IBS-CGP SPECIAL LECTURES ON SYMPLECTIC EMBEDDINGS IN DIMENSION GREATER THAN FOUR BY RICHARD HIND

GABRIEL C. DRUMMOND-COLE

1. March 5

[Okay, so I'm very happy to introduce Richard Hind from Notre Dame. He'll give three lectures about symplectic embeddings, and also another in the department in the geometry and topology seminar. Please sign on the sheet if you are interested in coming. I'll ask this question again in the intermission.]

I want to talk about the embedding problem in symplectic topology, focusing on ellipsoids, where we can actually do some calculations.

What can you do in the symplectic category as opposed to the smooth or Riemannian category. These questions break down into two parts. What can you do symplectically, the first part is examples or constructions, and I'll talk about those today. You have some examples of things you can do, and the next step is to show that you can't do anything else, these are as good as it gets. I'll talk about those tomorrow.

It turns out that these are kind of independent things. Looking at it naively, the constructions and obstructions seem unrelated. In the third lecture, when you have embeddings you can ask how many there are or what the space of them is, something like that.

I'll run through, remind you of some definitions. I think you know all this.

Definition 1.1. *A* symplectic manifold (M^{2n},ω) *is an even dimensional manifold with a closed non-degenerate* 2*-form* ω

The basic example, almost the only one I'll talk about, is $\mathbb{R}^{2n},\omega_0=\sum dx_i\wedge dy_i$. I'll often separate the variables like that. It's sometimes convenient to think of this as $\mathbb{C}^n$ with $z_j=x_j+iy_j$.

A symplectic ellipsoid is $E(a_1,\ldots,a_n)=\sum\frac{\pi|z_j|^2}{a_j}\le 1$. Usually we write these with $0<a_1\le\cdots\le a_n$. I think of the a_i as areas, not lengths. So $E(a,\ldots,a)$ is a ball of radius $\sqrt{\frac{a}{\pi}}$.

This notation will be best. Sometimes we consider $Z(a)\coloneqq E(a,\infty,\infty,\cdots,\infty)$ which is a disk of area a times $\mathbb{C}^{n-1}$.

We can write $\lambda E(a_1,\ldots,a_n)=E(\lambda a_1,\ldots,\lambda a_n)$ which is scaling by $\sqrt{\lambda}$, just to confuse you from the start.

Definition 1.2. *A smooth map between two symplectic manifolds* (M,ω) *xrightarrowf* (N,σ) *is* symplectic *if* $f^*\sigma=\omega$.

The basic exaple is $H:M\times[0,1]\to\mathbb{R}$, a time dependent function, you take (p,t) to $H_t(p)$. You can make a vector field X_t defined by $\omega(X_t,\quad)=dH_t$. There's a duality between 1-forms and vector fields.

Then time 1 flow is a symplectic diffeomorphism of (M,ω). An even more concrete example, if you take $\mathbb{C}^n \times [0,1] \to \mathbb{R}$, when you solve for X_t, it's $i\nabla H_t$, the gradient with respect to the Euclidean metric. At least in $\mathbb{C}^n$ you can visualize this thing. In Morse theory you'd study the gradient flow. In this case it's perpendicular to the gradient flow.

The embedding problem is, given (M,ω) and (N,σ), does there exist a symplectic *embedding* $f : M \to N$.

For me, M and N will have the same dimension, either open or with boundary. This is a different kind of thing. If they're closed, an embedding is a symplectic diffeomorphism and this is a classification problem.

In particular, open subsets of $\mathbb{C}^n$.

So there are some results.

Theorem 1.1. *Liouville's theoremA necessary condition is that the volume of N must be bigger than the volume of N. This is $\int_N \sigma^n$.*

This was in the 1890s. In about 1985 we realized this was not a sufficient condition, due to Gromov's non-squeezing theorem.

Theorem 1.2. *There exists an embedding from a ball of size a to a cylinder of size b if and only if $a \le b$.*

The nature of this is a continuous sort of problem. Say M and N are subsets of $\mathbb{C}^n$, say M is bounded. If $\lambda \gg 1$, there will be an embedding $(M,\omega_0) \to (N,\lambda\omega)$, which is something like scaling N. If you scale it enough, there will certainly be an embedding. So what's the minimal amount you can scale?

An embedding $f : M \to (N,\lambda\omega_0)$ is optimal if there exists no embedding for M into $(N,\mu\omega_0)$ for $\mu < \lambda$.

I'm looking for optimal embeddings. Really I mean, a symplectic embedding $M \to N$ I don't really mean it, I mean there exists a symplectic embedding $M \to (N,\lambda\sigma)$ for all $\lambda > 1$.

[some discussion of this point]

I want continuous invariants of domains. The continuous invariants are wound up with

Definition 1.3. Symplectic capacity *is a function $c : \mathcal{C} \to [0,\infty]$ (this $\mathcal{C}$ is some set of symplectic manifolds) such that*

1. *if $M \hookrightarrow N$ then $c(M) \le c(N)$,*
2. *the capacity $c((M,\lambda\omega)) = \lambda c(M,\omega)$,*
3. *we want $c(B) > 0$, and $c(Z(a) < \infty)$.*

This is what excludes volume. You'll never get the volume of a cylinder to be finite.

So someone didn't just write this down.

Because of Gromov's theorem, you can define $c_B(M)$ is the supremum of λ such that $B(\lambda) \hookrightarrow M$. Or you could define $c_Z(M)$ which is the infimum of λ such that $M \hookrightarrow Z(\lambda)$. These are obviously monotonic under embeddings. The main thing is to check that the capacity of a cylinder is finite, which in this case is Gromov's non-squeezing theorem.

