CGP DERIVED SEMINAR

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1. January 8,2019: Jun Yong Park: Notion of motives and Grothendieck's proposal

Sorry I'm late. Thank you all for coming, and thank you to the organizers for the kind invitation to speak.

Today I want to talk a little bit about motives, I can't do too much in one hour. Let me tell you something we already know very well. Let me talk about Weil cohomology. When you have a variety, say, a smooth projective variety X, you want to study its cohomology with some coefficients, $H^i(X, k)$. There are several variants. Maybe the most obvious is singular, but what we're really doing is Betti cohomology $H^i_{\mathrm{cl}}(X/\mathbb{C}; \mathbb{C})$. Then you could do (algebraic) de Rham cohomology $H^i_{\mathrm{dR}}(X/\bar{\mathbb{Q}}; \mathbb{Q})$. Then there are ℓ-adic versions, so you could do étale, $H^i_{\text{ét}}(X/\bar{\mathbb{F}}_{q=p^r}, \mathbb{Q}_\ell)$, or the crystalline, $H^i_{\mathrm{cris}}(X/\bar{\mathbb{F}}_q, \mathbb{Q}_\ell)$, where $\mathbb{Q}_\ell$ is the ℓ-adic rationals, the field of fractions of the ℓ-adic integers.

One can compute these things for X and the punchline is that these all coincide when you formulate and compute them.

So these are vector spaces and the Betti numbers are all the same. What one wants to know is what is going on, why are they all coinciding? To show that these coincide, you need to show that the various things know how to talk to each other.

Besides cup product there is extra structure. So if you compare singular and de Rham cohomology, you get the Hodge structure, the Hodge filtration, if you consider the projection $\pi : \mathcal{H}^1(E, \mathbb{C}) \cong \mathbb{C}^2$, then you can project this to $H^{0,1}(E; \mathbb{C}) \cong \mathbb{C}$. I pass from de Rham to Hodge and look at the singular class. We know that the integer lattice lies inside the $\mathbb{C}^2$ plane, and that tells you the isomorphism classes of elliptic curves via the so-called period mapping.

Why do you want to compute this cohomology, because you want to know about the variety. The extra structure is obtained via this "comparison" which lets you distinguish different elliptic curves via the period mapping.

So that's interesting, extra structure, the Hodge filtration, that's interesting stuff. If you compare the singular with the ℓ-adic, screw the crystalline, we're going to be human beings for today, so if you compare étale to singular, what do you get? The Galois action, the absolute Galois group, if X is over a finite field $\mathbb{F}_p$, then I take the closure, so $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ is what we're looking at, so this is $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_q)$. Understanding this is easy because this is finite, the real deal is doing this over $\mathbb{Q}$. This acts on the étale cohomology as long as ℓ is not p.

Now if you have a group action on a vector space you get a representation, and what do you want to do? you want to count the rational points of $X(\mathbb{F}_q) \to \mathbb{F}_q$. If I consider $X(\bar{\mathbb{F}}_p)$, I have a natural "geometric Frobenius", and this Frob_q raises the coordinate X to X^q, and the number of fixed points, which is a finite set, coincides with the number of points this variety has over this finite field. Why? Why?

Because this makes you do the American college student's dream: $(x+y)^n = x^n + y^n$, you don't need transversality, this is a bad explanation for why these two things coincide but it does. The number of fixed points can be captured by the trace formula, so what is the trace formula I want to talk about. You all learned, say, the Lefschetz fixed point formula when you were a child, which counts the fixed point of a continuous map. So I'll talk about the Grothendieck–Lefschetz–Hopf trace formula, which tells me that

$$|X(\mathbb{F}_q)| = \sum_{i=0}^{2\dim\ X} (-1)^i \operatorname{tr}(\mathrm{Frob}_q^* : \mathcal{H}^i_{\text{ét}}(X/\bar{\mathbb{F}}_q;\mathbb{Q}_\ell) \to \mathcal{H}^i_{\text{ét}}(X/\bar{\mathbb{F}}_q;\mathbb{Q}_\ell))$$

and this is the number of fixed points, that's what the trace formula does. That coincides with $\mathbb{F}_q$-rational points for finite fields. What's lying behind this is that the absolute Galois group is procyclic, profinite, and this is generated topologically by the Frobenius map. There's nothing else to look at because that's the generator.

