathrm{Poly}(X,n)).$$

If $f : L \to X$ is a morphism of derived stacks, then you can define relative polyvectors

$$\mathrm{Poly}(f,n) = \Gamma(X, \mathrm{Sym}(\mathbb{T}_X[-n-1]))[n+1] \oplus \Gamma(L, \mathrm{Sym}(\mathbb{T}_{L/X}[-n]))[n]$$

and there's a differential from the first to the second term. If you have a vector field on X, you can pull it back to a vector field on L, and then you can use pushforward, there's a sequence $\mathbb{T}_{L/X} \to \mathbb{T}_L \to f^*\mathbb{T}_X$. Again, this is a graded complex, and I claim that this is a graded differential graded Lie algbebra.

Let me say it's a quasi-theorem (Melani–S.) that there is a graded differential graded Lie structure on the relative polyvectors and their projection maps

$$\begin{tikzcd} & \mathrm{Poly}(f,n) \arrow[dl] \arrow[dr] & \\ \mathrm{Poly}(L,n-1) & & \mathrm{Poly}(X,n) \end{tikzcd}$$

and I claim that these are maps of graded differential graded Lie algebras.

So using this theorem (which I will assume) you can define coisotropic structures.

Definition 6.2. Suppose X is n-shifted Poisson. A *coisotropic structure* on the morphism $f : L \to X$ is a lift of the Poisson structure (a map $\mathbf{k}(2)[-1] \to \mathrm{Poly}(X, n)$) to $\mathrm{Poly}(f, n)$. This lift is my coisotropic structure.

Let me observe that if $L \to X$ is coisotropic and X is n-shifted Poisson, then L has has a natural $n-1$-shifted Poisson structure. This is coming from the other projection map from the theorem. You lift and push down as above.

This definition of coisotropics, for example, for $n = 0$ if L and X are smooth schemes, then this is the usual definition of coisotropic structures on submanifolds.

This is quasi–Poisson geometry, let me try to apply this to BG. Let G be a group, not necessarily reductive. Then I'll define (this is classical):

Definition 6.3. A *quasi-Poission structure* on G is a a bivector $\pi \in \Gamma(G, \wedge^2 \mathbb{T}_G)$ and a trivector $\varphi \in \wedge^3 g$, such that

$$\pi(g_1 g_2) = L_{g_1} \pi_{g_2} + R_{g_2} \pi_{g_1}$$

(multiplicativity), such that

$$[\pi, \pi] = \varphi^L - \varphi^R$$

and so that

$$[\pi, \varphi^L] = 0$$

If $\varphi = 0$ then this is called a *Poisson Lie structure.*

Such elements act by *twistings*, there is an action of $\wedge^2 G$ on the set of quasi–Poisson structures, and this defines a groupoid, quasi-Poisson structures mod twists.

Here's a theorem classifying quasi-Poisson structures on BG.

Theorem 6.2. *(S.) Consider n-shifted Poisson structures on BG. If $n > 2$ then there are no nontrivial Poisson structures. If $n = 2$, the space is a set, given by*

$$\mathrm{Sym}^2(g)^G,$$

call such elements Casimirs*. If $n = 1$, then the space is the groupoid of quasi-Poisson structures modulo twists. If $n < 1$ you can only get Poisson structures if G is not reductive. I will not talk about those.*

Here I gave a classification of shifted Poisson structures. Let me reinterpret this. Braided monoidal deformations of $\mathrm{Rep}\, G$ are controlled by Casimirs, elements in $\mathrm{Sym}^2(g)^G$, and monoidal deformations of $\mathrm{Rep}\, G$ are controlled by quasi-Poisson structures.

What else can you do with this?

Next I'll talk about coisotropic structures, let's talk about coisotropic structures on $\mathrm{pt} \to BG = \mathrm{pt}/G$. There are no nontrivial coisotropics if BG has a 2-shifted Poisson structure. Then for 1-shifted, coisotropic structures on $\mathrm{pt} \to BG_1$ are given by Poisson-Rie structures on G.

How can you think about this in the quantum context? This map from a point to BG corresponds to a forgetful functor from Rep G to Vect. The first statement tells you that there are no braided monoidal functors $\mathrm{Rep}_q\, G \to \mathrm{Vect}$. The second claim tells you that if you choose a Poisson structure rather than [missed some].

Okay, so let me reinterpret slightly this condition on coisotropic structures. Say the quasi-Poisson structure, assume the structure on G is given by $(\pi = 0, \varphi)$.

Then coisotropic structures on a pt $\to BG$ are given by twists of this quasi-Poisson structure to an honest Poisson structure $t \in \wedge^2 g$ such that $\frac{1}{2}[t,t] = \varphi$. This is a well-known equation in the theory of quantum groups, the modified classical Yang–Baxter equation.

Let me continue talking about coisotropic structures. Suppose $P \subset G$ is a parabolic subgroup of a reductive group. We can think about the case $G = GL_n$ and P upper triangular matrices. Then the claim is that $BP \to BG$, and give BG a 2-shifted Poisson structure (the standard Killing form)—then $BP \to BG$ is coisotropic. A one-shifted Poisson structure on BP, which we get from this, is the same as a quasi-Poisson structure on P. So say G is reductive, the same statement tells you that the diagonal subgroup $BG \to BG_2 \times BG_2$ is coisotropic. So there is a way to map 2-shifted Poisson structures on BG to 1-shifted Poisson structures on BG. This is true for any space. You can always drop the shift down by one. Start with some Casimir and let's trace which quasi-Poisson structure it goes to. In the reductive case, I claim that the first space is given by Casimirs, and the 1-shifted Poisson case is given by elements in $(\wedge^3 g)^G$. The natural map sends $c \mapsto [c_{12}, c_{13}]$, where $c_{12} = c \otimes 1$ and $c_{23} = 1 \otimes c$.

If you started with a c in $\mathrm{Sym}^2(g)^G$, and then you look at coisotropic structures on pt $\to BG$, they are given by $t \in \wedge^2 g$ such that

$$\frac{1}{2}[t,t] = [c_{12}, c_{23}]$$

and $r = t + c$ satisies $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$. This is the *Classical Yang–Baxter equation.*

This is saying that on the quantum level, quantum R-matrices give braided monoidal structures on $\mathrm{Rep}_q G$ such that the forgetful functor to Vect is monoidal.

Let me mention some general fact, that a G-space is the same as a space over BG, so if you have a G-space, then you can send it to X/G living over BG, and a space over BG can be sent to the pull back of $Y \times_{BG}$ pt and here you have an action of pt $\times_{BG}$ pt which is G. Then you can ask if you can upgrade this to Poisson geometry.

So say X is a smooth scheme and G is a quasi-Poisson group. Then the coisotropic structures on $X/G \to BG_1$ (getting the 1-shifted Poisson structure from the quasi-Poisson structure on G) are given by quasi-Poisson G-space structures on X (I won't define this).

Let me mention one more classical result. Say that r is a classical R-matrix (a solution to the classical Yang–Baxter equation). Then there exists a Poisson-Lie structure on G where the bivector π is the antisymmetric part $r_-^L - r_-^R$, this is a bivector at the unit, and you can translate on the left or on the right. Thys was defined by Sklyanin. How do you interpret this in shifted Poisson geometry? It's very natural using the following theorem that I'll call a quasi-theorem, joint with Melani, that says if you have L_1 and L_2 coisitropic structures in X which is n-shifted Poisson, then $L_1 \times_X L_2$ has a natural $(n-1)$-shifted Poisson structure So for pt $\to BG$ and the classical R-matrix, you get pt $\times_{BG}$ pt Poisson Lie, and the claim is that these two constructions agree.

Let me describe some phenomena that do not yet have quantum analogues.

Say that G is reductive and $B \subset G$ is Borel, and $H \subset G$ is Cartan. So think $G = SL_2$, B is upper triangular, and H is diagonal.

I mentioned that $BP \to BG_2$ is coisotropic, you can prove a similar statement, say E is an elliptic curve. Then you can look at a map from B bundles on E to G bnudles on it, $\mathrm{Bun}_B(E) \to \mathrm{Bun}_G(E)$. The codomain is 1-shifted Poisson and the map is coisotropic. So you have a Poisson structure on the moduli of B-bundles on E.

Now you can just go and compute in geometry what these spaces are. This moduli $\mathrm{Bun}_B(E)$ in our case has an open substack whose components are projective bundles over E. You can compute this Poisson structure, and this coincides with the Sklyanin or Feigin–Odesskii Poisson structure on $\mathrm{Bun}_B(E)$, and the quantization is known, [unintelligible], so what we call $\mathrm{QCoh}_\eta(\mathrm{Bun}_B(E))$, modules over the Feigin–Odesskii algebra, so far this is known, but now you have this coisotropic map to BG which gave a coisotropic map into $\mathrm{Bun}_G(E)$

Conjecture 6.1. Given $\eta \in E$ there is a monoidal category $\mathrm{QCoh}_\eta(\mathrm{Bun}_G(E))$ and there's an action of this monoidal category on modules over the Feigin–Odesskii-algebra, and fairly explicitly you can compute this category when G is a torus. I don't know how to produce the quantization when G is not a torus.

Let me stop here.

7. Claudia Scheimbauer: Fully extended semi-classical TFTs and Weyl quantization

Damien was so kind to set up the basic framework that we will need. He was talking about AKSZ in derived algebraic geometry, and the goal today is to explain how this leads to an oriented and even fully extended TFT, and our objects in the target will be derived stacks, with Lagrangian correspondences, and higher versions. I'll try to avoid techincalities about higher categories.

If time permits I'll explain what happens in the linear setting, just linear symplectic vector spaces and linear Lagrangian correspondences.

Let's recall from this morning the AKSZ/PTW construction. We started with Y a k-shifted sypmlectic derived stack and we take X a derived stack, which is nice, and by nice what I mean is has some compactness condition and some sort of a d-orientation, and we'll see a little later what this should be.

Then we can produce the mapping stack $\mathrm{Map}(X, Y)$ and if this is nice, it has a $k-d$-shifted symplectic structure.

One example is for X to be M_B, the Betti stack of an oriented compact d-dimensional closed manifold. This is the stackification of the constant prestack which sends $C^{*,\mathrm{sing}}(M)$ to any algebra A. The orientation will give us d-orientation of the stack X and compactness is compactness.

We want manifolds with boundary to get a TFT. So let's do a relative version.

If M is compact, oriented, d-dimensional, and with boundary. We can look at the same mapping stack as above, but we can also look at the mapping stack $\mathrm{Map}((\partial M)_B, Y)$, which will be $n-(d-1)$-symplectic, and we get a restriction map from $\mathrm{Map}(M_B, Y)$, and this restriction map, the theorem says, has a Lagrangian structure.

This theorem is due to Damien and has a nice corollary, if we have M an oriented d-bordism, with an in and an out part, then we can do the same game

$$\begin{tikzcd} & \mathrm{Map}(M_B, Y) \arrow[dl] \arrow[dr] & \\ \mathrm{Map}(\partial_0 M_B, Y) & & \mathrm{Map}(\partial_1 M_B, Y) \end{tikzcd}$$

and this is a Lagrangian correspondence, where you have a diagram $X \leftarrow L \rightarrow Y$ and the map $L \rightarrow \bar{X} \times Y$ is Lagrangian.

Let's try to understand first what this TFT should do, what the target should be.

If we start with a cobordism we get a correspondence, and classically, composition then becomes a problem. If you intersect non-transverse Lagrangians you don't get anything nice. But working in derived algebraic geometry we can always compose Lagrangian correspondences. If we have

$$\begin{tikzcd} & L \arrow[dl] \arrow[dr] & & L' \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

then we can form the homotopy pullback

$$\begin{tikzcd} & & L \times^h_Y L' \arrow[dl] \arrow[dr] & & \\ & L \arrow[dl] \arrow[dr] & & L' \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}$$

and this is a Lagrangian correspondence [some discussion].

Now we can try to form a category. Our source category is cobordisms, $d\,\mathrm{Cob}^{\mathrm{or}}$ with objects $(d-1)$-dimensional oriented closed manifolds and morphisms diffeomorphism classes of d-dimensional oriented bordisms.

Our targets will be Lagrangians. Our objects will be $n-(d-1)$-symplectic, and our morphisms will be Lagrangian correspondences.

Again, we need to identify some of these in a suitable sense.

Taking these diffeomorphism classes comes from the fact that we are dealing with $(\infty,1)$-categories. ∞ means that we have morphisms of morphisms, morphisms between those 2-morphisms, and so on. So we have $\mathrm{Bord}_d^{(\infty,1)}$, so objects are the same, morphisms are d-dimensional bordisms, 2-morphisms are diffeomorphisms, then isotopies, then isotopies of isotopies, and so on. The 1 means that starting from the diffeomorphisms, these are all invertible.

On the other side, we have something similar, in $\mathrm{Lag}_{k-(d-1)}^{(\infty,1)}$, so objects are as above, morphisms are Lagrangian correspondences, and then various equivalences. What if we do $d-2$, $d-3$, and so on? We should be able to do this, all the way down to 0-dimensional. So actually what we want to put here is (∞,d)-categories.

I'll put this as $\mathrm{Bord}_d^{\mathrm{or}}$, meaning $\mathrm{Bord}_d^{(\infty,d),\mathrm{or}}$. Then we'll get for our target category, the point will go to, it should be k-shifted symplectic, so the target is $\mathrm{Lag}_k^{(\infty,d)}$. Extending this down should be our goal, this should be a symmetric monoidal functor. So given Y k-symplectic, there exists a fully extended TFT (that's a symmetric monoidal functor of this sort), that's the goal. This is joint work in progress with Calaque and Haugseng.

Okay, what did we do? We chose an object [unintelligible]target and constructed a fully extended TFT. As a corollary, using the cobordism hypothesis (which tells you the field theories with any target) we get

Corollary 7.1. *Every object in* $\mathrm{Lag}_k^{(\infty,d)}$ *is fully dualizable.*

They even give oriented TFTs but never mind.

In the next ten or fifteen minutes I'll explain how to build this target category of Lagrangians.