Here you can take all symplectic manifolds but sometimes you have to restrict.

I'm interested in these kinds of functions. I'm interested in embedding capacities, and we'll see they're very difficult to compute. There exist other capacities, well, these are interesting, you can link these to dynamical systems, you get relations to other subjects. Typically the other capacities will be harder to define but easier to compute.

(1) Ekeland–Hofer capacities (92), $c_1, \dots$ of capacities, but for this, this will only work for open subsets of $\mathbb{C}^n$, these are related to choosing Hamiltonians.

Theorem 1.3. *On an ellipsoid $E(a_1, \dots, a_n)$, we have c_k is the kth number in the sequence ℓa_j where $\ell \in \mathbb{N}$ and $j = \{1, \dots n\}$. It's conceivable these numbers are the same, so you use repetitions.*

(2) Embedded compact homology capacities (ECH) due to Hutchings. The sequence is $e_0, e_1, e_2, \dots$ and these only work in dimension 4, these are 4-dimensional manifolds with boundary. We'll come back to the definitions. On an ellipsoid, $E(a_1, \dots, a_n)$, then e_k is the kth number in the sequence $\ell a_j + m a_k$ for ℓ, m nonnegative integers and j, k in the appropriate range.

So for some examples, let's look at $B^4(2) = E(2,2)$, you get

$$2, 2, 4, 4, 6, 6, \dots$$

What about $E(1,4)$? The volume of the ellipsoid is proportional to the product of the numbers. So $E(2,2)$ and $B(1,4)$ have the same volume. There's another point, there's no embedding $E(2,2)$ into $B(1,4)$ because that goes to $B(1,\infty)$ and Gromov tells you that can't happen.

It would have to be a funny thing going in the other way. It has to fill up the volume but it can't extend to the boundary.

The Ekeland-Hofer of this is

$$1, 2, 3, 4, 4, 5, 6, 7, 8, 8$$

and you can see that $c_k(B^4(2)) \geq c_k(E(1,4))$ for all k. So there's no obstruction.

What about the ECH obstructions? Now I can take all linear combinations, for the ball,

$$0, 2, 2, 4, 4, 4, 6, 6, 6, 6, \dots$$

If you worked it out for $E(1,4)$ you get

$$0, 1, 2, 3, 4, 4, 5, 5, 6, 6$$

and again it's unobstructed but there are some equalities so it certainly can't get smaller, that's reproving Liouville's theorem.

Okay there's one last thing, if you didn't know about capacities, it's tough to define a capacity because I'm stuck with the condition on the cylinder having finite capacity. You might think to yourself, it would be easier to let this be infinite, but have a few less factors that are infinite, you have that one be finite.

Definition 1.4. *(Hofer) a* capacity of order k *satisfies the first couple of conditions, but I let the cylinder have ∞ volume, but when I remove factors it's finite. I want the capacity of $B^{2k}(1) \times \mathbb{C}^{n-k} < \infty$ but $B^{2(k-1)}\mathbb{C}^{n-k+1} = \infty$.*

Presumably this is easier to define. So the Gromov width is a capacity of order 1. But $\sqrt[n]{vol}$ is a capacity of order n.

So given this, you could have written these capacities of order n in the nineteenth century, so you might think it was easier to get capacities of higher order.

The thing is, if there was a capacity of order k, that implies there does not exist embeddings $B^{2(k-1)}(1) \times \mathbb{C}^{n-k+1} \hookrightarrow B^{2k}(R) \times \mathbb{C}^{n-k}$ for any $\mathbb{R}$.

This is all background stuff. Now I want to tell the story, what do we know about ellipsoid embeddings?

I'll list out some general theorems

Proposition 1.1. (1) *Say q is a positive definite quadratic form and the set $E = \{q \leq 1\}$. Then there exists a linear symplectic map of $\mathbb{C}^n$ such that $\phi(E) = E(a_1, \ldots, a_n)$ for some a_j. So up to linear isomorphism all ellipsoids are of this form.*

(2) *There exists a linear symplectomorphism such that $\phi(E(a_1, \ldots, a_n)) \subset E(b_1, \ldots b_n)$ if and only if $a_j \leq b_j$ for all j.*

Theorem 1.4. *GromovFor nonlinear ϕ we still need $a_1 \leq b_1$.*

At this point, I suppose, you might think you need equality in all factors, even for non-linear maps. But that's wrong.

This is due to this construction, symplectic folding. My kind of claim is that this is supposed to explain a whole bunch of stuff. This says that the nonlinear situation is more complicated.

Even before ellipsoids, take $P(a,b)$ a polydisk $B^2(a) \times B^2(b)$

Proposition 1.2.

$$P(a,b) \hookrightarrow P(2a, \frac{b}{2} + a)$$

Then you can do $E(a,b) \subset B^4(\frac{b}{2} + 3a)$. I want to prove this.

Proof. Think of $P(a,b)$ as being fibered over $B^2(b)$. Symplectically, this is just area-preserving geometry. A disk of area b in the z_2 plane, I can replace it with anything of the right area, so I'll take two disks of area 1 with a thin strip on the other end.

So the fibers are disks of size a which I think of as being a square. Then look at $G(x_1, y_1) = x_1$ This is a Hamiltonian on $\mathbb{C}$. If you take $i\nabla G$, you get $\frac{\partial}{\partial y}$, the time one flow displaces the fiber, moves it just up, that's in a disk of size $2a$.