So if I want to count the number of points of $\mathbb{A}^1$, it's smooth but not projective. You need to modify, so count for $\mathbb{P}^1$. How many points does it have over $\mathbb{F}_q$? It's $q+1$. According to your little formula it should be

$$\sum_{i=0}^{2} \operatorname{tr}(\mathrm{Frob}_q^* : H^i \to H^i)$$

and so you know that this has $\mathbb{Q}_\ell$ in $i = 0$ and $i = 2$. You know the cohomology because you're a human being, and then you become more of a human being with the Artin comparison. So how does the induced operator act? It's q^0 in degree 0 and q^1 in degree 2. These are one dimensional vector spaces, so the trace is the sum of the eigenvalues. So λ_0 is 1 because Frobenius acts trivially on the point. So then λ_2 has to be q. There is a guy named Deligne, I think it's Pierre Deligne, and one thing he proved is that the étale cohomology is of Tate type and étale pure. So this gives an independent proof. You get weight k Hodge structures, what Tate type means, it's semi-simple, the representation is a direct sum of $\mathbb{Q}_\ell$. If it's étale pure, then the weight is what Kim said, its weight should be $i/2$.

So how are you going to write this up with a non-projective or singular variety.

What I'm saying is, back in the 60s, Alexander Grothendieck saw this. At the end of the day all of the cohomology theories have the same Betti numbers, and their extra structures talk to each other.

So what are the axioms of Weil cohomology?

(1) It should be a contravariant functor from smooth projective connected varieties over some field to graded vector spaces (or maybe k-algebras) (over some other field).
(2) $H^i(X,k) = 0$ when $i < 0$ or $i > 2\dim X$, and
(3) $H^{2\dim(X)}(X) \cong k$,
(4) Poincaré duality: there is a perfect pairing $H^i(X) \times H^{2\dim X - i} \to k$,
(5) The Künneth formula holds, $H^n(X \times Y) \cong \bigoplus_{i+j} H^i(X) \otimes_k H^j(X)$.
(6) The existence of a cycle map η^i, an Abelian group of algebraic cycles, this is a smooth connected subvariety of codimension i equipped with a map $\eta^i(X) \to H^{2i}(X)$.
(7) The weak Lefschetz isomorphism (I won't discuss this)
(8) The hard Lefschetz isomorphism (I won't discuss this)

Once you have a Weil cohomology, when they satisfy these conditions, they should coincide as vector spaces in terms of ranks, and even then each of the structures have their own stories, period isomorphisms, point counts, and so on.

So what Grothendieck asked was what is *behind* cohomology, he said there should be an object, a $\mathbb{Q}$-linear Abelian category in which all the Weil cohomology would factor through, $\mathcal{M}_k$, the category of motives over k.

So the $\mathcal{M}(\mathbb{P}^1)$ should be $\mathcal{M}(\mathbb{A}^1) \oplus \mathcal{M}(\mathrm{pt})$, so this is $\mathbb{L} + 1$ which tells you that $H^i(\mathbb{P}^1, k)$ is rank 1 in degrees 0 and 2. For morphisms, when $Z \subset X \times Y$, then we ask $\mathcal{M}(X) \to \mathcal{M}(Y)$ (over $\mathcal{M}(Z)$) to be the closed subvarieties of $X \times Y$ [sic].

This is a very bold statement. The cohomologies that we talked about, if I assume these things, then, well, the reason I'm struggling is that defining this correctly is hard. The period isomorphism and the trace formula should be seen through something as abstract as this.