The target category sits in $\mathrm{Span}_d(d\,\mathrm{St}/\mathcal{A}^2_{\mathrm{cl}}(k))$. The d-fold spans were constructed by Rune, and he showed that they are fully (d−)dualizable. It remains to show that it's true in the subcategory. For $d=1$ then he showed that every object is 1-dualizable. For $d=2$, Amorim–Ben Basset constructed in the last week, Lag_k^2 as a weak two-category. But we want to get all the way to d. We use complete d-fold Segal spaces. I'll restrict to $d=2$ for simplicity. Objects should be k-symplectic. Morphisms should be Lagrangian correspondences. Let me unravel the condition. We saw this morning the definition of Lagrangian. Induced from the span

$$\begin{tikzcd} & L \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

we get maps

$$\begin{tikzcd} & \mathbb{T}_L \arrow[dl] \arrow[dr] & \\ f^*\mathbb{T}_X & & f^*\mathbb{T}_Y \end{tikzcd}$$

and then we can go further:

$$\begin{tikzcd} & \mathbb{T}_L \arrow[dl] \arrow[dr] & \\ f^*\mathbb{T}_X \arrow[d, "\cong"] & & f^*\mathbb{T}_Y \arrow[d, "\cong"] \\ f^*\mathbb{L}_X[k] \arrow[dr] & & g^*\mathbb{L}_Y[k] \arrow[dl] \\ & \mathbb{L}_L[k] & \end{tikzcd}$$

and the condition is for this to be a pullback–pushout square (these are the same; this is a stable category).

What happens when we go to 2? So for 2-fold Lagrangian correspondences

$$\begin{tikzcd}
 & \Sigma \arrow[dl] \arrow[dr] & \\
L_1 \arrow[d] \arrow[drr] & & L_2 \arrow[d] \arrow[dll] \\
X & & Y
\end{tikzcd}$$

, we have L_1 and L_2 to $\bar{X} \times Y$ Lagrangian ,nd $L_1 \times^h_{\bar{X}\times Y} L_2$ is $(k-1)$-symplectic and then require that $\Sigma \to L_1 \times^h_{\bar{X}\times Y} L_2$ is Lagrangian. So if X and Y are a point, we just recover the ordinary notion.

Now I can write the version of my square, which we require to be a limit or colimit diagram. This is equivalent but easier to generalize.

$$\begin{tikzcd}
 & \mathbb{T}_\Sigma \arrow[dl] \arrow[dr] & \\
\mathbb{T}_{L_1} \arrow[d] \arrow[drr] & & \mathbb{T}_{L_2} \arrow[d] \arrow[dll] \\
\mathbb{T}_X \arrow[d] & & \mathbb{T}_Y \arrow[d] \\
\mathbb{L}_X[k] \arrow[d] \arrow[drr] & & \mathbb{L}_Y[k] \arrow[d] \arrow[dll] \\
\mathbb{L}_{L_1}[k] \arrow[dr] & & \mathbb{L}_{L_2}[k] \arrow[dl] \\
 & \mathbb{L}_\Sigma[k] &
\end{tikzcd}$$

so composition, I can put things down like this

$$\begin{tikzcd}
X_4 & L_{34} \arrow[l] \arrow[r] & X_3 \\
L_{14} \arrow[u] \arrow[d] & \Sigma \arrow[l] \arrow[r] \arrow[u] \arrow[d] & L_{23} \arrow[u] \arrow[d] \\
X_1 & L_{12} \arrow[l] \arrow[r] & X_2
\end{tikzcd}$$

then a fact is that $\mathrm{Lag}_k^{(\infty,d)}(*,*) \cong \mathrm{Lag}_{k-1}^{(\infty,d-1)}$ and this gives us the symmetric monoidal structure.

We can do this for higher levels, we get more complicated versions of the same correspondences and then we get a bigger diagram to ask to be a colimit, but it's the same idea.

Getting back to our functor, what will our strategy be? We can decompose the functor into simpler ones.

We wanted to start with $\mathrm{Bord}_d^{\mathrm{or}}$ and end in Lagrangian correspondences. So first we cut our bordism, and I'll explain in a minute, to get cospans of nice spaces (I won't go into the details of which spaces). So I have a cutting functor that cuts wherever things are possible, rememebering the intersections along which I glued. [pictures]

The next thing is, I take this functor $(\quad)_B$ to derived stacks, and then map into Y, which makes things into spans

$$\mathrm{Bord}_d^{\mathrm{or}} \xrightarrow{\text{“cut”}} \mathrm{Cospan}_d^{[]}(\mathcal{S}^f) \xrightarrow{(\quad)_B} \mathrm{Cospan}_d^{[]}(d\,\mathrm{St}^{\mathrm{cpt}}) \xrightarrow{(\mathrm{Map}(\quad,Y))} \mathrm{Span}_d(d\,\mathrm{St}/\mathcal{A}^2_{\mathrm{cl}}(t)) \supset \mathrm{Lag}_k^{(\infty,d)}$$

and we can lift. Let me make another less trivial example [picture].

Let me give some sort of proof, not in the fully extended case but maybe you can convince yourself that some similar argument should hold. We start with

$$\begin{tikzcd} & M & \\ \partial_0 M \arrow[ur] & & \partial_1 M \arrow[ul] \end{tikzcd}$$

and we want that

$$\begin{tikzcd} & L = \mathrm{Map}(M_B, Y) \arrow[dl] \arrow[dr] & \\ X = \mathrm{Map}((\partial_0 M)_B, Y) & & Y = \mathrm{Map}((\partial_1 M)_B, Y) \end{tikzcd}$$

is a Lagrangian correspondence.

So now I want to say something about my orientations. I clim that

$$\begin{tikzcd} & M_B & \\ (\partial_0 M)_B \arrow[ur] & & (\partial_1 M)_B \arrow[ul] \end{tikzcd}$$

has a relative d-orientation. The boundary guys $(\partial_1 M)_B$ and $(\partial_1 M)_B$ have $(d-1)$-orientations. That means say $\partial_0 M$ has a fundamental class $[\partial_0 M] \in H_{d-1}(\partial_0 M, \mathbf{k})$, so that

$$\cap[\partial_0 M] : \underbrace{C^*(\partial_0 M)}_{\Gamma((\partial_0 M)_B, \mathcal{O}_{(\partial_0 M)_B})} \to \mathbf{k}[-(d-1)]$$

gives a non-degenerate pairing (call $(\partial_0 M)_B$ by the name N_0) $\Gamma(N_0, E) \otimes \Gamma(N_0, E^\vee) \to \Gamma(N_0, \mathcal{O}_{N_0}) \xrightarrow{[N_0]} \mathbf{k}[-(d-1)]$ and nondeneracy here comes from Poincaré duality, for any E a perfect complex

This was absolute and you can probably guess what it means to have a relative d-orientation.

Now we also know that M in the ordinary sense has a relative fundamental class in the chains $C_d(M, \partial M, \mathbf{k})$, and note that this is the homotopy fiber of the map from the inclusion of the boundary and under this map, $[M]$ goes to $[\partial M]$

$$\begin{tikzcd} C_d(M,\partial M,\mathbf{k}) \arrow[r] \arrow[d] & 0 \arrow[d] \\ C_{d-1}(\partial M) \arrow[r] & C_{d-1}(M) \end{tikzcd}$$

the condition is, well,

$$\begin{tikzcd} & \Gamma(M_B,E^\vee) \arrow[dl] \arrow[dr] & \\ \Gamma((\partial_0 M)_B,E^\vee) \arrow[d] & & \Gamma((\partial_1 M)_B,E^\vee) \arrow[d] \\ \Gamma((\partial_0 M)_B,E)^\vee[-(d-1)] \arrow[dr] & & \Gamma((\partial_1 M)_B,E)^\vee[-(d-1)] \arrow[dl] \\ & \Gamma(M_B,E)^\vee[-(d-1)] & \end{tikzcd}$$

and now relative Poincaré duality tells you this is a pullback pushout square. This pattern continues as you move up in levels.

So one more thing I should tell you, for $L = \mathrm{Map}(M_B,Y)$, then $\mathbb{T}_{L_1x} = \Gamma(M_B \times \operatorname{Spec} A, \xi^*\mathbb{T}_Y)$ where $x : \operatorname{Spec} A \to L$ or equivalently $\xi_x : M_B \times \operatorname{Spec} A \to L$. [missed some].

In the last minute I will, as promised, explain an outlook. Damien mentioned this morning, these are the semi-classical TFTs. These should be the starting point for some quantization. In general there is no hope for functorial quantization, but there's something we can do. We got

$$\mathrm{Bord}_d^{\mathrm{or}} \to \mathrm{Lag}_x^{(\infty,d)}$$

and there's another fully extended TFT

$$\mathrm{Bord}_d^{\mathrm{fr}} \to \mathrm{Alg}_d$$

and here given any object in the target (in either case) we get a fully extended TFT. The target category in the second case has objects E_d-algebras, morphisms bimodules, 2-morphisms bimodules of bimodules in a certain sense, up until d, and if we take $d = 1$ we get the Morita category here. This is the thing you might hope to quantize to.

This is a non-existent thing, but you might want a quantization functor from Lag_k. This will not work, it's clear. But in the linear case there's something you can do.

Now in $\mathrm{Lag}_{d-1}^{(\infty,d),\mathrm{lin}}$ my objects are $(d-1)$-symplectic differential graded vector spaces, morphisms are linear Lagrangian correspondences. If we start at the bottom we should get a Weyl algebra and Foch module for the correspondences. There's a nice procedure to generalize this in some way.

Maybe to give one last ingredient, the thing we built in E_d was by using factorization algebras. We saw the definition yesterday in Brian's talk. If you take $\mathbb{R}^2$, we have a functor from open sets in $\mathbb{R}^2$ to our target category (chain complexes) and structure maps, multiplications, and the nice thing is that E_d algebras are locally constant factorization algebras, and bimodules can also be described in such a way, they are locally constant factorization algebras on a stratified space with a line.

The Weyl quantization can split into two steps, we start with $\mathrm{Symp}\,\mathrm{Vect}_{d-1}$ and we first produce a Lie algebra and then a chain complex. We want to land in $\mathrm{Alg}_d(\mathrm{Ch}_k)$. We can apply Alg_d throughout to this map, and we should get a map to this from our linear Lagrangian category. Owen did the bottom case if you do the objects. For $d = 1$ he showed on the objects this gives you back the Weyl quantization that you expect. You should get back the functor that you expect. I'm out of time, thanks for your attention.

8. Nick Rozenblyum: Additivity for Poisson structures and quantization

The thing I want to talk about is deformation quantization, the classical problem of deformation quantization is, of course, you have some Poisson algebra and you want to quantize it to an associative algebra, that's the basic problem, and the first variation on this that I want to describe is to follow Losev's suggestion from yesterday and rather than working with strictly Poisson and associative algebras, work with homotopical versions thereof. So I want to reformulate to start with a homotopy Poisson algebra and quantize to a homotopy associative algebra.

So there was this discussion in Losev's talk about this idea that you should really give up at least one of associativity or commutativity on the nose. In Chicago where I live, there's a tradition of voting early and often, and in that spirit I'll give up on both, vote for both. I'll let everything be up to homotopy.

This question still makes sense, of course, you have to be careful in defining this problem. In a homotopical setting, in the classical setting, if your Lie bracket vanishes, you automatically get a commutative algebra.

The rules of the game up to homotopy is that you can't specify equality, you need to specify a path, a homotopy. If I make the Lie bracket trivial, you might think that this gives you a commutative algebra, that's false.

It's not true that deforming a homotopy commutative to a homotopy associative algebra gives you a Poisson bracket. Instead, you have to rigidify the situation slightly, and the way to do that is the BD_1 operad. That's an operad (introduced by Ed Segal) over $\mathbf{k}$, really it's $\mathbb{C}$, over $\mathbf{k}[[\hbar]]$, and the generators are a multiplication $\cdot$ and a Lie bracket $\{,\}$ which satisfy the Leibniz rule and the assertion is that $a \cdot b \mp b \cdot a = \hbar\{a, b\}$, so if $\hbar = 0$, this is Poisson (P_1) and if $\hbar$ can be inverted, then this is exactly associative.

The problem of quantization is to deform a P_1-algebra to a BD_1 algebra.

This is the right formulation in that setting. That's a good way of formulating these things. That's the classical deformation quantization problem; I'll explain a few more and then explain why they are the same problem.

Two other instances of deformation quantization,

(1) BV quantization, so this is a formulation due to Kevin Costello and Owen Gwilliam, this is the following, start with a P_0 algebra. This is the operad, you have a commutative multiplication and the bracket lives in degree 1, and they satisfy the Leibniz rule (with the Koszul rule of signs), and Costello and Gwilliam explain in his book that BV quantization is the problem of deforming a P_0 operad to a BD_0 operad, also an operad over $\mathbf{k}[[\hbar]]$, it also has generators the commutative multiplication in degree 0 and the

bracket in degree 1, and then you say $d(\cdot) = \hbar\{,\}$. So BD here stands for Beilinson and Drinfeld, and this should be the BV operad, but there's already something called the BV operad.

Anyhow, so these sholud look different, BD_0 and BD_1, these should look different, although the Poisson parts look pretty regular. This is all I'll say about this for now except to say the upshot that all deformation quantization problems reduce to BV quantization.

(2) Before expanding on that slogan, let me give the second example, which comes from conformal field theory. I have to apologize a bit, because I'm probably only talking about half a conformal field theory, I'm talking about vertex algebras, and there's a notion of *Poisson vertex algebras*, and you quantize those to vertex algebras, and if you've seen the definition of a Poisson vertex algebra, first of all vertex algebras are complicated by Poisson vertex algebras are even more complicated. I'll give a conceptual definition later.

So this is the goal, three somewhat different looking problems, and there's a context in which all of these become the same problem. We'll have to introduce some geometry. I'll introduce a background *spacetime*, using what is called *factorization algebras*. An associative algebra is a factorization algebra on the real line. One advantage is that you can consider other manifolds, not just Euclidean spaces, and you can prove things about the algebra using the geometry.

So X is either a manifold or a smooth algebraic variety, corresponding to topological or algebro-geometric field theories.