Now we look at a Hamiltonian of z_1, z_2, which will be $\chi(x_2)G(x_1)$. What's χ? It's a cutoff function that goes from 0 to 1. This is a function on $\mathbb{R}^4$ and you work out the gradient, multiply by i. If x_2 is negative, nothing happens. On the other region, χ is identically 1 and the motion displaces the fibers.

In the middle you have a derivative bounded more or less by $(1+\epsilon)a$. If I take the partial derivative with respect to x_2, you get 1 times $G(x_1)$ which is bounded by a.

So the flow, ψ^1, the final step, let g be an immersion of the z_2-plane which maps $D_{RHS} \to D_{LHS}$. The claim is that f, it's the identity in the z_1 components and $g(z_2)$, I claim it's an embedding. You just gotta check that there are no double points. In these points the fibers are disjoint. You can choose g to be symplectic.

Then $f(\psi^1(P))$, the fibers started in a disk of size a, they're not leaving $2a$. The area of the other is $a + \frac{b}{2}$. □

Theorem 1.5. *(Schlenk) You can sort of fold a polydisk in half, there's a little bit left over at the end. Schlenck showed that for M^{2n} symplectic, as long as the* *polydisk is thin enough, you can approximately get a volume-filling embedding into M.*

$$\lim_{\epsilon\to 0}\sup \frac{vol(\lambda E(\epsilon,\dots,\epsilon,1))}{vol(M)} = 1$$

where the supremum is over all λ such that there is an embedding $\lambda E(\epsilon,\dots,\epsilon,1) \to M$.

The intuition is you start with something very thin, you fold and fold and can kind of fill up everything.

The difference between folding and what you can actually do is not big.

Theorem 1.6. *(Buze-Hind) If M is closed and $[\omega]$ is rational in cohomology, then there exists a k such that there exists an embedding $\lambda E(1,\dots,1,\ell) \to M$ for any $\ell > k$ where $\lambda^n = \frac{vol(M)}{vol(E(1\dots,1,\ell))}$.*

These are close but here it's saying that you can actually get there precisely, not within ϵ, but we only know this when the class is rational.

The other intuition is once the domain is thin, these embeddings are more flexible.

Another modification. I'll just sort of show you, imagine we understood the polydisk case so well, we can just wave our hands and see what's happening. We'll view everything from the moment projection. We can take (z_1, z_2) to $\pi|z_2|^2, \pi|z_1|^2$, call this map $\mathbb{C}^2 \to \mathbb{R}^2$, call it p. We get a triangle in the plane. If you did it with a ball you get a triangle with two equal sides.

So I fold at $\frac{b}{2}$. The fibers I displace, I only actually need $\frac{a}{2}$, if you open things out and apply the moment projection, this is what you'd see [picture].

Now I'd like to displace fibers, but I'll only do it to level $\frac{a}{2}$. Now you want to fold this guy back, I want disjoint fibers to go to disjoint fibers, which will fit.

When I was folding for the polydisk, I put a block on top of another block. When I'm starting with an ellipsoid, you'd leave a lot of space for free there.

Anyhow, where does this hit the axes, it goes to $(\frac{b}{2}, a)$ in the corner and $(\frac{b}{2}+\frac{a}{2}, \frac{a}{2})$.

I'm missing ϵs as well but let's leave that.

The other thing I've got, so $\frac{b}{2} + a < b$, so b is what I'd get by inclusion.

In other words, folding beats the inclusion exactly when $2a < b$. Folding will be more effective when the domain gets thinner.

What's the answer? In general we don't know, but two cases have been worked out.

The known sharp result is for 4-dimensional ellipsoids into balls. This is due to McDuff-Schlenck. They compute $c(x) = \inf\{R|E(1,x) \hookrightarrow B^4(R)\}$ for x in $[1,\infty)$.

So for free, Liouville says that $c(x) \geq \sqrt(x)$. On the other hand, I know that $c(x) \neq x$. On the other hand $\frac{x}{2} + 1$ may be less than x.

To describe the answer, they define a sequence $g_0 = 1$ and g_n is the odd Fibonacci numbers, $1, 1, 2, 5, 13, 34, \dots$, and $a_n = \left(\frac{g_{n+1}}{g_n}\right)^2$ and $b_n = \frac{g_{n+2}}{g_n}$.

Proposition 1.3. *$a_0 < b_0 < a_1 < b_1 < \cdots$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \tau^4$ where τ is the golden ration $\frac{1+\sqrt{5}}{2}$.*

We can check a few of these, we get $1 < 2 < 4 < 5 < 6.25 < 6.5 < \cdots <\sim 6.85$

Theorem 1.7. *If $x < \tau^4$, then $c(x)$ is linear between a_j and b_j, and $c(a_j = \sqrt{a_j}$, volume preserving maps $E(1,a_j) \to B^4(\sqrt{a_j})$. Then $c(b_j) = c(a_{j+1})$. If $x > 8 + \frac{1}{36}$*

then $c(x) = \sqrt{x}$. Once your ellipsoid gets thin enough you can always fill a ball of the volume.

There's a gap. This explains up to 6.85 *and then you got something from* 8 *roughly, so in between $\tau^4, 8\frac{1}{36}$ you get eight exotic steps.*

[picture] If you draw the square root, [pictures]

You can check that between 1 and 2, the inclusion is optimal, the line is $y = x$, you know that $E(1,2) \subset B^4(2,2)$, the inclusion map is optimal exactly until folding is useful.

So that's the solution to the four-dimensional equation.

It raises various questions. Is there some way to intuitively understand this and also, these embeddings are very obscure, they need to build these from Seiberg-Witten invariants. We'd like something explicit and some explanation for this.

This is an independent theorem, also due to McDuff.