This failed. So but you can talk about Chow motives, these all factor through that but this is not Abelian. This is the subject of motivic cohomology.

So what one can do is this. There is a triangulated category in which all the Weil cohomologies factor through and if I have a map $X \times \mathbb{A}^1 \to X$, then $\mathcal{M}(X \times \mathbb{A}^1) \cong \mathcal{M}(X)$, the homotopy axiom. This is called MT_k, the Voevodsky motive. I'm skipping a whole lot of things.

People want to find so-called "mixed motives" that work for non-smooth (singular) and non-projective cases. That's what Voevodsky and his school did.

So Tate motives, DMT_k, this is a triangulated category of mixed motives, and this says that a motive of X is polynomial in $\mathbb{L}$, it's Tate type, this triangulated category

[some discussion of possible future directions]

2. January 15: Damien Lejay: Introduction to spectra

This is going to be informal, so this has two meanings. First, stop me whenever you want, ask a question. On my part, it means that I can lie and not give you the full details. All right? So last time we saw an algebraic cohomology in the sense of Weyl. I'll remind you of the definition and then we'll see what the definition tells us.

A *cohomology theory* is a functor H^* from smooth projective varieties, contravariant, it should go to graded Abelian groups, so they'll be graded commutative algebras with a non-degenerate trace,

$$H^* : SPV^{\mathrm{op}} \to K - \mathrm{alg}^{\mathrm{Tr}}.$$

So K here is characteristic zero field. All my varieties are over $\mathbb{C}$, an algebraically closed field. In the domain, you have the symmetric monoidal structure of taking the product, and on the right you have the tensor product, and you require this to be symmetric monoidal, taking the product to the tensor product. You should have some shifts, Tate twists, and blah blah blah, because it's complicated.

If you only satisfy this, you know nothing from this, where is the algebraic geometry. What you always require on top of that is a transformation $H^*_{\mathrm{cl}}(-, k) \to H^*$. Then every time you give yourself a variety, you should have something in H^*, and this is compatible with pullback, and non-degenerate, the trace of a point is 1, so you can recover something non-trivial.

You can think of this as a kind of sheaf of rings, and you have a morphism of rings from the classical object to any of your homology theories. Anything is below this initial object, the classical cohomology. It's like the way $\mathbb{Z}$ controls a lot about rings, this classical cohomology gives this kind of control.

One of the examples of such a thing is the topological cohomology, the cohomology with values in any field that you like. I was thinking that this is very related to this, but maybe this is not a cohomology theory. It's a bit as for quantum field theories, maybe it's too hard and you look at those that are topologically invariant. So we know that $H^*(-,K)$ is topologically invariant. People have asked for a long time, how do I make this representable, say that this is the same as the maps into some object, $H^*(-,K) \cong \mathrm{Hom}(-,?)$.

In algebraic topology the answer is via spectra.

You have what is called a generalized cohomology theory and what is called a generalized multiplicative theory. You could ask to have cohomology groups, and how they behave linearly, and this is spectra. If you add a multiplicative structure, you get something called ring spectra. Then later we can add a multiplicative structure.

There's a very famous paper in 1945 of Eilenberg and Steenrod that I invite you to read, it's only 6 pages and absolutely marvelous. I'll write the axioms that they give for a homology theory. Then we'll simplify the axioms and change the category a bit. A (co)homology theory is a functor that takes pairs (X,A) (where X is a space and A a closed subspace) to $H_n(X,A)$ (I put no coefficients), which should be functorial, it means that for $f:(X,A)\to(Y,B)$ (that is, a map $f:X\to Y$ so that $f(A)\subset B$) I get maps

$$H_n(X,A)\xrightarrow{f_*}H_n(Y,B)$$

and a boundary map

$$H_n(X,A)\xrightarrow{\partial}H_{n-1}(A)=H_{n-1}(A,\varnothing),$$

and I want this data to be functorial too.