We'll consider factorization algebras on X. Let me try to give an idea of the definition. The kind of formulation that is relevant here, let me introduce the *Ran* space. You glue configuration spaces together so that they become contractible. It's the space of finite nonempty subsets of X, there are two cases. If X is a manifold this is a topological space. I want the minimal topology such that X^n mapping into it is continuous. If X is a variety there is a natural way to put this into algebraic geometry.

A *factorization algebra* on X is a "sheaf" on the Ran space of X together with coherent isomorphisms $\mathcal{A}_{\{x,y\}}$, so the fiber over $\{x, y\}$ should identify with $\mathcal{A}_x \otimes \mathcal{A}_y$. I should say what I mean by "sheaf." I put this in quotes because of our two contexts. For manifolds I mean a constructible sheaf (with respect to the natural stratification by number of points of the Ran space). For varieties, instead of constructible sheaves I want D-modules.

Theorem 8.1. *(Lurie) Factorization algebras on $\mathbb{R}^n$ are the same thing as E_n-algebras, e.g., E_1 is associative.*

I want to emphasize, everything is up to homotopy, if you insist on particular models, associative should be A_∞ or something. But higher E_n don't have good explicit algebraic models.

Theorem 8.2. *(Beilinson–Drinfeld) Vertex algebras are the same thing as factorization algebras on algebraic curves.*

This is our entry point into conformal field theory, whether in manifolds or in algebraic varieties.

That's the quantum story, but what about the classical story? First let me state the thing for clarity, the situation in the topological case. There are these operads

P_n and it looks like, there's a commutative product in degree 0 and a Lie bracket satisfying the (appropriately graded) Leibniz rule in degree $1-n$, then you should quantize this to an E_n algebra, this is the n-dimensional version of deformation quantization. I told you this for $n=0$ and $n=1$. For $n \geq 2$, the homology of the E_n operad is the P_n operad (this goes back to Arnold and Fred Cohen), and we can combine these in an ad hoc way, to say that for $n \geq 0$, E_n has a filtration with associated graded P_n, for $n \geq 2$ it's the Postnikov filtration and otherwise the one I applied before. Applying Rees to this filtration I get the BD_n operad over $\mathbf{k}[[\hbar]]$, so this gives P_n by quotienting by $\hbar$ and E_n over $\mathbf{k}((\hbar))$ by inverting $\hbar$.

Theorem 8.3. *(Dunn) E_k-algebras and, well, E_ℓ-algebras, you can tensor them, and it's additive, and so E_k algebras in E_ℓ algebras are $E_{k+\ell}$ algebras, filtering in the ground category of E_ℓ and in the target, not in E_k.*

Theorem 8.4. *(R.) This is compatible with filtration.*

Corollary 8.1. *E_k-algebras in P_ℓ-algebras are the same thing as $P_{k+\ell}$-algebras. This is another reason it's a good idea to pass to a homotopy version.*

Here's a really weird formulation of Poisson algebras. Up to homotopy, though, a Poisson algebra is an associative algebra in P_0-algebras. Just to give you a sense of how weird this is, all the interesting stuff, all the stuff is in the associative structure, the Poisson bracket is all in the associative algebra on the commutative thing, a sort of trivial P_0-algebra.

We saw that associative algebras are the same as factorization algebras on the real line. So Poisson algebras are the same as factorization P_0-algebras on $\mathbb{R}$. Now we can study factorization P_0-algebras on anything, and as long as our P_0-algebras can be valued in other categories, then our problem of quantization reduces to the problem of quantization of P_0 algebras.

The geometric connection will be the key to this proof.

There's this notion of "coisson algebras" where the "c" is for compound. Now sheaves on X have two tensor structures, we could call them $\otimes^*$ or $\otimes^!$. You could form their exterior product, and that's a sheaf on the product, and you pull back, you could take the star restriction or the shreik restriction Δ^* or $\Delta^!$, I don't know how familiar these are. If X is an oriented manifold and these are locally constant sheaves, then these differ by a cohomlogical shift. Then the differ by an orientation sheaf. The shift is by the dimension of the manifold.

So the notion of a coisson algebra, very roughly speaking, is a kind of Poisson algebra, you have a commutative multiplication, using the shriek tensor product, and you have a bracket using the other tensor product. There's a compatibility between the two that lets you consider such a thing.

In the relevant example, manifestly, in an oriented manifold these differ by a degree shift. So coisson algebras on $\mathbb{R}$ are P_1 algebras, the degrees are built in using the tensor products.

The second example, Beilinson and Drinfeld's motivation, is that coisson algebras on a curve are the same as Poisson vertex algebras. That's a first reformulation, although you might argue that's not any better.

Let me state this part of the theorem.

Theorem 8.5. *(R.) There is an equivalence between coisson algebras on X and factorization P_0-algebras on X compatible (I don't have time to explain) with quantization.*

[Some monkeying with the indexing]

This gives an obstruction deformation theory for Poisson vertex algebras. The other thing in the last few minutes, I want to describe this equivalence and what it says about the AKSZ/PTVV construction. So far I was talking about P_n algebras, but you can globalize and talk about shifted Poisson structures and derived stacks.

Before I do that, let me say this: we're going to look at more interesting background spacetime. If A is an associative algebra, well, let me say a BD_1-algebra, a deformation quantized P_1-algebra, nothing necessarily homotopical. The consequence is that we obtain from A a factorization P_0-algebra (I said, a BD_1-algebra is a factorization BD_0-algebra on the real line. Then we can put it on the circle) on the circle. Then we can take the factorization homoolgy of A along the circle. This is $H^*(\mathrm{Ran}(S^1), A)$. Because A is a factorization BD_0-algebra, this naturally becomes a BD_0 algebra. Let's look at what it is. There's another name for this, $\int_S^1$ is Hochschild homology HH_*. This is something like the Hochschild chains for me because HC looks cyclic. So $HH_*(A/\hbar)$ should be Poisson, and this is the ring of functions on the derived mapping space from the circle to $\operatorname{Spec} A$ mod $\hbar$. So this is $\mathcal{O}_{\mathrm{Maps}(S^1, \mathrm{Spec}\ A/\hbar)}$. If we have a commutative factorization algebra, forgetting to Poisson and then forgetting the Lie bracket, then a theorem of Beilinson and Drinfeld says that then factorization homology on a compact space is the ring of functions on the derived mapping stack.

So the upshot is that a BD_1-quantization, deformation quantization of A gives a BD_0-quantization of $HH_*(A)$. This is a geometric reformulation of a known statement in deformation quantization, this computes trace in deformation quantization.

Just to give an example, suppose $A = \mathcal{O}_{T^*Y}$ for Y a smooth variety. I guess first of all, if Y is a smooth variety then $\mathrm{Maps}(S^1, T^*Y)$ is identified with $T^*[-1]\,\mathrm{Maps}(S^1, Y)$, this comes with a -1-shifted symplectic structure and a fact in this context due to Costello, although in the usual BV formalism this is the beginning of the subject, is that equivariant quantizations of $T^*[-1]\,\mathrm{Maps}(S^1, Y)$ is the same thing as (projective) volume forms on $\mathrm{Maps}(S^1, Y)$ which is the same thing as (projective) differential forms on Y.

So this implements the trace function, and so this volume form turns out to be the Todd class (Markarian).

Let me give a few more examples. If you take the next level up, study maps from an elliptic curve, factorization algebras there, then for quantization to exist, you need a trivialized second Chern character, and this quantization gives the factorization version of chiral differential operators and following through the version of this trace construction gives the Witten genus. Doing this for [unintelligible]gives B-model operations.

The last thing I wanted to point out is that this gives a kind, this is like a local-to-global version of the AKSZ construction. If you do [unintelligible]you get something $P(m-n)$ where n is the dimension of the manifold. In this version, start with X an oriented "manifold" then you can form, and Y is a shifted Poisson target, then you can form a local version of this Maps, there's a factorization space, I hope it's not confusing, I passed from algebras to factorization objects over the Ran space of X. It means that the fiber over two disjoint points is the product of the fibers. In this case, this space has a shifted Poisson structure, the factorization space, and the shift is the dimension of X. When X is compact, taking sections gives the same shifted Poisson structure. I'll stop here. Any questions?

9. January 13: Alastair Hamilton: Noncommutative Geometry and the BV–formalism in moduli spaces of Riemann surfaces

[I do not take notes on slide talks]

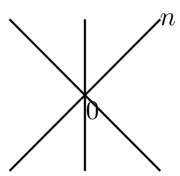
10. Branislav Jurco: Operads, homotopy algebras and strings

Thank you very much for the invitation and the opportunity to give the talk. The talk is based on some work done together with M. Doubek and K. Münster. What I would like to talk about is how homotopy algebra appears in string field theory and how it actually, how the construction of string field theory of Zwiebach can be reinterpreted in the language of operads.

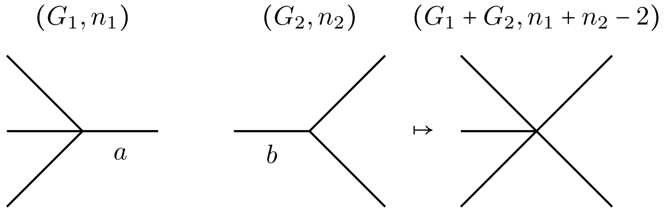
Essentially, the setting is that we start with a modular operad $\mathcal{O}$, I hope to explain what it is, and then take the Feynman transform $\mathcal{FO}$ and get a twisted modular operad. Then what Zwiebach does is takes an appropriate moduli space $\hat{\mathcal{P}}$ and look at chains on this moduli space, and you can read off, he is secretly equipping the space of chains with the structure of a twisted modular operad, and what he does, what he calls the decomposition of moduli space is actually a morphism of these two twisted modular operads. Then there is another map of twisted modular operads. You have your Hilbert space in conformal field theory, and you can look at the endomorphisms of this state space, and this is again a twisted modular operad, and the arrow $C_*(\hat{\mathcal{P}}) \to \mathcal{E}_V$ is what you could call TCFT. then this gives an arrow $\mathcal{FO} \to \mathcal{E}_V$. In principle any string field theory [unintelligible]

$$\begin{tikzcd} \mathcal{FO} \arrow[r, "\text{decomp. mod. space.}"] \arrow[dr, "\text{algebra}"'] & C_*(\hat{\mathcal{P}}) \arrow[d, "\text{TCFT}"] \\ & \mathcal{E}_V \end{tikzcd}$$

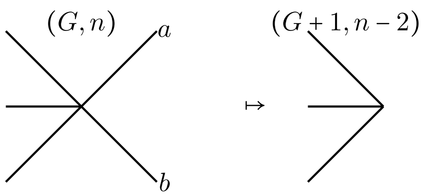
So let me start, look at the corolla with genus, with n labelled legs



and you can join legs and contract edges



and for ξ_{ab}:



So what sort of structure do you have for $\mathcal{P}$? You have maps $\mathcal{P}(\rho)$ for permutations, you have composition maps ${}_a\circ_b$, and you have maps ξ_{ab} to contract. What should this satisfy? You have

(1) ${}_a \circ_b (X \otimes Y) = (-1)^{|X||Y|} {}_b \circ_a (Y \otimes X)$
(2) $\mathcal{P}$ should be equivariant and compatible with ${}_a\circ_b$ and ξ_{ab}
(3) $\xi_{ab}\xi_{cd} = \xi_{cd}\xi_{ab}$
(4) starting with two different corollas with a and c on one and b and d on the other, you can do join and then contract in a couple of ways

$$\xi_{ab\,c}\circ_d = \xi_{cd\,a} \circ b$$

(5) If you have two corollas with a, c, and d on one, you can first join and then contract or vice versa,

$$\xi_{cd\,a}\circ_b = {}_a \circ_b \xi_{cd}$$

(6) associativity if you have three corollas

So this is roughly a definition of a modular operad. [missed something about $G = 0$]

Let me give some examples.

(1) So take $\mathrm{Mod}(\mathrm{Com}^c)$, this is coming from Riemann surfaces, you have a single corolla for every g and n, then your symmetric action is trivial, the gluing is also trivial, there's nothing to describe.
(2) The next one is the classic open strings, Ass^c cyclic associative, so you have $g = 0$ and n leaves. Open strings have only cyclic symmetry, so you get all the cyclic orderings, this is the generators of your vector space, and the action of the permutation group acts by permutation of the legs. Now the circle operation ${}_a\circ_b$, I can use cyclic permutation to put a and b in the first place, so

$${}_a \circ_b ((a, x_1, \dots, x_m)) \otimes (b, y_1, \dots, y_n) \mapsto ((x_1, \dots, x_m, y_1, \dots, y_n))$$

(3) so the next example is the modular version $\mathrm{Mod}(\mathrm{Ass}^c)$, so now I should have an arbitrary number of boundaries and also boundaries with no open insertions. If I have a corolla, I have $G = 2g + b - 1$, and I have [unintelligible]as my generators.

So in this case I would be looking at

$$
\begin{aligned}
{}_a\circ_b\,(((1))\cdots((a,x_1,\dots x_n))\cdots((k)))& \\
\otimes\,(((k+1))\cdots((b,y_1,\dots,y_m))\cdots((N)))& \\
\mapsto (((x_1,\dots x_n,y_1,\dots y_m)),((1))\cdots((N)))&
\end{aligned}
$$

and

$$\xi_{ab}\,(((1))\cdots((a,x_1,\dots x_{k-1},b,x_{k+1},\dots x_m))\cdots((n)))\mapsto(((x_1,\dots,x_{k-1}))((x_{k+1},\dots x_m))((1))\cdots((n)))$$

so that's my operations. It takes some work to prove that this is a modular thing. that is another example.

Since I mentioned superstring, for example in type II superstring theory, what you can use is a kind of modular analogue of a colored version of the cyclic commutative. You can imagine it would be the same as $\mathrm{Mod}(\mathrm{Com}^c)$ except now you have four colors.

These are like NS-NS, NS-Ramon, Ramon-NS, and Ramon-Ramon, your colors, and then your symmetric group action should not interchange different colors.

This was about modular operads but maybe now I can give you a rough idea of the Feynman transform, but before I do that I should say something about a twisted modular operad, where the ${}_a\circ_b$ and ξ_{ab} relations get some additional signs.