Theorem 1.8. *An ellipsoid $E(a_1, a_2) \hookrightarrow E(b_1, b_2)$ if and only if $e_k(E(a) \leq e_k(E(b))$ for all k. So ECH is a complete invariant for ellipsoid embeddings.*

We have formulas for these invariants. But I don't think anyone can prove McDuff-Schlenck using this as the ingredient because the combinatorics are nightmarish. It's very hard to use, this is infinitely many inequalities. Nevertheless, we got one case

Proposition 1.4. *(Buze–Hind)(combinatorics) $E(1,a)$ into $\sqrt{\frac{a}{b}}E(1,b)$, volume preserving, multiply to match volumes, this is only possible if $a \geq \frac{(5b+16)^2}{16b}$.*

This is slightly better than folding, which says you can roughly fill by elementary constructions. The key ingredient is this bound which comes from the ECH capacities.

This is the answer for 4-dimensional ellipsoids.

What I want to do next, rather than explaining how they prove this, the question is, can you think about these results, maybe next time, I want to look at the function $inf\{R|E(1,x) \times \mathbb{C}^{n-2} \hookrightarrow B^4(\mathbb{R}) \times \mathbb{C}^{n-2}\}$. This is the simplest higher dimensional analogue. The first question is what you can do with folding? Well $f(x) \leq c(x)$, it's a smaller function, but you might imagine it's exactly the same, what I get here next time, it's less than what you can do by folding, and it turns out that the folding is never optimal, these are constructed by Seiberg-Witten. For the f-graph, the folding graph clips the tops of these and then crosses and is underneath past τ^4. The red line follows McDuff-Schlenck until τ^4 and then is equal to what you get from folding at least for $3d - 1$. Compared to McDuff-Schlenck, you get sharp results in higher dimension even though it's weaker in dimension 4.

2. March 6

Maybe I'll just recap. Last time, there's this McDuff-Schlenk capacity function $c(x) = \inf\{R|E(1,x) \hookrightarrow B^4(R)\}$. You can always reverse factors and rescale and so this covers all cases.

You have infinitely many steps before the fourth power of the golden ratio and then a few sporadic steps and then you become the square root.

So with elementary methods, folding you can never do as well as the optimal embedding after 2. So these are non-explicit embeddings.

In very special cases, take $\mathbb{CP}^2$, let lines have area 2. Take a degree two curve in here. The area is four. In other words, there's an embedding of a disk, more or less, of size four into C. This has the Fubini-Study form. So if you restrict this to the complement of $\mathbb{C}$ it's exact, this is $d\lambda$. So λ is dual to some X and let's compute $\mathcal{L}_X\omega$. By Cartan's formula this is $d(X \lrcorner\, \omega) = d\omega = \omega$. Then

$$E(1,4) \cong \{x \in (\mathbb{CP}^2 \backslash C) | \lim_{t\to\infty} \phi_X^t(x) \in D(4)\} \cup D(4).$$

You could start writing the primitives down pretty precisely. The X is a gradient flow a pleuriharmonic exhaustion. The log of a section of a line bundle whose zero is exactly this divisor.

Then we had this function $f_n(x) = \inf\{R | E(1,x) \times \mathbb{C}^{n-2} \hookrightarrow B^4(R) \times \mathbb{C}^{n-2}\}$.

So we clearly have $f(x) \le c(x)$ but it's not necessarily an equality. The motivation is that this is a first step to higher dimensional results. The problem with higher dimensions is, there's no Seiberg-Witten theory and no embedded contact homology.

Maybe also you get some insight back into the McDuff-Schlenk function.

So Ekeland-Hofer says, you could look at $E(1,x,\infty,\dots,\infty)$, you take multiples of factors and write them in order, you get

$$1,2,3,\dots,x,x,x+1,\dots,$$

On the other hand, on the other side, for $B^4(\mathbb{R}) \times \mathbb{C}^{n-2} = E(R,R,\infty,\infty,\dots)$, so here you get

$$R,R,2R,2R,3R,3R,\dots$$

If there's an embedding, these have to dominate, you see that $R \ge 2$ multiple times and then you don't get any other information after you hit x. So if $x > 2$, you only find out that $f(x) \ge 2$. These are very weak obstructions.

Theorem 2.1.
(1) *There exists an explicit embedding (folding style) for any T, $E(1,x,T,\dots T) \hookrightarrow B^4(\frac{3x}{x+1}) \times \mathbb{C}^{n-2}$.*
(2) *If $x = 3d-1$ and $T \ge x$ then this is optimal (joint with Kerman).*
(3) *If $x \le \tau^4$ and $T \gg 1$ then $E(1,x,T,\dots,T) \hookrightarrow B^4((c(x)) \times \mathbb{C}^{n-2}$ is optimal (joint with Cristofaro–Gardiner)*

We'll have to use a bit of machinery for the second and third but the first we can write down.

Corollary 2.1. *By Pelayo–Ngöc, you can take a limit as $T \to \infty$ to get an embedding of $E(1,x,\infty,\dots,\infty) \to B^4(\frac{3x}{x+1})$. So $f(x) \le \frac{3x}{x+1}$. If you let x run to ∞ you get $E(1,\infty,\infty,\dots) \hookrightarrow E(3,3,\infty,\dots)$. These kinds of embedding were found by Gouth, called catalyst maps.*

In physics, there's a Hamiltonian flow from one to the other. You could embed all of Euclidean space into a cylinder. With a mechanical system, you add in an extra parameter, a particle that's a catalyst, you can cheat non-squeezing.