I also want to posit long exact sequences

$$\cdots\to H_q(X)\to H_q(X,A)\to H_{q-1}(A)\to\cdots$$

and I ask that this be a (long) exact sequence.

The final axiom is excision, whenever you have $U\subset A\subset X$ with U open and its closure contained in an open set V inside A, you have

$$H_q(X\smallsetminus U,A\smallsetminus U)\cong H_q(X,A)$$

which I don't understand but let me say something I do understand that's equivalent

$$H_q(X,A)\cong H_q(X/A,A/A)$$

in good cases, and the final axiom that is no longer used is

$$H_q(*)=0$$

if $q\neq 0$.

One thing you can see with some computation, you need $H_*(\varnothing)=0$, so you can factor this through the category of pointed spaces. So you can go to pointed topological spaces, where every space has a specified point. The point is initial and is still the terminal object, and $\mathrm{Ab}^{\mathbb{Z}}$ is pointed too, by zero, so $\mathrm{Top}_*\to\mathrm{Ab}^{\mathbb{Z}}$

is the *reduced homology*. The axiom becomes a bit more manageable if we write them for reduced homology theories. This you can think of as a baby example of a factorization, we go toward more representability by doing this. Then H_* we say is a functor and when we take a map of pointed spaces, you can take the quotient which should give you exact sequences in good cases $\widetilde{H}_q(X) \to \widetilde{H}_q(Y) \to \widetilde{H}_q(Y/X) \to \cdots$

One axiom I forgot, not always included, is that homology should commute with disjoint unions. Another property that you can derive from this is, there is something called a suspension, and one property of a suspension is that $\widetilde{H}_q(\Sigma X) \cong \widetilde{H}_{q-1}(X)$, and this is something that fills this square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$

and we do this homotopy invariantly, so this is a cone with its end glued to a point.

So instead of the boundary, you can ask $\widetilde{H}_q(\Sigma X) \cong \widetilde{H}_{q-1}(X)$. I forgot to say that if $X \sim Y$ is a homotopy equivalence then you get an induced equivalence.

So the things you get are homotopy invariant, shift along the suspension functor, and give you exact sequences. If you want you also have $\widetilde{H}_n(\bigvee X_i) = \bigoplus H_n(X_i)$.

Nom I want to talk about representability. If A is nice and you have a functor $A^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}$ which commutes with all small limits. Then F gets a (left) adjoint G. Because the category of sets is generated by a point, you have a single object of A^{op} which is the value at a point. Then $F(X) = \mathrm{Hom}(X, G(*))$.

The exact sequence for quotients and for disjoint unions is about commutation with limits, but the homotopy axiom and the suspension axiom are not.

Let's say you have a map of groups between a group G and an Abelian group A. Because A is Abelian, you can always factor through the quotient $G/[G,G]$. We have functors that respect the homotopy thing and suspension thing. So we should take topological spaces and go to something where we have inverted or quotiented out by homotopies, where the objects are homotopy types of topological spaces.

One way to build this is to keep the same objects and add inverses of homotopy equivalences. If you do this in a universal way, you get this homotopy category. Now suppose you have a functor F from Top to $\mathcal{C}$ so that when you take a homotopy equivalence it goes to an isomorphism, then you can always factor F through the localization to this homotopy category.

What this means is that our homology or cohomology theories are homotopy invariant and should factor through this homotopy category.

The only thing is that by doing this you destroy all your understanding of the category. So what people do to understand this, they put a model structure on Top, that's a lot of extra data to add but then we can have a fine-grained understanding when we invert all the things.

Because you're invariant by the shift functor, you have to localize again to say that the shift is invertible. So now $\mathrm{Top}[he^{-1}][\Sigma^{-1}]$ is the category of spectra.