The endomorphisms are twisted because the pairing is degree -1 for the symplectic form. So right, maybe now I can say what the Feynman transform is, so I have to tell you what I associate with a corolla, for a corolla with n legs and genus G, I associate a decorated graph with n legs and genus g, what does it mean? I have some graph, always some stability condition which I didn't mention, earlier I needed to say this $2(g-1)+n>0$. Now we have this for every vertex. Each vertex has a genus, and the genus is the sum of the topological genus of the graph plus the genus of every vertex.

So I have genus g and n-leg graphs, and we cut it into vertices, and decorate such graphs, for each edge, say here I have five edges, and I take a Grassmannian variable for every edge, $e_1\wedge\cdots e_5\otimes P_1\otimes P_2\otimes P_3$, these are elements, and since I have, if I cut my edges, I have this vertex [picture] with five legs, and so then if I label them, this, I know what I can associate to a corolla with genus g_1 and these five legs, so I have an element in the dual space of that. This is what my prescription associates to a corolla.

The operation is quasifree, just joining graphs. But what makes it nontrivial is you have a nontrivial differential on it. So just graphically, you think of all possible ways to split the corolla to give something with one internal edge, which I do with circle operations. [pictures]

I can also add loops to the differential, which I do with ξ_{ab}

So an example is End_V, you need V a differential graded vector space and a symplectic form ω of degree -1 compatible with d, and then $\mathcal{P}(n,G)=Hom(V^{\otimes n,\mathbb{C}})=\mathcal{E}_V(n,G)$ and then ${}_a\circ_b$ and ξ_{ab} are just contracting inputs with ω. In the colored version you just have different pairis. The Q is some subspace of the state space and ω is the [unintelligible]pairing with insertions.

I should have said, trivially you can combine your modular Com and Ass to a two-colored modular operad I'd call *quantum open-closed*.

So algebras over $\mathcal{FP}$ are just morphisms $\mathcal{FP}\to\mathcal{E}_V$, the algebras with underlying vector space V.

There is a very nice theorem due to Barannikov which gives a nice description of an algebra over the Feynman transform, this is equivalentto $m(C,G)$ in $P(C,G) \otimes \mathcal{E}_V(C,G)$, and I take the invariants with respect to the symmetric group. This should satisfy some condition in order to define an algebra, and first of all, we have a differential on our modular operad induced from V, and a differential induced by by P, in our examples this was always trivial. One is from the left and the other from the right, and I can apply this

$$\underbrace{d_{\mathcal{E}_V} - d_{\mathcal{P}}}_{a} m(C,G) = \underbrace{(\xi_{ab})_{\mathcal{P}} \otimes (\xi)_{\mathcal{E}_V}}_{\Delta} m(C \cup \{a,b\}, G-1) +$$
$$\frac{1}{2} \sum_{C=_1 \cup C_2, G_1+G_2=G} \underbrace{({}_a\circ_b)_{\mathcal{P}} \otimes ({}_a\circ_b)_{\mathcal{E}_V}}_{\{,\}} (m(C_1 \sqcup \{a\}, G_1) \otimes m(C_2 \sqcup \{b\}, G_2)$$

and this is like $(\mathcal{P}, d, \Delta, \{,\})$ make a differential graded Lie algebra.

You can collect together these m into a generating function. You have a genus zero thing, you have here a solution to the quantum master equation $dS + \hbar\Delta S + \frac{1}{2}\{S,S\}$. Then you have a solution to a non-commutative Batalin–Vilkovisky master equation.

Now let's start to compare it with physics, do I still have some time left? We're just going to write this BV equation, since we're working in characteristic zero, if I take $(\mathcal{P}(n,G) \otimes Ee_V(n,G))^{\Sigma_n}$ I can identify invariants and coinvariants, taking this to

$$((\mathcal{P}(n,G) \otimes V^{\# \otimes n}))_{\Sigma_n}$$

via

$$\sum p_i \otimes \psi_i \mapsto \sum_i \sum_I \psi^i(a_I)(p_i \otimes_{\Sigma_n} \phi^I)$$

where I have a basis a_i for V and ϕ^i in $V^{\#}$, and $a_I = a_{i_1} \otimes \cdots a_{i_n}$, and I can just use this and look at what the master equation gives, and I'll show you on examples. So this ϕ is the string field. If you are in the Mod(Com^c) case, then S under this mapping is something that lives in the symmetric algebra of $V^{\#}[[\hbar]]$, and this is

$$S = \sum_{n,G} \frac{\hbar^G}{n!} \sum_I f_n^G(A_I)\phi^I$$

, the sum over multi-indices, and these are graded symmetric. The other way is that you can permute the products of ϕ^I and you have the graded commutative product in ϕ, using these indices. There you have a product, this is the usual Batalin–Vilkovisky equation. This is the case of closed string field theory.

Things become a bit more involved in the Ass case but what happens now you can imagine. I'd call Mod(Ass^c) things quantum A_∞ algebras, now

$$S = \sum g, b, \vec{m}\hbar^G \frac{1}{b!} \frac{1}{m_1 \cdots m_i} f^{g,b}(a_{I_1}, \ldots a_{I_b})\phi^{I_1} \cdots \phi^{I_B}$$

and now a_{I_i} is symmetric, graded, with respect to cyclic permutations within the boundary and permuting the boundaries. You can imagine what happens if you combine the commutative with the associative.

I don't know if other actions of symmetric algebras would be useful for things, here you just have Com and Ass. I erased my diagram of operads, but I guess the same basic statement should be true for a quantum field theory, but you put a different thing, graphs, metric, tropical, it should be related of course to the [unintelligible]approach to quantum field theory.

11. Ben Ward: Gravity algebras as obstructions

[I do not take notes on slide talks]

12. Jesse Wolfson: Higher determinants and double loop groups

This is joint ongoing work with J. Kaad and R. Nest.

The goal here is for the classical theory of loop groups, there's a really rich interplay between the representation theory and the operator theory related to scattering theory or gauge theory on the circle. We know very little for maps of any manifold of dimension greater than one into a Lie group.

Let me remind you of the one-dimensional story. I'll think of G as a Lie group compact or its complexification). So the loop group LG is smooth maps $S^1 \to G$. Then there's a central extension $\widetilde{LG}$ by $\mathbb{C}^\times$, and this arises from the restricted general linear group, the master group GL_∞ whenever you have a central extension, and $\widetilde{LG}$ arises from pulling back $\widetilde{GL}_\infty$.

Typically we'll be considering a Hilbert space, and have in mind $L^2(S^1, \mathbb{C})$, equipped in this setup with a polarization $\mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{H}_+$ will be the hardy space $H^2(S^1)$. We have $M_A = \{A \in \mathcal{L}(H) | [A, \pi_+] \text{ is Hilbert–Schmidt}\}$. Then GL_∞ is the units of this algebra. This is central to a number of things, and this is the starting point for investigations I want to move to higher dimensions. The animating question of this project is, if X is a manifold of dimension greater than 1, what can we say about smooth G^X.

It's been known for some time that if G is simple, then any central extension of G^X basically arises by pulling back the Kacs–Moody extension of the loop group for some $f : S^1 \to X$. But higher central extensions are much richer. This goes by an AKSZ construction where if we consider a class in, say $H^5(BG, \mathcal{O}^\times)$, then we can do this transgression construction,

$$\begin{tikzcd} X \times BG^X \arrow[r, "\mathrm{ev}^*"] \arrow[d] & BG \\ BG^X & \end{tikzcd}$$

and you can consider $\int_X \mathrm{ev}^* \omega \in H^3(BG^X, \mathcal{O}^\times)$ and its known for a class here you can associate a higher central extension. This is a group up to homotopy with fiber [unintelligible]lines, $1 \to B\mathbb{C}^\times \to \hat{G}_q^\times \to G^\times \to 1$.

So the goal is to define $L_{\infty,\infty} \subset M_{\infty,\infty} \subset \mathcal{L}(H)$ and give an operator theoretic construction

$$1 \to B\mathbb{C}^\times \to \widetilde{GL}_{\infty,\infty} \to GL_{\infty,\infty} \to 1.$$

As differential geometers we can do smooth loops; as algebraic geometers we can do formal loops. It's known recently, some things about operator theory for formal loop groups.

Let me say a little more in one dimension, bring out the operator theory a little more, and then turn to two dimensions.

The basic $1D$ story says, we have $\mathcal{H}$ equipped with a polarization $\mathcal{H}_+ \oplus \mathcal{H}_-$ and we'll look at invertible operators such that the commutator with π_+ is Hilbert–Schmidt. I'll use the notation of these Schouten ideals.

(1) First off, if you have two operators in the master group then the operator you get by projecting $B\mathcal{H}_+ \xrightarrow{AB} A\mathcal{H}_+$ is Fredholm.
(2) In this setup we also have that the commutator of these translates of this projection, this commutator $[A\pi_+A^{-1}, B\pi_+B^-1]$ and $A\pi_+A^{-1} - B\pi_+B^{-1}$ are L^2.

As consequences

(1) there's a holomorphic line bundle $\mathcal{D} \to GL_\infty \times GL_\infty$ with $\mathcal{D}_{A,B} = \det(B\mathcal{H}_+ \xrightarrow{AB} A\mathcal{H}_+)$
(2) Due to Carey–Pincus there's a torsion isomorphism $\tau : \mathcal{D}_{A,B} \otimes \mathcal{D}_{B,C} \xrightarrow{\cong} \det(C\mathcal{H}_+ \xrightarrow{ABC} A\mathcal{H}_+)$ nad there's also a perturbation isomoprhism ρ from this to $\mathcal{D}_{A,C}$.

Just to say off the bat, these isomorphisms have really nice properties.

The resulting map $\mathcal{D}_{A,B} \otimes \mathcal{D}_{B,C} \to \mathcal{D}_{A,C}$ is holomorphic in A, B, and C, is associative, and is natural with respect to multiplication by GL_∞.

Then you can build a $\mathbb{C}$-linear category in complex manifolds $\mathcal{C}$ which is going to give us this central extension. The objects are $\{A\pi_+A{-}1 | A \in GL_\infty\}$ and morphisms will be $\mathcal{D} \mapsto GL_\infty \times GL_\infty$ and $\rho \circ \tau$ as composition. And we see GL_∞ acts on $\mathcal{C}$.

Observe

(1) This category is a $\mathbb{C}^\times$-gerbe, that is, any A gives $B\mathbb{C}^\times \xrightarrow{\cong} \mathcal{C}$ by $[unintelligible] \mapsto A$
(2) (Brylinksi) If G acts on $\mathcal{X}$ then we get $1 \to \mathbb{C}^\times \to G_{\mathcal{X}} \to G \to 1$, and
(3) in our case $\widetilde{GL_\infty} \cong (GL_\infty)_{\mathcal{C}}$.

A lot of this is encoded in τ and ρ. A lot of this [unintelligible]is kind of deep. There's a theorem that computes the higher energy limit of Toeplitz operators on Hardy space, we only look at this on the first n eigenvalues, and let n get large. Using a careful study of the perturbation isomorphism you can give a good answer to that.

All of this in the background is a noncommutative differential geometry perspective. We know that this is essentially a one-dimensional story from Connes. The basic example is the circle $L^2(S^1, \mathbb{C})$ with its' diract operator $id\theta$ and a basic non-example is $L^2(\mathbb{T}^2, \mathbb{C})$ with $\pi_+ = \chi_{[0,\infty]}\not{D}$.

Algebraic geometry is going to be our guide. Let G be an algebraic group, reductive and let's say affine. A close cousin to $G^{\mathbb{T}^2}$ will be the formal double loop group $G((t_1))((t_2))$, which if this is a matrix group, you can think of matrices with entries two variable Laurent series. Then if V is a G-representation, then we can split $V((t_1))((t_2)) \cong V_{\infty,+} \oplus V_{\infty,-}$ where this is $V((t_1))[[t_2]]$ and $V((t_1))[t_2^{-2}]$. The operator theory here is increasingly well-understood, building on Tate, Beilinson, Yekutieli, Braunling–Groechenig–W. Let me say I started in smooth loop groups talking about AKSZ, and there's a nice paper of Pavel that says that Weyl invariant forms on the [unintelligible]lattice, cubic ones, classify not just extensions of the formal double loop group but also the third [unintelligible]K-theory sheaf, which is richer. So we have some idea that this should have some chance to have some good stuff going on.

So $M^{\mathrm{alg}}_{\infty,\infty}(R)$ is $\varphi \in \mathrm{End}_R(V((t_1))((t_2)))$ such that for all i and j, there is a $k > j$ and $\ell < i$ such that

$$\begin{tikzcd} t_2^{\ell}V_{\infty,+} \arrow[r] & t_2^{j}V_{\infty,+} \\ t_2^{i}V_{\infty,+} \arrow[u, "\subset"] \arrow[r, "\varphi"'] & t_2^{k}V_{\infty,+} \arrow[u, "\subset"] \end{tikzcd}$$

such that $\bar{\varphi} : t_2^i V_{\infty,+}/t_2^{\ell}V_{\infty,+} \to t_2^k V_{\infty,+}/t_2^j V_{\infty,+}$ is continuous.

So we can construct $\widetilde{GL^{\mathrm{alg}}}_{\infty,\infty}$ by analogy with [unintelligible].

So we construct a 2-category $\mathcal{C}^{\mathrm{alg}}$ such that

(1) the choice of $A \in \mathbb{C}^{\mathrm{alg}}$ gives $B^2\mathbb{G}_m \xrightarrow{\cong} \mathcal{C}^{\mathrm{alg}}$ taking $*$ to A
(2) $GL_{\infty,\infty}$ acts on $\mathcal{C}^{\mathrm{alg}}$, so by Brylinski, we have

$$1 \to B\mathcal{G}_n \to \widetilde{GL^{\mathrm{alg}}}_{\infty,\infty} \to GL_{\infty,\infty} \to 1$$

The objects are $A \in GL_{\infty,\infty}$.

Theorem 12.1. *(Braunling–Groechenig–W.) for A and B in $GL^{\mathrm{alg}}_{\infty,\infty}(R)$ there is a $\mathbb{Z}/2$-graded topologiacl R-module $\mathcal{D}_{A,B}$ which is smothing like the index of $BV_{\infty,+} \xrightarrow{AB} AV_{\infty,+}$ (up to diagonal summands)*

So $\mathcal{D}^{\pm}_{A,B} \cong_R R((t))$ and there's a canonical equivalence $\mathcal{D}_{A,B} \oplus \mathcal{D}_{B,C} \xrightarrow{\cong} \mathcal{D}_{A,C}$ and the assignment $A, B \mapsto \mathcal{D}_{A,B}$ is natural with respect to multiplication by elemnts in our group.