In particular, what we're getting is $B^2(1) \times \mathbb{C}^{n-1} \to B^4(3) \times \mathbb{C}^{n-2}$. This means there are no intermediate capacities.

A corollary of the second part of the theorem says that $f(x) = \frac{3x}{x-1}$ for $x = 3d-1$.

A corollary of the third part is that $f(x) = c(x)$ for $x \le \tau^4$.

So it's enough to show $E(1,x,T) \hookrightarrow B^4(\frac{3x}{x+1}) \times \mathbb{C}^{n-2}$, you can actually do better, look at $\pi|z_1|^2 \le \frac{2x}{x+1}, \pi|z_2|^2 \le \frac{2x}{x+1}$.

You can't, in McDuff Salomon, you can't flow from a ball of area 1 to another ball of area 1 passing through a slit of size less than 1.

So there are balls, you can ask about $E(1,S,S)$ passing through a slit in $\mathbb{C}^n$ of the shap $B^4(R)\times B^{2(n-2)}(T)$ for arbitrary T, you could ask if they're isotopic for $R<3$. You'd like to push them into a torpedo and push them through, and that's not going to work.

I'm going to focus on getting into $P(\frac{2x}{x+1},\frac{2x}{x+1})$, maybe you believe that then getting it into the ball is not such a big deal.

Think of $E(1,x,T)$ as fibered over a disk of area x, set $\lambda=\frac{x}{x+1}$, so what's the ellipsoid? It's $\pi|z_1|^2+\pi|z_2|^2/x+\pi|z_3|^2/T\le 1$. So if $\pi|z_2|^2\ge\lambda$, this means $\pi|z_1|^2\le 1-\frac{\lambda}{x}=\lambda$.

So I'll write my disk as being a disk of area λ and then a very long strip. The points going on the sliver, I can say the fibers are in $P(1,T)$ over the disk and $P(\lambda,T)$ on the sliver.

I'll have to do a multiple symplectic fold. Call $P(\lambda,T)$, call it P_0, we'll wind things up. We'll really try to use the $\mathbb{C}$-factor.

Let me draw a picture. In the image I just need to control the factors. [pictures, missed some.]

So how to get obstructions?

Definition 2.1. $\Sigma^{2n-1}\subset(M,\omega)$ *is of* contact type *if there is a Liouville vector field X (i.e. $\mathcal{L}_X\omega=\omega$) near Σ transverse to Σ.*

Define $\lambda=X\lrcorner\omega$. Set $\lambda_0=\lambda|_\Sigma$. Then λ_0 is never zero. It's $\omega(X,\quad)$, since the ω is nondegenerate this can't be vanishing. It's transverse to the hypersurface so it should evaluate on something tangent to the hypersurface.

We can also use the formula $d(X\lrcorner\omega)=d\lambda$. If you differentiate λ_0, you get $d\lambda_0=\omega|_\Sigma$. So $\lambda_0\wedge(d\lambda_0)^{n-1}$ is $\lambda_0\wedge\omega^{n-1}$, and again by nondeneracy this is a volume form.

We call λ_0 a contact form. The kernel of λ_0 is what you call a contact structure.

This is the odd dimensional analogue. A contact form has a *Reeb vector field* corresponding to the form given by $R\lrcorner d\lambda_0=0$, this defines this up to a multiple since ω is nondegenerate. So then we normalize it so that $\lambda_0(R)=1$.

So $E(a_1,\ldots,a_n)$, the boundary of this is of contact type by, I have the radial vector field transverse to this. The Reeb vector field is parallel to $i\vec{n}$ which is something like, a level set of a function, a gradient —

$$i\left(\frac{x_1}{a_1},\frac{y_1}{a_1},\ldots,\frac{x_n}{a_n},\frac{y_n}{a_n}\right)=\left(\frac{-y_1}{a_1},\frac{x_1}{a_1},\ldots,\frac{-y_n}{a_n},\frac{x_n}{a_n}\right).$$

In complex coordinates, the flow of R, we have $\dot{z}_j=\frac{iz_j}{a_j}$. So $z_j(t)=z_j(0)e^{\frac{it}{a_j}}$.

Let me stick a 2π in so that I get a 2π in the exponent. That's the Reeb flow. So $z_j(t)=z_j(0)$ either if t is an integer multiple of a_j or $z_j(0)=0$.

Now let me assume $\frac{a_j}{a_k}$ is irrational. If t is a multiple of a_j then it's not a multiple of a_k. This will let us write down closed orbits. There are then exactly n closed orbits. So $\gamma_k=\{z_j=0,j\neq k\}$.

Definition 2.2. *A* symplectic cobordism M *is a symplectic manifold with contact type boundary, let's say compact.*

For example, the main example, if $\phi:E(a_1,\ldots,a_n)$ into the interior of $E(b_1,\ldots,b_n)$, then, call them $E_2\backslash\phi(\mathring{E}_1)$ is a sypmlectic cobordism.

Theorem 2.2 (Weinstein theorem)**.** *If Σ is of contact type with contact form λ_0, then a neighborhood of Σ is symplectomorphic to $(\Sigma \times (-\epsilon, \epsilon), d(e^t \lambda_0))$.*

Then the boundaries of the symplectic cobordism are *positive* if the X is outward-pointing and *negative* otherwise.

So symplectic cobordisms have canonically defined inward and outward ends.

We want to study these, and the idea is that a symplectic embedding gives you a symplectic cobordism, so let's work out properties of these.