There are many models for these spectra, but I'll give one model. I'll go back to generalized cohomology theory and do that step by step in a kind of stupid way.

So when I write $H^n(X)$ I imagine I have a special space so that this is $[X, E_n]$, equivalence classes of continuous maps up to homotopy. Let's say for every n I can represent that functor. Then since

$$\widetilde{H}^n(\Sigma X) cong \widetilde{H}^{n-1}(X).$$

So I get $[\Sigma X, E_n] \cong [X, E_{n-1}]$, and $[\Sigma X, E_n]$ is isomorphic to $[X, \Omega E_n]$. So I can ask that $E_0 \cong \Omega E_1$ et cetera, so that $E_{n-1} \cong \Omega E_n$.

This is called an Ω-spectrum, a bunch of pointed topological spaces and homotopy equivalences like this. Then we'd say $E^n(X) \cong [X, E_n]$. In the literature when people start speaking of spectra, they use E^* for the cohomology theory.

You can put a model structure on spectra (sort of) and the best example is the Eilenberg–MacLane spectrum, there's a space $K(n, \mathbb{Z})$, where e.g., $K(1, \mathbb{Z}) \cong S^1$, and $K(n-1, \mathbb{Z})$ is equivalent to $\Omega K(n, \mathbb{Z})$. You could do the same thing with A.

Now I'll mention one of the coolest things about Ω-spectra. I should tell you about homotopy groups. I give myself an Ω-spectrum E_*, with $E_0 \cong \Omega E_1$ etc.

I know that $\pi_i(E_0) \cong \pi_i(\Omega E_1) = \pi_{i+1}(E_1)$, and so this is also the same as $\pi_{i+2}(E_2)$, and is also $\pi_{i+n}(E_n)$. I can then define $\pi_0(E)$ to be $\pi_0(E_0)$. For $\pi_1(E)$ I can define $\pi_1(E)$ to be either $\pi_1(E_0)$ or $\pi_1(E_1)$.

Now there's something I can do, I can define $\pi_{-1}(E)$ to be $\pi_0(E_1)$. So now I have the negative groups that I can define only starting at a certain point, $\pi_{-n}(E) = \pi_0(E_n) \cong \pi_1(E_{n+1}) \cong \cdots$, so now this has π_i in each direction. Now a homotopy equivalence is a map of pointed spectra that induces an equivalence of all π_i.

This is the same as Ω-spectra where I've inverted homotopy equivalences. So now I want to be able to invert homotopy equivalences. I have a concrete category, and once I invert these maps, I get the category of spectra that I want, the localization of Ω-spectra.

Now I have the Brown representability theorem, which says that generalized cohomology theories, up to homotopy, are the same as Ω-spectra, up to homotopy. You can also represent maps between cohomology theories as maps between spectra.

So say you define a reduced cohomology theory $\mathrm{Top}_*^{\mathrm{op}} \xrightarrow{\widetilde{H}^*} \mathrm{Ab}^{\mathbb{Z}}$, and there is a functor from $\mathrm{Top}_*^{\mathrm{op}}$ to $\mathrm{Sp}^{\mathrm{op}}$, called Σ^∞, and

$$\widetilde{H}^*(X) \cong \mathrm{Hom}_{\mathrm{Sp}}(\Sigma^\infty X, E)$$

Well, this needs shifts so let me say instead that $\widetilde{H}^n(X) \cong [X, E_n]$, and that is the solution to the problem of motives in this case.

This is a linear version, we didn't have the monoidal structure. There is a monoidal structure on spectra, topological spaces have the product structure. For pointed topological spaces, then the product becomes something called the smash product $\wedge$. One way is that you take two pointed topological spaces, which is the quotient of the product $X \times Y$ by the (pointed) sum $X \wedge Y$. This has the property that you want which is that $\mathrm{Hom}_*(X \wedge Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$, and this smash product can be defined on the category of spectra. If you take Ω-spectra it's not easy to define.