For fields, if we chose R a field, the knowledge was in the air for a long time. Our contribution was to figure out how to do this in families and still retain control of the operator theory.

For every pair of objects in this two category, we need a 1-category $\mathcal{M}(A,B)$, which will be given as a pair of automorphisms, one on the positive part and one on the negative part, $\mathrm{Aut}_{\mathrm{cts}}(\mathcal{D}^+_{A,B}) \times \mathrm{Aut}_{\mathrm{cts}}(\mathcal{D}^-_{A,B})$. If you have two such operators, there exists an R-line $d_{\varphi,\psi}$ for any $\mathcal{D}^{\pm}_{A,B} \cong R((t))$ so that

$$d_{\varphi,\psi} \cong \det(\varphi R[[t]] \xrightarrow{\varphi\psi} \varphi R[[t]])$$

This allows us to define our 2-morphisms

$$\mathrm{Hom}_{\mathcal{M}(A,B)}(\psi^{\pm}, \varphi^{\pm}) = d_{\varphi^+,\psi^+} \otimes d^{\vee}_{\varphi^+,\psi^+}.$$

Then this is a 2-category, you can check for yourself it's a two-gerbe of the appropriate type and $GL^{\mathrm{alg}}_{\infty,\infty}$ acts on it.

That sort of gives us our construction.

(1) the composition of one and two-morphisms encode baby analogues of the torsion and perturbation isomorphisms.
(2) At the level of sets, for $G = GL_n$, consider the subgroup $\mathcal{G}$ of the smooth double loop group whose terms are meromorphic in the second variable, whose coefficients are meromorphic functions on the disk,

$$A_{ij} = \sum_{n \geq N} f_n(\theta_1)\theta_2^n, f_n(\theta_1) \text{ meromrphic on } \mathbb{D}$$

and then we get a discrete 2-extension:

$$1 \to BC^{\times} \to \hat{\mathcal{G}} \to \mathcal{G} \to 1.$$

And if you switch from t-adic to C^∞ this turns out to be continous (this is more delicate but checks out)

Some problems:

(1) Smooth double loop groups are much more delicate. Pushing through continuity might work but it's not ultimately going to be the right way to consider this problem.
(2) Our class of operators is far too restricted. We have fewer operators, but we want something on the full set of operators on Hilbert space. From Connes we know we're missing lots of operators.
(3) You could ask about a surface other than the 2-torus.

The third is the least of our problems. The setup we expect is what Connes called a [unintelligible]Fredholm module, and locally, all these look like products $L^2(X), \not{D}$ so products plus gluing gives us this last part.

This is a heuristic for going from the second to the third. So let me end by discussing the current status.

For these first two problems take the formal case as a model. So I want to combine this with 3-summable Fredholm modules. I should put up a definition here, you have a C^*-algebra A and you have a Hilbert space with an idempotent $(\mathcal{H}, F)$ with $F^2 = 1$ and you require that for all $a \in A$, that $[a, F]$ is in the third Schouten ideal $\mathcal{L}^3$, they're cube summable.

We want to consider things in practice, F will be built from the spectrum of an unbounded operator. We'll consider product Hilbert spaces equipped with an operator $\not{D}$ which decompose as $\not{D}_1 \otimes 1 + 1 \otimes \not{D}_2$. Consider $E(H, \not{D}) = \{A \in \mathcal{L}(\mathcal{H}) | A - \not{D}^{-1} A \not{D} \in \mathcal{L}^3\}$.

Now we have two families of idempotents $\{P_a\}$ and $\{Q_b\}$ where these are bounded perturbations of the projection onto the spectra of $\not{D}_1$ and $\not{D}_2$ respectively.

Now construct $\mathcal{C}_\mathcal{H}$ with objects $\{Q_b\}$ and let one-morphisms (Q_a, Q_b) be pairs $\underline{P} = (P_{\text{ker}}, P_{\text{coker}})$ where P_{ker} acts on the kernel of $(Q_b\mathcal{H} \xrightarrow{Q_b Q_a} Q_a\mathcal{H})$. Then for $\underline{P}$ and $\underline{P}'$ we have $F_{\underline{P},\underline{P}'}$ and two-morphisms will be $\det(F_{\underline{P},\underline{P}'})$. This is work in progress, the analysis isn't pinned down, but with tweaking this should constrain our choices and guide us to a set of operators.

13. January 14: Sam Gunningham: Categorical harmonic analysis on reductive groups

Thanks for the invitation. It's the last day and it's the morning after the conference dinner so I'll start with something gentle. My toy example is gauge theory for a finite group.

How does this go? If you have Γ a finite group, then I can consider $\mathcal{M}_\Gamma(\Sigma)$; here Σ is, say, a topological space and $\mathcal{M}_\Gamma(\Sigma)$ is the space of Γ-local systems on Σ, maps from $\pi_1(\Sigma, \Gamma)/\Gamma$, modulo conjugation. This is the correct formula if Σ is connected; otherwise I could take functors from the fundamental groupoid to the classifying space of Γ.

By quotient I mean "take the action groupoid" so this is a groupoid, and as long as the fundamental group is finitely generated, this is a finite groupoid, that is, it has a finite number of isomorphism classes and a finite automorphism group of each.

Given any finite groupoid $\mathcal{G}$, we can define the *size* of $\mathcal{G}$, denoted $\#\mathcal{G}$, to be the number of objects scaled by automorphisms.

$$\sum_{x\in \mathrm{Ob}(\mathcal{G})/\sim} \frac{1}{|\operatorname{Aut}(x)|}$$

so the size of $\mathcal{M}_\Gamma(\Sigma)$, takes the size of the space of homomorphisms and dividing by Γ,

$$\frac{|\operatorname{Hom}(\pi_1\Sigma,\Gamma)|}{|\Gamma|}$$

and the case we're interested in, Σ is a closed oriented surface, and in that case, this $\mathcal{M}_\Gamma(\Sigma)$, this fundamental group has $2g$ generators and one relation. So I get $2g$ elements of Γ satisfying that relation,

$$\mathcal{M}_\Gamma(\Sigma) = \{(a_1,b_1,\dots,a_g,b_g) \in \Gamma^{2g} | [a_1,b_1]\cdots[a_g,b_g] = 1\}/\Gamma.$$

There's a really nice formula for this due to Frobenius, an early application of character theory.

Theorem 13.1. *(Frobenius)*

$$\#\mathcal{M}_\Gamma(\Sigma) = \sum_{V\in \underbrace{\hat{\Gamma}}_{\text{irreps of }\Gamma}} \left(\frac{\dim V}{|\Gamma|}\right)^{\chi(\Sigma)}$$

This is counting Γ-bundles on a closed surface, there are things you could do for punctures, with symmetric groups these are Hurwitz numbers, so this is counting those.

How can we derive this formula? I want to present a modern formulation.

There is a topological field theory (a functor from a bordism category to a target category, I'll explain in detail later, it eats topological spaces and spits out invariants) which to a closed surface Σ gives this number

$$\mathcal{Z}_\Gamma(\Sigma) = \#\mathcal{M}_\Gamma(\Sigma).$$

The key to understanding is cutting and pasting, so we want to understand what happens with a manifold with boundary. Associated to Σ with boundary $\partial_0\Sigma$ and $\partial_1\Sigma$, I get a span (induced by restriction)

$$\begin{tikzcd} & \mathcal{M}_\Gamma(\Sigma) \arrow[dl, "r_0"'] \arrow[dr, "r_1"] & \\ \mathcal{M}_\Gamma(\partial_0\Sigma) & & \mathcal{M}_\Gamma(\partial_1\Sigma) \end{tikzcd}$$

and we've seen similar things in lots of other talks. This is nice because everything actually works.

So these are my spaces of fields and my size function is like my path integral. So I want to take a function on the boundary, pull back and then push forward, to get a map

$$\mathbb{C}[\mathcal{M}_\Gamma(\partial_0\Sigma)] \xrightarrow{(r_1)_*(r_0)^*} \mathbb{C}[\mathcal{M}_\Gamma(\partial_1\Sigma)]$$

Let me remind you, a Γ bundle on the circle is a monodromy, up to Γ, this is the adjoint action of Γ on itself, $\mathcal{M}_\Gamma(S^1) = \frac{\Gamma}{\Gamma}$ so $\mathbb{C}[\mathcal{M}_\Gamma(S^1)] = \mathbb{C}[\Gamma]^\Gamma$.

So pushing forward is integration along the fibers, with this measure I just defined. If Σ has no boundary at all, I get a point on either side and I just get a map from the complex numbers to themselves.

I shouldn't spend too long on this as you're all familiar with it. I can look at the pair of pants, the cup, the cap and so on, and I get a commutative Frobenius algebra, and it's a familiar one, $\mathbb{C}[\Gamma]^\Gamma$, these numbers are determined by the data of this Frobenius algebra. The representation theory tells us the structure. It's also semisimple, it has a basis of idempotents e_V for $V \in \hat{\Gamma}$, and that determines its structure as an algebra. I need to tell you the trace of each idempotent. The trace of e_V, these are almost the characters of V, I believe it's $\left(\frac{\dim V}{|\Gamma|}\right)^2$. You can say, there's an isomorphism between $\mathbb{C}[\Gamma]^\Gamma$ and $\mathbb{C}[\hat{\Gamma}]$ which takes e_V to δ_V. So o each irreducible representation I associate this number.

Then computing the number is just an exercise in Frobenius algebras, just decomposing your surface into pairs of pants.

I want to think about this situation by thinking that $\mathcal{Z}_\Gamma$ can be refined to a topological field theory relative to $\hat{\Gamma}$, a sheaf over $\hat{\Gamma}$. To a closed surface I don't have just a number but an assignment of a number to each thing in $\hat{\Gamma}$. I don't get just a vector space for a circle but a vector bundle over $\hat{\Gamma}$.

This isn't what I really want to talk about, I want an analogous situation where I replace the finite group Γ with a reductive group.

Okay, so what's going to happen now? I'll replace Γ by G, a reductive group over the complex numbers, and it'd be fine to keep in mind $G = GL_n(\mathbb{C})$. Then $\mathcal{M}_\Gamma(\Sigma)$, I can still do $\mathcal{M}_G(\Sigma)$. Before, this has the same definition, $\mathrm{Hom}(\pi_1(\Sigma), G)/G$. This is in the same way a subvariety of a bunch of copies of G cut out by an equation, modulo the action of G, and I'll think of this as a stack, this is the character stack. You could call it the Betti moduli stack of G-local systems modulo Σ.

So before I could take a volume of this finite groupoid, in the finite group case. In this case, at least naively, G is noncompact so it won't make sense to take the volume of the group. There's a version for a compact group, we were talking about this yesterday, there's something you can do, described by Witten, I think it was called $2D$ Yang–Mills, then take the topological limit, this has the same formulas, that's a nice story but I'm not going to talk about it.

What I'm going to do is replace the size of $\mathcal{M}_\Gamma(\Sigma)$ with the Borel–Moore homology $H_*^{\mathrm{BM}}(\mathcal{M}_G(\Sigma))$. Just like size is additive with respect to a decomposition, Borel–Moore has a long exact sequence for a decomposition of your space, so that's some kind of analogue.

Let me at this stage mention a different approach, taken by Hausel–Rodriguez-Villegas, and other collaborators at different stages. What these guys do is count the points of $\mathcal{M}_G(\Sigma)$ over a finite field $\mathbb{F}_q$. They use the Frobenius formula I just mentioned for $G(\mathbb{F}_q)$ and G is always like GL_n or SL_n or whatever. They use all type A together, do a bunch of combinatorics. They are using the character table of $GL_n(\mathbb{F}_q)$ to say something about the Euler characteristic of [unintelligible]. The philosophy is that the number of $\mathbb{F}_q$ points is related to the complex homology.

Computing that character table in general often requires you to go to geometry. One way is due to Lusztig that uses character sheaves on the corresponding algebraic group. In some sense I want to circumvent the middleman and always work in the categorified setting.

[some questions about history]

For the symmetric group, maybe these formulas are due to Hurwitz a little earlier. I also should have mentioned, Dan Freed and also Dijkgraff–Witten based on Migdal (1975) for finite groups. The way I've expressed it here I learned from papers of Dan Freed.

I have no particular agenda so feel free to keep asking questions.

Before, $\mathcal{Z}_\Gamma$ was a functor from the bordism category of two-manifolds to vector spaces.

Proposition 13.1. *(Ben-Zvi–G.–Nadler–Oren) There is a topological field theory* $\mathbb{Z}_G$ *which assigns to a surface* $H_*^{\mathrm{BM}}(\mathcal{M}_G(\Sigma))$ *and assigns to* S^1 *instead of the vector space of class functions,* $D(G)^G$*, the category of* G*-equivariant* D*-modules on* G*. These are like class functions for* D*-modules. I'll explain in a second.*

This is a fully extended TFT. Previously to a point I'd assign maybe modules for the group algebra. Here to a point I assign $D(G)$*, all* D*-modules on* G*, with* $*$*, before that was an algebra but here this is a monoidal category. The* $*$ *is convolution.*

Let me say more precisely what I mean by this. This will be a functor from a bordism category $\mathrm{Bord}^{\mathrm{or}}_{(0,1,2)}$ to Alg(dgCat), the Morita category, with algebras, bimodules, and intertwiners. There should be an unoriented version as well.

Let me, before talking about D-modules, make some remarks about the proposition. Actually this has been a motivating example for me for a really long time. The idea of this was taught to me by David Ben-Zvi and David Nadler, we talked about it a lot. Until recently we didn't think there really was such a TFT, this is a little embarrassing, because the monoidal category is not rigid. The philosophy is if you start with something dualizable enough you get a TFT, and we didn't know but then it's Morita equivalent to something rigid, Harish–Chandra bimodules on G, these are a subcategory of $U\mathfrak{g}$-bimodules such that the diagonal action of $\mathfrak{g}$ is integrable, comes from an action of G. This is due to Gaitsgory, written up in Beraldo's thesis. This particular statement is being written up with Ben-Zvi, Nadler, and Oren (a student of Ben-Zvi). Both of these things are equivalent to a universal Hecke category $D(N\backslash G/N)$ (here for G a reductive group with B its Borel, this N is the unipotent radical so for $G = GL_n$ then Borel is upper triangular and N is upper triangular with 1 on the diagonal), this is in our forthcoming paper. This relates the monoidal category to things people are more used to studying in geometric representation theory.