It's customary near a positive end to choose a diffeomorphism, a neighborhood of the end, our expanding vector field is $x = \frac{\partial}{\partial t}$. So a positive end will look like $\Sigma \times (-\epsilon, 0)$, and it's helpful to just choose a diffeomorphism $(x, t) \to (x, f(t))$, so you identify with $\Sigma \times [0, \infty)$, and now it's like $\phi(t)$ is increasing and you want $\lim_{t\to\infty} \phi(t) = 1$.

Somehow the analysis is clearer in this picture.

All right, so why do we want to think of the interval as being long?

Definition 2.3. *An almost-complex structure on $\circ M$ is* compatible *if*

- *$g(X, Y) = \omega(X, JY)$ is a Riemannian metric,*
- *on $\Sigma \times (0, \infty)$, we have $J(\xi) = \xi$ where ξ is the kernel of λ_0,*
- *$J(\frac{\partial}{\partial t} = R$ the Reeb field, and*
- *J is translation invariant.*

[Pictures]

Note that the map from the punctured disk to $\Sigma \times (0, \infty)$(write it as $S^1 \times (0, \infty)$ but use the complex structure from the complex plane) so $i\frac{\partial}{\partial t} = \frac{\partial}{\partial s}$.

Before we make this, suppose γ is a closed Reeb orbit in Σ parameterized by $x : [0, T] \to \Sigma$. In other words $x(0) = x(T)$ and $\dot{x}(t) = R$.

So the map takes (s, t) to $(x(Ts), Tt + c)$. I claim this is a holomorphic map.

You have to check, in other words, it satisfies the Cauchy-Riemann equations, so $du \circ i = J \circ du$, to check this, you should check $\frac{\partial u}{\partial s} = J(\frac{\partial u}{\partial t})$.

So anyway, if I have a closed Reeb orbit in Σ I can look at it as a holomorphic punctured disk.

You do this in closed manifolds, define these structures, now I can get holomorphic curves that are cylinders over Reeb orbits. Let's get something more global.

Definition 2.4. *A (genus 0)* finite energy curve *is a map $u : \mathbb{CP}^1 \backslash \Gamma \to \mathring{M}$ such that $du \circ i = J \circ du$ and u is asymptotic to a closed Reeb orbit near each puncture.*

So Γ is a finite set of points in $\mathbb{CP}^1$, so what does it mean to be asymptotic to a closed orbit? You can write Γ as $\Gamma^+ \sqcup \Gamma^-$, so $\Gamma^\pm$ will be asymptotic to positive or negative ends.

If z a positive end and γ is a closed orbit in a positive boundary component.

For u to be asymptotic to γ at z means I can choose coordinates $S^1 \times (0, \infty)$ as before on a neighborhood U of the puncture such that $u(U)$ is contained in $\Sigma \times (0, \infty)$ and $u(s, t)$ is $(x(Ts), Tt + c)$ for some parameterization x of γ plus an error term $e(s, t)$ which is exponentially small in t, so $e^t |e(s, t)|$ is bounded.

So a holomorphic curve might look like [picture]. In the symplectic picture they look like surfaces with boundary. You could compactify to get a symplectic surface with boundary. Holomorphically they look like punctured disks. The claim is they have properties like closed holomorphic curves so we can use them to prove things.

3. March 9

Last time we finished off wth the idea of a finite energy curve, a holomorphic map $u : \mathbb{CP}^1 \backslash \Gamma \to (M, J)$.

It's a holomorphic map asymptotic to Reeb orbits at the punctures.

The idea is to prove things using these as the tool.

There's a theory of these things. There's a theory analogous to closed curves. Let me mention some names: Hofer–Wysocki–Zehnder wrote four or five papers in the mid-nineties, and for compactness it was joint with these guys and Bourgeouis-Eliashberg.

I want to mention a few points about this.

(1) There's a Fredholm theory that says, for generic J, you get a moduli space $\mathcal{M}^s(\gamma_1^+, \dots \gamma_r^+, \gamma_1^-, \dots, \gamma_s^-, A, J)$ with specified behavior at the punctures, and this is a manifold of the expected dimension. We mean that $u \sim u \circ \phi$ where ϕ is a Möbius mapping. Hree γ_i are the orbits at the boundaries. A is a relative homology class. S means that this is injective, meaning that if you restrict away from more finite points, it shouldn't be a branched cover. So throw out the multiple covers or the statement is false.

    Let me tell you briefly why you need this. The idea is to look at a universal moduli space $\widetilde{M}^S = \{(u, J) | \bar{\partial}_J u = 0\}$. This is a subset of $\mathcal{B} \times \mathcal{J}$, a Banach space of maps cross a space of compatible almost-complex structures.

    There's a vector bundle over this with fiber $(0,1)$-forms valued in the pullback of the tangent bundle of M. There's a section of this bundle $(\bar{\partial})$ and I claim that it's transverse to the 0-section, which implies the zero set is a manifold and once you know that you look at the projection onto $\mathcal{J}$ and this is a Fredholm map and you can use the Sard-Smale theorem for J of the second category. If you take J and pull it back you get $\mathcal{M}(J)$, which is a finite dimensional manifold (of the dimension you want).

    Why is the claim true? I'm mentioning it, this is why you need the somewhere injective condition. If you differentiate $\bar{\partial}$, take the vertical component, you want to show that's onto, you look at $D^v \bar{\partial}$ and apply it to $(0, y)$, where $0 \in T_u \mathcal{B}$ and $y \in T_J \mathcal{J}$. You want to choose J supported near $u(z)$ where z is an injective point. [discussion]

(2) Compactness. First of all, this is supposed to be a generalization of Gromov's compactness theorem.