I'll tell you about another model, which is called just a pre-spectrum. A prespectrum is a bunch of pointed spaces and instead of having homotopy equivalences you just have maps $\Sigma E_n \to E_{n+1}$, what is absolutely clear is that an Ω-spectrum gives you a spectrum, because you can use the adjunction to get from $E_n \cong \Omega E_{n+1}$ to $\Sigma E_n \to E_{n+1}$, but we don't ask about anything being a homotopy equivalence. With prespectra you can easily define the smash product. Then I can tell you the suspension spectrum. Take X a pointed space and then suspend it n times. This is a prespectrum. This is a priori not an Ω spectrum. The nice way to go from pointed topological spaces to spectra goes to prespectra, which might not land in Ω-spectra, to do that you want to localize at homotopy equivalences. This is what people call Σ^∞. If you suspend S^0 many times you get the sphere spectrum $\mathbb{S}$, probably the most important spectrum, so this one represents the stable homotopy groups.

I have other things to say, but anyway that's another model. People also use other models. I wanted to introduce those to give the smash product of prespectra.

If you have E and F you want to define $(E \wedge F)_{2n}$, which will be $E_n \wedge F_n$ and I'll define $(E \wedge F)_{2n+1}$ as $E_n \wedge F_n \wedge S^1$, so this is the suspension $\Sigma(E_n \wedge F_n)$. With a lot of work you can see that this is compatible with homotopies. So you can say what is a monoid object in this homotopy category. I want a monoid in the category of spectra, this is a *ring spectrum*, so you have $E \wedge E \to E$ with associativity, unit, so on, up to homotopy. Then the cohomology theory becomes multiplicative.

So this gives an equivalence between multiplicative cohomology theories and ring spectra. Up to now we were talking about non-multiplicative theories. These are central to people doing stable homotopy theory. The sphere spectrum is the unit for the smash product, so it has the same universal property that $\mathbb{Z}$ has for rings, it's hard to understand, we don't know its homotopy groups. But it acts everywhere.

With the Eilenberg–MacLane spectrum, you can turn a ring into HR, the ring spectrum, and you have a morphism of ring spectra $\mathbb{S} \to HR$, including $H\mathbb{Z}$, so you can go below $\mathbb{Z}$ to $\mathbb{S}$, and this is what people want to do to go to *spectral* algebraic geometry. This is stuff, K-theory, cobordism, between $\mathbb{S}$ and $H\mathbb{Z}$.

I could speak about topological K-theory, maybe, with my remaining time. So K-theory has things in all directions, not just in positive degrees.

So I'll define K-theory as a spectrum, I'll define an Ω-spectrum. Because of my equivalence that's the same as defining the cohomology theory. If X is compact, then I'll define, and then I'll extend by universal constructions to other spaces.

So $K^0(X)$ is the set of equivalence classes of finite dimensional vector bundles on X, complex vector bundles and because I can always take the sum of vector bundles, I get the direct sum, and I want an Abelian group, and there's a universal construction to give a group, with pairs of vector bundles (V_1, V_2) with the relation $(V_1, V_2) \sim (V_1 \oplus Z, V_2 \oplus Z)$ where Z is anything.

Since X is compact, any vector bundle is a factor of a free bundle. I can always find a Z so that $V_1 \oplus Z \cong \mathbb{C}^n$. So my Abelian group of $(\mathrm{Vect}(X), \oplus)$ as (V, n) with $n \in \mathbb{Z}$, I think of this as $V - \mathbb{C}^n$. This is representable already by a topological space, there is a candidate for that. People in algebraic geometry can say that this is the same as the homotopy classes of maps from X to $BU \times \mathbb{Z}$, this is BU the classifying space for linear bundles of varying dimension and n is the virtual class.