Let me say some more about D-modules. The easiest example to think about, and my favorite example, is a torus, the reductive case. So the character variety or character stack is not very interesting, but it's helpful to have this example in mind. So let X be a smooth affine variety over $\mathbb{C}$ and $\mathcal{D}_X$ denote the ring of polynomial differential operators, for example for X an affine space this is called the Weyl algebra, this is a deformation of functions on the cotangent bundle. Then $D(X)$ is the derived category or a dg version of this, $D(\mathcal{D}_X - \mod)$. I should also talk about $D(X)^G$ which is $D(X/G)$, and if G is an affine algebraic group acting on X, so this is the G-equivariant derived category.

This kind of thing, this is not the derived category of G-equivariant $\mathcal{D}$-modules on X. Bernstein and Lands talked about this, also a recent book by Gaitsgory and Rozenblyum.

Let's say I have a torus $\mathbb{T}$ and for simplicity let's say it's $\mathcal{C}^\times$. Then what's $\mathcal{D}_T$L we have functions and differentiation, so it's $\mathcal{C}[x, x^{-1}, \partial_x]$ subject to the relation $[\partial_x, x] = 1, and so$ what's $\mathcal{D}_T x^\lambda$? (here λ is in $\mathfrak{t}^*$). This is $\mathcal{D}_T/\mathcal{D}_T(x\partial_x - \lambda)$.

I meant to mention that, what's another way of thinking about a $\mathcal{D}_X$-module? it's an $\mathcal{O}_X$-module with an action of vector fields in a compatible way, that's a quasicoherent sheaf on X with a flat connection. This $\mathcal{D}_T x^\lambda$ is a vector bundle with flat connection.

So generally the story I wanted to tell, I have this function x^λ and it gives you a $\mathcal{D}$-module. If you look at x^λ and $x^{\lambda+1}$ they give the same $\mathcal{D}$-module, there's a gauge equivalence. This only sees the functions up to some gauge equivalence.

Another thing, I can take $\mathcal{D}_T.\delta(x-a)$ for $a \in T$. Then the delta function satisfies a zero order differential equation $\mathcal{D}_T/\mathcal{D}_T(x-a)$.

This is an Abelian group so you expect a kind of Fourier transform. I'm running out of time, I don't know how much more I can say. The ring $\mathcal{D}_T$, I can think about it as first $\mathbb{C}[x\partial_x][x,x^{-1}]$, I put this subject to the same relations, but think about this as a polynomial ring $\mathcal{O}(\mathfrak{t}^*)$ with difference operators, acting by translation by integers. I can think about this as a quasicoherent sheaf on t^* with an action of translation operators.

In other words, $D(T)$ is equivalent to $QC(t^*)^\Lambda$ with $\Lambda = \mathrm{Hom}(T,\mathbb{C}^\times)$.

As you might expect, well, this thing is sometimes called the Mellin transform, and here I have a monoidal product given by convolution on $D(T)$, and on the other side $\otimes$ for quasicohorent sheaves. So $D(T)$ is a monoidal category under convolution and I want to say that its spectrum is $QC(t^*)$. This is like $\hat{\Gamma}$ from before, the place that the TFT lived over, it lives over t^*/Λ. The monoidal category I assign to a point is quasicoherent sheaf on this object. That's the sort of thing I want to do but for a non-Abelian group.

Let's see, what can I do in five minutes. Maybe I'll just sort of mention the basic idea of how it'll work.

For G non-Abelian, I fix a Borel subgroup, like upper triangular matrices in GL_n, and then N is like my upper triangular with 1 on the diagonal, and B/N is H, my diagonal matrices.

So I'm looking for modules for my group algebra, I want to think of $D(G)$ acting on, the easiest thing is $D(G/B)$, and I could also twist these, like actions on G/N with specified monodromy in the H direction. I'll write this $D(G/B)_\lambda$. For each $\lambda \in \mathfrak{h}^*$ I get the following:

- $\mathcal{H}_\lambda$ which is $\mathrm{End}_{D(G)}(D(G/B)_\lambda)$
- $e_\lambda \in D(G)^G$, an idempotent,
- $e_\lambda D(G)^G e_\lambda$ are character sheaves of Lusztig, and this is the central [unintelligible]
- I get a vector space $H_*^{\mathrm{BM}}(\mathcal{M}_G(\Sigma))$, and I can consider the "λ-part of it" and this is just a differential graded vector space.
- there's actually a TFT $\mathcal{Z}_{G,\lambda}$ that controls all this data. One λ at a time you get a TFT in this way. These Hecke categories here are more amenable to studying by the usual techniques of studying monoidal categories in representation theory, so there's a hope that these are more computable.

One thing to do is try to compute these invariants, and another is to see if you can recover the full moduli space from these λs, the spectrum is continuous, that's ongoing with Ben-Zvi and Nadler but I'm out of time so I'll stop here.

14. Dan Berwick-Evans: Effective field theories and elliptic cohomology

Thank you, it's been a great week here, thanks to the organizers.

I want to describe a relation between field theories and homotopy theory, specifically elliptic cohomology. The connection to field theory started, as it so often does, when Witten made an observation in the 1980s. He said you should think of elliptic cohomology as being roughly related to the S^1-equivariant K-theory of the loop space, that's like vector bundles on loop spaces, that's like quantum mechanics on the loop space, and that's like a $2D$ quantum field theory.

I just want to motivate this by playing games between these two. One nice thing about K-theory is everyone has their own way to think about it. I'll connect real K-theory and the universal elliptic cohomology TMF. To say what TMF is, you should say something about derived algebraic geometry.

How do you get KO? The multiplicative group has a $\mathbb{Z}_2$ action, and so you can think of complex K-theory as sitting over a point mod $\mathbb{Z}_2$, and then you have real K-theory. Similarly, for TMF you do the same thing but you have elliptic group laws, and global sections of some sheaf give you topological modular forms. This story is due to Hopkins and Miller with recent contributions due to Lurie.

Maybe that's not how you think of K-theory. Maybe you think of vector bundles. We don't know what does this for TMF, and this will be to try to get hints for what should go in this square.

To continue along, representation theory, equivariant K-theory reads off the representation theory of a group. Let me do complex with the caveat that you can deal with the $\mathbb{Z}_2$ action.

The representation theory on TMF is a little spooky [missed some] and there's a 2-group representation interpretation that is even spookier.

For analysis, what we want for the K-theory is twisted Dirac operators. Atiyah–Singer tells you how to compute the index of a dirac operator [unintelligible]. Witten's suggestion is to compute the Dirac operator on the loop space. This isn't meant literally, there seems to be something like the Dirac operator on the loop space and if you make it precise you land in physics.

	(derived) alg. geo.	diff. geo	rep. theory
KO	formal multiplicative group	vector bundles	$\mathrm{Rep}_G \cong K_G(\mathrm{pt})$
TMF	elliptic group laws	?	$TMF^\ell_G(\mathrm{pt}) \to \mathrm{Rep}^\ell(LG)$
	analysis	physics	
KO	Twisted Dirac	$1d$ $N=1$ SUSY QM	
TMF	Dirac on loop spaces	$2D$ $N=1$ QFT	

I'm going to start by talking about modular forms.

They keep giving back, most notably with Fermat's last theorem.

So MF_{2k} is functions in $\mathcal{O}(\mathcal{H})$ such that,

$$\mathrm{MF}_{2k} = \{f \in \mathcal{O}(\mathcal{H}) | f(\frac{ac+b}{cz+d}) = (cz+d)^k f(z); \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})\}$$

and I'm not enforcing conditions at ∞, some would call these weak modular forms.

I could also write this as sections of line bundles on a stack, $SL_2(\mathbb{Z})$ acting on the upper half plane,

$$\Gamma(\mathcal{H}//SL_2(\mathbb{Z}); \omega^{\otimes k}) \cong \Gamma(\mathcal{M}_\ell; \omega^{\otimes k})$$

So there's periodicity, there's a modular discriminant $\Delta \in \mathrm{MF}_{24}$ which is invertible so we have a periodicity

$$\mathrm{MF}_\bullet \xrightarrow{\Delta} \mathrm{MF}_{\bullet+24}$$

which will come back in TMF.

Let me say something about q-expansion, we have a map of stacks $\mathcal{H}//\mathbb{Z} \to \mathcal{H}//SL_2(\mathbb{Z})$, this takes $n \mapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ which gives a map $MF_\bullet \to \mathbb{C}((q))$ where $q = e^{2\pi i\tau}$ for $\tau \in \mathcal{H}$.

For integral modular forms, I can make $\mathrm{MF}^{\mathbb{Z}}$ as the pullback

$$\begin{tikzcd} MF^{\mathbb{Z}} \arrow[r] \arrow[d] & \mathbb{Z}((q)) \arrow[d] \\ MF \arrow[r] & \mathbb{C}((q)) \end{tikzcd}$$

Now let me talk about topological modular forms, first as a ring. These refine $\mathrm{MF}^{\mathbb{Z}}$ by a *ring spectrum* (in the sense of homotopy theory) via a map $\pi_* \mathrm{TMF} \to \mathrm{MF}^{\mathbb{Z}}_*$, this is a rational isomorphism but has kernel and cokernel, lots of 2-torsion and 3-torsion, homotopy theorists go wild for this, and there's maybe interesting number theory connections, so an interesting thing to study.

One of the more striking things is what happens to periodicity. It turns out that Δ is not in the image of π_* but Δ^{24} is and gives a Bott element, i.e., TMF is $24^2 = 576$-periodic. You're supposed to think in analogy that real K-theory is 8-periodic. This 576 is much bigger than 8 but we think QFT is more complicated than quantum mechanics.

So let me talk about a topological version of q-expansion, that was studied by [unintelligible]. So you can take a generalized Chern character, which goes $\mathrm{TMF} \to \mathrm{TMF} \otimes \mathbb{C}$, and this has a reasonable de Rham interpration. Then you can evaluate at the Tate curve, and you have a formal neighborhood, and so evaluation there gives you a map $\mathrm{TMF} \to K((q))$, and there's a compatibility condition that says if you take the usual Chern character in K-theory, or take q-expanison in $\mathrm{TMF} \otimes \mathbb{C}$, I get a map to $H\mathbb{C}((q))$. Life would be very easy if this were a pullback. I'll name the pullback, but this is just my name for it, I'll call it KMF and by the universal property receives a map from TMF. I'll try to understand KMF, $\mathrm{TMF} \otimes \mathbb{C}$, and hardest, TMF, using field theory.

$$\begin{tikzcd} \mathrm{TMF} \arrow[dr] \arrow[drr, "\mathrm{ev}_{\mathrm{Tate}}"] \arrow[ddr, "\mathrm{Ch}"'] & & \\ & \mathrm{KMF} \arrow[r] \arrow[d] & K((q)) \arrow[d, "\mathrm{Ch}"] \\ & \mathrm{TMF} \otimes \mathbb{C} \arrow[r, "q\text{-expansion}"'] & H\mathbb{C}((q)) \end{tikzcd}$$

The big picture of the big picture is that I want to interpret something like this:

$$
\begin{tikzcd}
 & \{\text{string mflds}\} \arrow[ddl, "(D)"'] \arrow[dddl, "(C)"'] \arrow[ddr, "\text{Witten}"] \arrow[dr, "\sigma"] & \\
\color{red}{\{2-\text{extended } 2|1-\text{EFT s}\}} \arrow[rr, dotted, "(D)"] \arrow[d, dotted] & & \text{TMF} \arrow[d] \\
\{1-\text{extended } 2|1-\text{EFT s}\} \arrow[rr, "(B)"'] \arrow[d] & & \text{KMF} \arrow[d] \\
\{0-\text{extended } 2|1-\text{EFT s}\} \arrow[rr, "(A)"'] & & \text{TMF} \otimes \mathbb{C}
\end{tikzcd}
$$

There's one more ingredient, Witten gave something that goes from string manifelds via quantization of nonlinear SUSY sigma models to the 1 and 0 extended versions. On the homotopy side, there is the σ orientation in TMF due to Ando–Hopkins–Strickland. That's one thing that's an important aspect of this proposal. Witten gave the KMF version. I should say this red part was suggested and worked on by Segal and Stolz–Teichner.

So let me start with a sketch of (A) and (B). So my fields will be maps $\phi : \mathbb{R}^{2|1} \to M$. You have coordinates z and $\bar{z}$ and some vector fields θ related to $\bar{\partial}$ and with this you can write down an action, I'll do a Lagrangian density; I don't really mean to take a non-compact integral

$$\int_{\mathbb{R}^{2|1}} \langle \partial_z \phi, (\partial_\theta + \theta \partial_{\bar{z}})\phi \rangle$$

For (A), consider energy zero tori, denote this by $\Phi_0^{2|1}(M)$ this is a lattice Λ and a map ϕ from $\mathbb{R}^{2|1}$ mod the lattice to M with the condition that the energy is zero.

This looks roughly like a moduli stack of elliptic curves. The odd part maps to M, this is the odd tangent bundle whose functions are differential forms. So this is like $\mathcal{M}_\ell \times \underline{\text{SMfld}}(\mathbb{R}^{0|1}, M) \cong \mathcal{M}_\ell \times \pi TM$

Theorem 14.1. *(B.-E.)*

$$\mathcal{O}(\Phi_0^{2|1}(M))/ \sim \cong \text{TMF}(M) \otimes \mathbb{C}$$

For (B), you can also take a category of energy zero annuli $\phi : \mathbb{R}^{2|1}/\mathbb{Z} \to M$. Denote this by $\Phi_1^{2|1}(M)$. There's some compatibility with the tori that you have before if you cut a torus into annuli. You consider a filtration of finite dimensional representations and only do geometry on finite dimensional pieces.