    So C^∞ on compact subsets is not enough. Maybe we're looking at $\mathcal{M}(\gamma^+, \gamma^-)$. We're looking at positive and negative ended cobordisms. Near the end, it looks like $\partial M^+ \times [0, \infty)$. Maybe there's a family of curves in the moduli space that look asymptotic to σ and then they skew over. You could imagine taking a limit that converges uniformly on compact subsets and in the limit you get something converging to the wrong thing.

    Say (M, ω) is closed with Σ of compact type and its removal divides M into two parts. For U a neighborhood of Σ it can be made to look like $(\Sigma \times (-2\epsilon, 2\epsilon), de^t \lambda_0)$. Choose J_0 such that on U it preserves $\xi^{2n-2} = \ker \lambda_0 \subset T\Sigma \times \{t\}$ and $J_0(R) = -\frac{\partial}{\partial t}$, also transaltion invariant.

    So we can then define J^N equal to J^0 off U and on ξ. But $J^N(R) = f(t) \frac{\partial}{\partial t}$. [pictures and discussion]

Theorem 3.1. *(Bourgeois, Eliashberg, Hofer, Wysocki, Zehnder) Let U_N be a sequence of J_N-holomorphic maps $\mathbb{CP}^1 \to (M, J^N)$. Let $[u_N(\mathbb{CP}^1)] = A$, a fixed class. Then a subsequence of u_N converges á la Gromov to a so-called holomorphic building.*

Definition 3.1. *The domain of a* holomorphic building *is a nodal Riemann surface with punctures. F gives a holomorphic map on each component which is a finite energy curve with image in M^+, M^-, or $\Sigma \times \mathbb{R}$. This F has a* level *k, the components are labeled 0 through $k+1$. The level 0 components map into M^-. Level $k+1$ go to M^+, and the other levels go to $\Sigma \times \mathbb{R}$.*

If nodes connect components in the same level, they are removable singularities and the two components intersect at the node. Otherwise, if the labels don't coincide at the node, then they differ by exactly one, say they're r and $r+1$. Then level r has a positive puncture asymptotic to γ and level $r+1$ has a negative puncture.

[picture, discussion]

4. Ellipsoid embedding obstructions from finite energy curves

Say you want to study $E_1 \to E_2$. For any $\lambda > 0$ such that there exists an embedding $\lambda E_1 \to E_2$, you get a symplectic cobordism $E_2 \backslash \psi(\lambda E_1)$. There you can study finite energy curves.

Given a moduli space, $\mathcal{M}(\gamma_1^+, \ldots \gamma_r^+, \gamma_1^-, \ldots, \gamma_s^-, \psi, J)$. Say the moduli space is of dimension 0. Then I expect the cobordism class of $[M]$ (which is a number of points with orientation) to be independent of all choices (ψ, λ, J).

This is a special thing about ellipsoids. Why? The input data is a connected set. The set of ψ, λ contracts onto a smaller choice where it's just linear. You can interpolate between ψ_0, λ_0 and ψ_1, λ_1 via ψ_t, λ_t.

Generically $M = \{(u,t) | u \in \mathcal{M}(\psi_t, J_t)\}$ is generically dimension 1. I need M to be compact so that points need to cancel out.

The point about this whole claim. It'll be impossible to distinguish ellipsoids directly. The moduli space will not help for isotopy questions. Why is M compact? Suppose not. Then when we run this compactness theorem and get a building, the building will be non-trivial. The curves will have to converge to something other than a cylinder. The only thing you could do is lose information at the top or the bottom. [discussion]

…so the key is that any moduli space associated to ellipsoids has even index. So I get some spaces of index -2 which is a contradiction because I have only a one-parameter family to increase the dimension from the index.

I want to push that scheme through in a few cases in order to prove that theorem.

Theorem 4.1. *(with E. Kerman) Suppose $E(1, x, S) \overset{\hookrightarrow}{\psi} X = B^4(R) \times \mathbb{C}$. I'll compactify the four-ball to $\mathbb{CP}^4(R) \times \mathbb{C}$. Write $\mathbb{CP}^2(R) = B^4 \cup \mathbb{CP}^1(\infty)$. We want $x = 3d - 1 + \epsilon$ and $S > 3d - 1$. We'll think of X as a cobordism form ∂E to the empty set.*

Then $\mathcal{M}(X \backslash \psi(E), \gamma^{3d-1}, d, J)$ is not equivalent to the empty set as a cobordism class. Hree γ is a $3d-1$-cover of a short orbit on ∂E. We have d the class given by taking u of $\mathbb{CP}^1(\infty) \times \mathbb{C}$.

Corollary 4.1. *Let U be a finite energy plane in $\mathcal{M}$. Then $\int u^*\omega$, if I fill in a $3d-1$-cover, I should get $d \times R$, and then subtract the cover of the disk, so you get $dR-(3d-1)$. This has to be positive, so $R \geq \frac{3d-1}{d}$, which is roughly $\frac{3x}{x+1}$.*

So $f(x) \geq \frac{3x}{x+1}$.

So we just need to show that $[M]$ is nonempty, let's do it for the inclusion map $\epsilon(E(1,x,S)) \to \mathbb{CP}^2(R) \times \mathbb{C}$.

First we need some curves in $\mathbb{CP}^2$. Then I'll get curves in the complement of $E(1,x)$, and then we'll move to the product.

So first of all I need curves in $\mathbb{CP}^2$. I'll blow it up at one point and work in $Y = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ I'll let E be the exceptional divisor. I'll look at $N(Y, d[\mathbb{CP}^1(\infty)] - (d-1)[E], p_1, \ldots, p_{2d})$. I claim this has a single element (and that you know this) if J is integrable. Then $Y \to E$ is holomorphic with $\mathbb{CP}^1$ fibers. This takes all of $\mathbb{CP}^2$, lines through the origin project to the exceptional divisor.

Its degree is either ± 1, I'll say -1, and the homology class is a class of sections with d poles, intersections with the $\mathbb{CP}^1$ at ∞, and $(d-1)$ zeroes.

A section of a line bundle over a $\mathbb{CP}^1$ like this is determined by the position of the poles, the position of the zeroes, and one parameter for the phase. So that's $2d$ point constraints. Then general Gromov-Witten stuff says you have just one for any J.

Now what about $\mathbb{CP}^2$ minus the ellipsoid. I'll blow up a very small ball inside the ellipsoid. Let me fix the point constraints also inside the ellipsoid. I want to stretch the neck along the boundary of E. I get a curve in N for all J^N, this is an almost complex structure stretched along the boundary of E.

Then we stretch this to a holomorphic building. It'll have top-level components in $\mathbb{CP}^2 \backslash E$. What are they? This is where you just have to analyze this.

Proposition 4.1. *There is a single top level component (of degree d) asymptotic to γ_1^{3d-1}*

This is not, okay, somehow this is very lucky that this works, in a way. We'll see why but it's very fortunate.

First you calculate that the $(k+1)$ level is connected and that plus index 0 means that it winds $3d-1$ times.

It's very fortunate that there's only one end. It's conceivable, you could get something with two negative ends and that would be very bad but somehow it doesn't happen.

So we've produced curves in $\mathbb{CP}^2 - E(1,x)$. Then we could choose a J on $\mathbb{CP}^2 \times \mathbb{C} - E(1,x,S)$ such that the slice $z_3 = 0$ is holomorphic. Then we can lift our curves here. Now all I have to do is count them.

The thing about one end, we're producing curves in dimension 4 and saying they exist in dimension 6. But that mainly works with one end, because more than one end tends to lower your index.

So we have curves in here. The problem is to count them. In dimension four you have automatic regularity (due to Gromov, Wendl). That means the curves in $\{z_3 = 0\}$ count with sign, positively.

Further choose J to be invariant under rotation in the z_3 coordinate. If there were any other curves, they'd come in one-parameter families by rotating in z_3. It's index 0 so it should come in zero-dimensional families. That's sort of a contradiction to curves outside $z_3 = 0$ but it requires one thing, which is J being regular, which is kind of the hardest thing.

How do you choose J to be S^1-invariant and regular? You want to show that the universal moduli space is a manifold, and you can do that with somewhere injective curves, way back in the beginning. Here you have these orbits, so you want to replace this with "orbitally simple" meaning they intersect typical S^1-orbits at most once.

Proposition 4.2. *There exists a regular J such that the curves are orbitally simple.*

This is a delicate proof. It's not obvious that this should be true. If the curve is not orbitally simple, you'd project to a multiple cover on the zero section. You study the projection, so how would you do that, you'd somehow say, so in dimension four, for index zero, you've got kind of your, if you look at your degree and how much you have to wind around the orbit, the graph is $y = 3d - 1$. We're sort of sitting on this graph. If these curves are projected to a multiple cover, the degree will be k less, d/k. If you intersect k times you have degree d/k. The same thing will happen with the y-value. So you'll end up on a line to the origin. That's the underlying curve. Then you need to have more negative ends than your degree allows.

If you believe all that, the only curves are in $z_3 = 0$ and that's the end of the proof.

There's a few details that I'm skipping. Anyway, let me give some similar results.

This whole game works if you can find moduli spaces, index 0, with a single negative end. The point about the single negative end, it means the four dimensional index is the same as the higher dimensional index.

So an example of such a thing is, remember, if we look at $E(1, b_n, S) \hookrightarrow B^4 \times \mathbb{C} \subset E(R, R + \epsilon) \times \mathbb{C}$. (remember that b_n is, if g_n are the odd Fibonacci numbers, well, it's g_{n+2}/g_n).

The moduli space (no homology class because it's two ellipsoids) $\mathcal{M}(\underbrace{\gamma_2^+, \dots, \gamma_2^+}_{g_n+1}, \gamma_1^{-g_{n+2}})$ has index zero.

This is in $X \backslash E(1, b_n, S)$.

Theorem 4.2. *(with Cristoforo and Gardiner) $[M]$ is not cobordant to the empty set*

The corollary, you can compute the area by Stokes.

Corollary 4.2. *The area is $g_{n+1}R \geq g_{n+2}$. This gives $R \geq \frac{g_{n+2}}{g_{n+1}} = c(b_n)$.*

This is the McDuff-Schlenk numbers, it's nice that it works out.

This is supposed to say that folding explains optimal embeddings, but based on the argument you know nothing about isotopies. I claim this also tells us folding gives us isotopies as well.

Remember there's an embedding $P(a, b) \hookrightarrow B^4(\frac{b}{2} + 2a)$. This starts with $P(a, b)$, you fold at $\frac{b}{2}$, so you make room for the fold. Initially this is in a ball of size $a + b$. I pick up the inside and sort of fold the thing back.

If $2a < b$, then this is less than $b + a$, so this does a squeezing on the polydisk, but the thing expands before it shrinks. If you draw a ball, it goes to a ball of size $b + 2a$. Halfway through folding you've kind of gone out.

Theorem 4.3. *Let's suppose $a + b < R < 2a + b$.*

The initial polydisk $P(a, b)$ and the folding are not isotopic in $B^4(R)$.

The moduli spaces are a little different but the argument is the same. Folding is sharp here as well.

I'll leave it there.