Definition 14.1. A 1-extended 2|1-Euclidean effective field theory over M is a filtered representation of this category $\Phi_1^{2|1}(M)$, called $F_\bullet \in \text{Fun}(\Phi_1^{2|1}(M), \text{Vect}_{\text{fd}})$ such that

(1) (effectivity) the filtration has increasing energy with respect to the S^1-action on annuli, and
(2) (modularity) the character of this representation gives a function on energy zero tori, $Z \in C^\infty(\Phi_0^{2|1})(M)$.

Let me give an idea about how these look and then state a theorem for you.

- I have
$$\bigoplus_{n=-N}^{\infty} q^n V_n$$
such that $\sum a^n \dim(V_n)$ is an integral modular form, gives a 1-extended effective field theory over M a point.
- (subexample) the moonshine module, and you get the J-invariant [unintelligible].
- More generally, you could have these sequences with an odd operator (now they're $\mathbb{Z}_2$-graded spaces), like a cutoff of an odd operator
- More generally, you have vector bundles now, over a space, with a Quillen superconnection for each n. The modularity condition is no longer an integral modular form, now the Chern character has to be a differential [unintelligible]valued in modular forms.

This matches up with the definition of KMF, you have this agreement, this condition of the Chern character. Let me emphasize that q is not a formal parameter, an actual size of a bordism. There's a lot of geometry behind the scenes. I might not have time to talk about it but it's there. The theorem, you can probably guess.

Theorem 14.2. *(B.-E.)*
$$\{1 - \mathit{extended} 2|1 - \mathrm{EFT}\, s/\sim\} \cong \mathrm{KMF}(M).$$

I'm flexible about quantization but let me tell you about what I mean. I learned my ideas from Costello, he'll tell you they're older. If I phrase these things as energy zero fields, there's a normal bundle to that and you can do perturbative quantization, let me tell you more about what I mean.

There is an inclusion
$$\Phi_*^{2|1}(M) \overset{i}{\hookrightarrow} \{\text{all fields}\}$$
and you get a normal bundle which I'll call $\nu(i)$, but you get more than that, you get a linearization of the classical action. Then you get a free field theory. The action is quadratic. In each fiber we can quantize although things can be interesting in families. So we get a $\Phi_*^{2|1}(M)$-family of free field theories. You use determinants, in fact ζ-determinants, and these are sections of a line bundle, and you need to globalize this, which is where the string condition comes in.

Now for (C):

Theorem 14.3. *(B.-E.) For M oriented and $P_1^{\mathbb{C}}(M) = 0$ (the first complex Pontryagin class), then a 0-extended fiberwise quantization (ζ-determinant) constructs a relative volume form on $\Phi_0^{2|1}(M)$ such that*

$$\begin{tikzcd} C^\infty(\Phi_0^{2|1}(M)) \arrow[r, "\int"] \arrow[d] & C^\infty(\Phi_0^{2|1}(\mathrm{pt})) \cong \mathrm{MF} \arrow[d] \\ \mathrm{TMF}(M) \otimes \mathbb{C} \arrow[r, "\sigma\otimes\mathbb{C}"'] & \mathrm{TMF}(\mathrm{pt}) \otimes \mathbb{C} \end{tikzcd}$$

so this map commutes by which you can see that $\sigma \otimes \mathbb{C}$ takes 1 to the Witten genus of M.

You're modifying integration of forms by some function on the stack and that's what gives you the integration map on the top.

Finally, for (D), this is in progress but I'm confident it will work out

Theorem 14.4. *(B.-E., in progress) Fer M spin and $P_1^{\mathbb{C}}(M) = 0$, a 1-extended fiberwise quantization (a linear geometric quantization) constructs the Witten genus of M as an* integral *modular form. The spin property gives you exactly that thing.*

Great, I have a few extra minutes.

Let me finish talking about something a little different, talking about something less obvious from the homotopic side, which is an equivariant version. TMF is homotopy theory, constructing equivariant extensions is difficult, but from the field theory point of view, there's something really obvious to try, gauging your theory. There are some sophisticated candidates from other points of view, and it would be interesting to compare them to what this outputs.

Let's play the game again, starting with a gauged sigma model. Let's do the version for G a finite group, the version for compact is *way* more complicated.

We have our fields principal G-bundles over M with a map to $\mathbb{R}^{2|1}$. That's the same as maps from $\mathbb{R}^{2|1}$ into the quotient stack $M//G$. There's a computation that's not totally trivial but it's a follow-your-nose sort of thing, sub $M//G$ into the previous theorems and get new ones. We can also twist by $\alpha \in H^4(BG;\mathbb{Z})$.

Theorem 14.5. *(B.-E.)*

$$C^\infty(\Phi_0^{2|1}(M//G))/\sim$$

is Devoto's equivariant elliptic cohomology over $\mathbb{C}$. Classes α give twists for this theory.

Let me mention just one more, the one-extended case.

Theorem 14.6. *(B.-E., in progress) 1-extended $2|1$-EFTs over $M//F$ give an equivariant refinement of* KMF(M) *compatible with Ganter's model of equivariant Tate K-theory.*

Let me mention that $K_G^{\mathrm{Tate}}(\mathrm{pt})$ contains representation theory of the twisted Drinfeld double (twisted by α). I hope I've convinced you that some of this will eventually change from red to white.

15. Dev Sinha: Hopf invariants, rational homotopy theory, and physical integrals

So thanks to the organizers, especially because, I don't know who misread the title of the conference, them or us, for me, it's more like quantum field theory methods in homotopy theory, just a simple permutation, I don't know if there's a sign we need to introduce.

- I'll talk about a classical perspective, homotopy and linking invariants, and we'll see that
- this is one of many problems where combinatorial descriptions of homology and cohomology of configuration spaces come into play, and that's of independent interest, because in various cases the Feynman diagrams come up in the combinatorics of these configuration spaces, really the disks operad.
- I'll talk about Koszul duality between Lie and commutative algebras, and
- being a little future-focused I'll focus on Chern–Simons theory, connected with work that Alberto and Pavel have done and are doing, maybe with L_∞ models,

- and then what we think of should be the replacement of the Koszul duality of Lie and commutative, so commutative becomes E_∞ and the Lie cooperad is the rational manifestation of the Goodwillie tower of the identity. At the end I'll talk about how I maybe see things as playing out.

The basic question, given $f, g : S^n \to X$, I want to know if f is homotopic to g, let's say rationally, after multiplying by some integer.

We can compute $\pi_*(X) \otimes \mathbb{Q}$ if X is simply connected (I'll make that assumption throughout) as the indecomposables V in a Sullivan model $(\wedge V, d)$ for X. You know how many they are, so it's like doing knot theory, you can make a table, but then trying to identify something in that table, that's a different question.

I want to talk about something a little different motivated by work I've done in knot theory.

What's the first example. If I have $[\omega] \in H^*(X)$, then I can evaluate it

$$\int_{S^n} f^*(\omega)$$

after pulling back. That's an invariant. If I have two maps they'll pull back cohomologous cocycles, but here's another example due to J. C. Whitehead (37), this has many more citations in the physics literature than the math literature. To such an $f : S^3 \to S^2$, I think of $[\omega]$ which generates $H^2(S^2)$, and I pull it back to S^3, I get an antiderivative, and then I can make a number

$$\int_{S^3} d^{-1} f^*(\omega) \wedge f^*(\omega)$$

and Whitehead said this is the Hopf invariant. This tells you if f and g are homotopic.

Let me give one more example on similar lines. Let X be a 4-manifold, and W^2 and V^2 in X, with no boundary, and they may intersect, but that can be trivialized, $W \cap V = \partial T$. I'll take forms on a tubular neighborhood. So take forms ω, ν, and θ Then I claim that the integral

$$\int_{S^3} \phi^{-1} f^* \omega \wedge f^* \nu + f^* \theta$$

is invariant.

So if I look at the preimages of V and W, these are 1-manifolds. Then $d^{-1} f^*(\omega)$ will live on a disk bounded by this preimage. Then around some particular point of intersection of this with $f^*\nu$ you get the concentration of the form $d^{-1} f^* \omega \wedge f^* \nu$. When I pull back θ it's concentrated on a point.

When I do this first bit here, by transversality, the preimage of $W \cap V$ would be empty, it's codimension 4. In $S^3 \times I$, at some point, there's a cobordism here, and the links can unlink, because this is the preimage of W, of V, and that can happen if four degrees of freedom. But the condition that the boundary of T is the intersection, that comes in and cancels the term at that point. We have a +1 and a −1 that come in and cancel. This computes a linking with correction, that's the geometry of this story. With a choice of trivialization, you get something invariant.

Okay. In order to iterate this well, you want to keep track of everybody, and that's where this language of the Lie operad and cooperad come into play. Let me tell you, give you a digression, about H^* and H_* of the disks operad. This is well-known to be the Poisson operad, but let me tell you how that works out geometrically. I have labelled points in the disk, or in Euclidean space, this is configuration of three points by $\mathbb{R}^d$, take the product and remove the fat diagonal where any two points are equal.

An element of the Poisson operad could look like this

$$[[x_2, x_4], [x_1, x_5]]x_3$$

and how does that look as a homology class? It will be the image of a homology class of a torus. The circles will parameterize angles. So points 2 and 4 will orbit a midpoint with parameter S^1, something similar happens with 1 and 5, and then the midpoints orbit each other, which is parameterized by the third S^1. Then point three doesn't get to dance with anyone else. This is a 3-dimensional manifold, and take its fundamental class and you get this Poisson element.

Let me also tell you about *cohomology.* This goes back to Arnold, you see these basic forms wedged together. I'd rather take the numbers and look at them as a graph.

$$\alpha_{14} \wedge \alpha_{42} \wedge \alpha_{15}$$

$$\begin{tikzcd}
 & 4 \arrow[dr] & \\
1 \arrow[ur] \arrow[dr] & & 2 \\
 & 5 & 3
\end{tikzcd}$$

and that's Poincaré dual to a picture that looks like this

$$\begin{tikzcd}
2 \arrow[d, no head] & \\
4 \arrow[d, no head] & \\
5 \arrow[d, no head] & \\
1 & 3
\end{tikzcd}$$

and the evaluation involves a map $\beta_{g,t}$ from the edges of g to the internal vertices of t called the meet or greatest lower bound of leaves. Then

$$\langle g, t\rangle = \left\{ \vphantom{\begin{matrix} a \\ b \end{matrix}} \right\} \begin{matrix} \pm 1 & \beta_{g,t} \cong \\ 0 & \text{otherwise} \end{matrix}$$

So for example, [pictures]

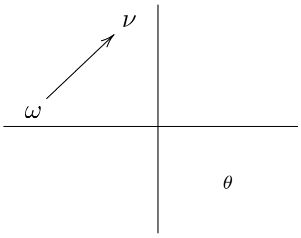
With the geometry you have, you can just work this out. You look for special planetary alignments, and those satisfy this combinatorial rule. Then this civilization and their astrology have the Jacobi identity at the center of their belief system. If $t_1 + t_2 + t_3$ is 0 by Jacobi, then $\langle g, t_1 + t_2 + t_3\rangle = 0$ and similarly if $\langle g_1 + g_2 + g_3, t\rangle = 0$.

This respects the quotient $\langle \text{Trees}\rangle \to \text{Poiss}$ and $\langle \text{Graphs}\rangle \to \text{Poiss}^\vee$. This gives a perfect (non-Kroneker) pairing. I emphasize this to say that some things are easier to see on one side than the other. So Poiss contains the Lie operad Lie, whereas Poiss has Eil, there's a presentation, not a formal duality.

Now let me tell you about the Hopf invariants more generally. The right context is a standard duality due to Quillen that fits into Koszul–Moore duality. You have differential graded commutative algebras and differential graded Lie coalgebras. You have bar constructions $\mathrm{Bar}_{\mathrm{Com}}$ and $\mathrm{Bar}_{\mathrm{Lie}^\vee}$, and I'll use $\mathrm{Bar}_{\mathrm{Eil}}\, C^*_{\mathrm{dR}}(X)$. The pieces will be

$$\bigoplus_n \mathrm{Eil}(n) \otimes_{S_n} (C^*(X))^{\otimes n}$$

and I claim that my example from before looks like [picture]



This is a double complex and both of these have differential $\omega \wedge \nu$. Then I pull back by f^* everywhere. I can evaluate $f^*\theta$ on the fundamental class. I don't know what I do with the pullback of the other thing. If I take $d^{-1} f^*\omega \to f^*(\nu)$. The other term is [unintelligible], which is homologous to $f^*\theta$, and now I can integrate this.

So in other words what I want to do is define integration in the bar complex

$$\int_{\mathrm{Bar}_{\mathrm{Eil}}[S^n]} \gamma \coloneqq \int_{S^n} \tau_\gamma$$

where $\tau_\gamma \sim \gamma$ is of weight one. Then the theorem is

Theorem 15.1. *(S.–Walter) The map*

$$HI : H_*(\mathrm{Bar}_{\mathrm{Eil}}\, C^*_{\mathrm{dR}}(X)) \to \mathrm{Hom}(\pi_*(X), \mathbb{Q})$$

given by

$$\gamma \mapsto (f \mapsto \int_{\mathrm{Bar}_{\mathrm{Eil}}[S^n]} f^*\gamma)$$

is a $\cong$

The cobracket structure is remove edges. By the Arnold identity, if you don't have a tree, you get zero. If you remove an edge you get two terms. That cobracket is linear dual to the Whitehead bracket on rational homotopy groups. The evaluation on a bracket is given by this pairing with homology cohomology evaluation.

Let me make another remark. We can perform this for $H_*(\mathrm{Bar}_{\mathrm{coAss}}(C^*(X, \mathbb{Z})))$. Define $HI_1(X)$ as $\mathrm{Hom}(\pi_*(X), \mathbb{Z})/\mathrm{im} HI$. This is a homotopy invariant and it should be difficult.

In standard linking theory we can choose the d^{-1} in the manifold setting, concentrated about submanifolds.

So I'll take $S^4 \to S^2 \vee S^2 \vee S^2$, and we'll pull back three surfaces in S^4 (or really $\mathbb{R}^4$), disjoint and what we can do then is count the number of times that points on each surface sit right underneath each other. It counts these kinds of coincidences. That's a very geometric picture of telling maps apart up to homotopy.

[some discussion of stable homotopy theory]

So Alberto and Pavel have essentially written these integrals down in a very different context. I am not qualified to discuss the context, but I have some idea. They want Chern–Simons invariants for general three-manifolds, the input is a Lie algebra, and they work in the BV formalism, get some solutions of the master equation and eventually they write down integrals where the data is very similar kinds of things. Cocycles come from a graph complex. The operations are either d^{-1} or wedge, and they write down integrals that seem to coincide with these.

Let me write down some speculations. I want mapping invariants. Most applications of field theory to topology has been at the homeomorphism or diffeomorphism level, so I want something a priori easier, using something like Chern–Simons theory with $\mathfrak{g}$ some L_∞ model for the target but an arbitrary domain. What I wonder, given that their things specialize to mine in some cases, maybe these can tell us about the maps $[X, Y]$. So field theory always starts with a space of sections. One technical difficulty is that $[X, Y]$ is a set, not a vector space. When you have $[S^n, Y_{\mathbb{Q}}]$, by adjoint you have a loop space and then that gives a linear structure. You could do naive things like take linear spans, but I think you'll have to do something different.

Then you do some perturbative things, then the conjecture is that you should get something that for the sphere specializes to this, and for $[X, Y]$ gives you something else.

So my dream, I hesitate to write things down, but if we get a good picture through this technology, which is speculative, then you could take $C^*_{E^\infty} \otimes$Goodwillie Tower of Y, and one could maybe start seeing something that's not strictly rational. A very modest start for a sphere in Y. But a concrete question, I'd be very interested to say, can one set up some field theory to give these integrals that I know are sharp.

Iterated integrals give complete invariants but

- The domain of the integral is a sphere cross an auxilliary space.
- The indeterminacy is "external;" philosophically you're using the coassociative operad, not coLie. I think that was Haynes' observation.

I wanted to hint that if this works very nicely, it's a different question, you have charts to compute homotopy groups of spheres. There's lots of framed bordism invariants that we spit out in industrial quantities. This is the start of a geometric interpretation.

16. Ryan Grady: Perturbative QFT from derived stacks

I thought we should say 감사합니다 to the organizers and the staff for a nice week. Si said that the tradition of the last talk is to summarize the others. I'll refuse to do that but hopefully I'll make contact with some of the other talks through the week. I'll talk some about a particular type of derived stack, L_∞ spaces.

(1) I'll present derived stacks.
(2) Then I'll talk about geometric constructions, like vector bundles, characteristic classes, sypmlectic structures, and so on, and this is a good place to play for that.
(3) Then you can feed this stuff into BV theories. These are some good reasons to consider this kind of thing.

The first thing I want to define, I'll fix a differential graded commutative algebra and a choice of nilpotent ideal, we've seen L_∞-algebras show up, you could define a curved L_∞ algebra over such a pair.

Definition 16.1. A *curved L_∞ algebra* over $(\mathcal{A}, I)$ is a graded vector space V (a locally free module over $\mathcal{A}$) with a cohomologically degree 1 operator on $\mathrm{Sym}(V[1])$ such that

(1) $d^2 = 0$ and
(2) (mod I), the differential d vanishes on Sym^0.

For us let $(\mathcal{A}, I)$ be $(\Omega_X^*, \Omega_X^{\geq 1})$. I should say that L_∞ often show up as extensions of differential graded Lie algebras, controlling formal moduli problems. So you can think that I'm pasting together formal moduli problems over a base X.

That's sort of the first connection to point 1. So:

Definition 16.2. An *L_∞ space* consists of a $\mathbb{Z}$-graded vector bundle V on X and an L_∞ structure on $g = \Gamma(V)$ over $(\Omega_X^*, \Omega_X^{\geq 1})$.

Let me give some examples.

0 An L_∞ algebra that's not curved is an L_∞ space on a point.
1 So we could have $\mathfrak{g}_X$ as $\Omega_X \otimes_{C_X^\infty} T_X[-1]$. Any time you have an L_∞ space you have a notion of cochains, and in this case, cochains on $\mathfrak{g}_X$ are a resolution of smooth functions on the manifold. These is a resolution of X as a smooth manifold over its own de Rham complex. You might think of this as a way of encoding smooth manifolds into L_∞ spaces.
2 Now let X be complex. then $\mathfrak{g}_{X_{\bar\partial}} = \Omega_X \otimes_{C_X^\infty} T_X^{1,0}[-1]$ and then this is a resolution of $\mathcal{O}_X$.
3 Say $L \xrightarrow{p} T_X$ is a Lie algebroid, then I get an L_∞ space and $\mathfrak{g}_L = \Omega_X \otimes_{C_X^\infty} (T_X[-1] \oplus L)$, adding in the extra bundle L. I'm not telling you the extra differentials, but this is a resolution of $C^*(L)$.

So all the others are specializations of this third example.

Let me give you a bit of flavor for how to construct these, a sort of amuse-bouche so on a long plane flight you can cook up your own. How do I demonstrate that something is an L_∞ space? I need a differential on the L_∞ cochains of this guy, let me do it for example (1). So I need a differential on $\Omega_X \otimes_{C_X^\infty} \widehat{\mathrm{Sym}}_{C_X^\infty}(T_X^\vee)$. How do I define this? Choose some connection ∇ on the tangent bundle T_X, then what does this do? It gives me a splitting of a conical quotient map from the first filtration of jets (by the order of vanishing) $F^1\mathcal{J} \to \Omega_X^1$. This connection gives me a splitting (in fact the space of splittings is modeled on the choice of connection) which leads to an isomorphism between $\widehat{\mathrm{Sym}}(T_X^\vee)$ and $\mathcal{J}$ the infinite jet bundle. Why does this help me solve my problem? $\mathcal{J}$ has a canonical connection, the Grothiendieck connection ∇^{Gr}, so that flat sections are smooth functions. What can I do? Use the de Rham construction, tensor with forms, and put the de Rham differential on the connection, to equip the tensor product with a connection, and the result is that $\Omega X \otimes_{C_X^\infty} \widehat{\mathrm{Sym}}(T_X^\vee)$ has this d built out of the connection and the de Rham differential.

If you know, you can try to do something similar using the holomorphic jet bundle for the complex case or jets for the third.

Remark 16.1. I actually could have used a different commutative differential graded algebra and nilpotent ideal in this picture. So you could have chosen C_X^∞ and 0 or various quotients of this thing, $(\Omega_X^{0,*}, \Omega_X^{0,\geq 1})$ if X is complex, or $(C^*L, C^{\geq 1}L)$. All of these lead to different notions of L_∞ space. Choose depending on what structure you want on your formal moduli problems. So in the first case you have a smooth parameter space and you have points for all your formal moduli problems. Now here things fit into a flat family. In the Dolbeault resolution you get a holomorphic family.

Let me give you more examples.

4 Suppose you have an embedding of complex manifolds $(X, \mathcal{O}_X) \hookrightarrow (Y, \mathcal{O}_Y)$. Then there's a construction of Shilin Yu that says that $\Omega^{0;n}_X(N[-1])$ (where N is the normal bundle) is an example, where this controls infinitesimal deformations of the normal bundle.

5 Take $\mathfrak{g}_{X_L}$ as $C^{\#}L \otimes_{C^\infty} L[-1]$ given a Lie algebroid. This is an L_∞ space over the pair $(C^*L, C^{\geq 1}L)$, and I'm resolving this thing over the base of the Lie algebroid itself.

Many of these examples are inspired by Kapranov. Anything that hasn't been done forty years ago has been done with Owen. There's a ton of examples, many probably already preexisting. Hopefully I've convinced you this isn't a vacuous thing. There's a lot of wiggle room.

I want to address point one in my outline, how do these things lead to derived stacks? We can evaluate a functor of points, they have points not only for their own L_∞ spaces $(X, \mathfrak{g})$ but on [unintelligible]. Let's go back to the original definition where the pair is the de Rham complex and the ideal generated by one-forms. Then we can define a functor from a category of nilpotent dg maniflods to simplicial sets

$$B_{(X,\mathfrak{g})} : \mathrm{dgMan}^{\mathrm{op}}_{\mathrm{Nil}} \to \mathrm{sSets}$$

which to $(M, \mathcal{O}_M, I_M = \ker(\mathcal{O}_M \to C^\infty M))$, this $\mathcal{O}_M$ is a unital differential graded algebra over de Rham complex of M. Just put whatever you want as your test spaces for [unintelligible]and that's what you put there, and check compatibility.

Let me give you the n simplices, I should check degeneracies but I'm not going to do that, this is a pair (f, α) where $f : M \to X$ is a smooth map and α is a Maurer–Cartan element in $f^*\mathfrak{g} \otimes_{\Omega^*_M} I_M \otimes_{\mathbb{R}} \Omega^*(\Delta^n)$.

You can see the simplicial structure from the n-simplex. You don't have to worry about an infinite sum since I is nilpotent. The theorem, which goes at least back to Costello, although I like to think we added some clarity, is that this defines a derived stack

Theorem 16.1. *(Costello, G.–Gwilliam) $B_{X,\mathfrak{g}}$ defines a derived stack.*

This is what I meant by presenting a derived stack. I should say that over a point lots of other people had already done this stuff, Getzler, Hinich, others. Smearing it out is slightly new, I gess.

So I want to talk about geometric constructions on $(X, \mathfrak{g})$. So for vector bundles, so that should be W sitting over X and an L_∞ splitting of the identity $g \hookrightarrow \mathfrak{g} = \ltimes W \to \mathfrak{g}$ where the brackets vanish if you have more than V. So the example, there are a slew, but you might be interested in $T_{B_\mathfrak{g}} = \mathfrak{g}[1]$ and $T^\vee_{B_\mathfrak{g}} = g^\vee[-1]$. This allows you to define forms, so you can define forms, and they're sections of the exterior powers

$$\Omega^k_{(X,\mathfrak{g})} = C^*(\mathfrak{g}, \wedge^k g^\vee[-k]).$$

For Lie algebroids, there are graded vector bundles for Lie algebroids. If you study representations up to homotopy on the Lie algebroid, there is a faithful functor to vector bundles over the corresponding L_∞ space. I'll define symplectic and shifted symplectic structures and that will become even more apparent.

So an n-shifted symplectic structure on $(X,\mathfrak{g})$ is a closed 2-form (I should put in quotes here that closed two-forms, it's truncated and shifted but being closed is not a property, it's data), and it's nondegenerate, the underlying two-form gives me a weak equivalence between $T_{(X,\mathfrak{g})}$ and $T^\vee_{(X,\mathfrak{g})}[n]$, the tangent and shifted cotangent bundles and one nice output, there were these examples of n-symplectic Lie algebroids, those are rigid, not homotopical.

Proposition 16.1. *If $L \xrightarrow{p} T_X$ is an n-shifted symplectic Lie algebroid (e.g., the Lie algebroid associated to a Poisson manifold, where L is a Courant algebroid) then $(X,\mathfrak{g}_L)$ is an n-shifted symplectic L_∞ space. There's potentially more in the L_∞ space.*

This gives one place to study n-shifted symplectic things in L_∞ algebroids in an invariant way. People in the Weinstein school, [unintelligible], [unintelligible], discussed this in terms of [unintelligible]and that embeds into our setup.

Let me give you an example of a similar flavor to what Dan talked about earlier. One reason you want to use this stuff is to try to understand mapping problems.

It's a common problem that you want to understand spaces of maps between smooth manifolds. You can choose various models of infinite dimensional manifolds and you may or may not be able to do enough analysis there, and you can cook up, often, a substack represented by an L_∞ space. I want to describe one such example.

I want to study maps $S^1 \to X$, and the free loop space is wild but not that wild. It's presented people with a lot of trouble over the years. I want to look at it as a functor of points, so, what do I want to call this guy? Imagine $\mathrm{dgMan}_{\mathrm{Nil}}^{\mathrm{op}} \to \mathrm{sSets}$, and to $\mathcal{M}$ I want to associate, maybe maps $S^1 \times \mathcal{M} \to X$ or rather families of these parameterized by a simplex. But instead of studying X, I could replace it with $\mathfrak{g}_X$, so study a functor instead

$$B_{\mathfrak{g}_X}^{S^1\,\mathrm{dR}} : \mathrm{dgMan}_{\mathrm{Nil}}^{\mathrm{op}} \to \mathrm{sSets}$$

defined by

$$\mathcal{M} \mapsto B_{(X,\mathfrak{g})}(S^1_{\mathrm{dR}} \times \mathcal{M})$$

I want to study the de Rham circle.

There's a subguy $\widehat{B_{\mathfrak{g}_X}^{S^1\,\mathrm{dR}}}$, which is the same, it just forces, it's the subspace where the underlying smooth map $S^1 \times \mathcal{M} \to X$ factors through the projection to $\mathcal{M}$. So the functor of points, there are pairs, and I had a smooth map $S^1 \times \mathcal{M} \to X$. This is a formal neighborhood of the constant loops. It turns out that

Proposition 16.2. (1) *(Costello) $\widehat{B_{\mathfrak{g}_X}^{S^1\,\mathrm{dR}}}$ is presented by an L_∞-space $\mathcal{L}(X,\mathfrak{g}_X)$ which is $(X,\Omega^*(S^1)\otimes\mathfrak{g}_X)$*

(2) *(G.–Gwilliam) This $\mathcal{L}(X,\mathfrak{g})$ is -1-symplectic when X is symplectic.*

That's the type of thing you might want to say, it's the type of thing the BV formalism is built to eat.

Let me make a little table

L_∞ space	Mapping problem	Obs^q (maybe obstructed)
$\mathcal{L}T^*[[\mathit{unintelligible}]]X$	$S^1 \to (X \hookrightarrow T^*X)$	$\mathrm{Diff}_X^\hbar$
$\mathcal{L}(X,\omega)$	$S^1 \to X$	$(C_X^\infty[[\hbar]], [\mathit{unintelligible}])$
$\mathcal{L}_E X$	$\Sigma \to X$	$CDO_X^\hbar$

The first line is G.–Gwilliam and G.–Gwilliam–Williams. The second line is G.–Li–Li. The third is Kevin, and the quantum observables is Gwilliam–Williams.