I said that I want a spectrum, so I need to define the other guys. I could say that $K^{-1}(X)$, this has to be $[X, \Omega(BU \times \mathbb{Z})]$ and K^{-2} is $[X, \Omega^2(BU \times \mathbb{Z})]$. Then Bott periodicity says that $K^{-2}(X) \cong K^0(X)$. So this has non-trivial groups in both directions.

There are variants, like with real bundles, where you have BO, and then you have eight-fold periodicity. That's an example of a spectrum, in fact a ring spectrum, which comes from the external tensor product.

The literature is enormous on that.

3. January 29: Yong-geun Oh: reminder on triangulated categories

This talk will be very elementary and may bore some of you, but going slowly may be helpful for some of the audience.

Let me start with category theory, with Abelian categories, what this is about. Let me first introduce additive categories, which have some kind of linear structure. We exploit the knowledge of modules where we know what linearity is. Many definitions are using some kind of functor which maps something in the given category to a category of modules. Then many objects in this abstract category are defined by representatives of functors satisfying certain properties.

Definition 3.1. An *additive category* $\mathcal{C}$ is a category such that

(1) for any pair (X, Y) of objects, the space of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of an additive group, an Abelian group, or $\mathbb{Z}$ modules, and the composition map is linear, so given two such groups, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$, this map is bilinear,
(2) there is a zero object 0, such that the space of morphisms $\mathrm{Hom}_{\mathcal{C}}(0, 0) = 0$,
(3) for any pair (X, Y) in the objects of the category, the functor $\mathcal{C} \to \mathrm{Mod}(\mathbb{Z})$
$$(-) \mapsto \mathrm{Hom}_{\mathcal{C}}(X, -) \times \mathrm{Hom}_{\mathcal{C}}(Y, -)$$
is representable, meaning that there exists $X \coprod Y$ such that for any W, we have
$$\mathrm{Hom}_{\mathcal{C}}(X \coprod Y, W) \cong \mathrm{Hom}_{\mathcal{C}}(X, W) \times \mathrm{Hom}_{\mathcal{C}}(Y, W),$$
(now this definition works for any category),
(4) a similar statement for the other way, $X \prod Y$ represents
$$(-) \mapsto \mathrm{Hom}_{\mathcal{C}}(-, X) \times \mathrm{Hom}_{\mathcal{C}}(-, Y)$$
as well—and in fact $X \coprod Y$ and $X \prod Y$ are isomorphic and so can be denoted $X \oplus Y$ which has natural maps $i_1 : X \to X \oplus Y$ and $i_2 : Y \to X \oplus Y$.

This is the object level, given maps from $X \to W$ and $Y \to W$ then there is a unique map making the diagram commute:

$$\begin{tikzcd}
X \arrow[dr, "i_1"] \arrow[drr, "f"] & & \\
& X \oplus Y \arrow[r, dotted, "h"] & W \\
Y \arrow[ur, "i_2"] \arrow[urr, "g"'] & &
\end{tikzcd}$$

Now the next thing we are going to do, we're given an additive category, but working with a module category, w can always define kernels and cokernels, and we want the same thing in this additive category. We'll state the universal property, defining kernel and cokernel as representatives of a certain functor.

So say $\mathcal{C}$ is an additive category, and Z an object of it. Then we are going to, for a given morphism $f : X \to Y$, we'll associate a similar morphism in the category of modules, we associate $\mathrm{Hom}_{\mathcal{C}}(-, f)$, this is again a functor, so we have
$$\mathrm{Hom}_{\mathcal{C}}(Z, f) : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$$

So we'll consider this $\mathrm{Hom}_{\mathcal{C}}(-,f)$ as a functor from $\mathcal{C}$ to $\mathrm{Mod}(\mathbb{Z})$. So we can talk about kernels. Then we have a functor which I denote

$$(-) \mapsto \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(-,f)).$$

More explicitly, for each object W we look at the kernel of $\mathrm{Hom}_{\mathcal{C}}(W,f)$, which is a map from $\mathrm{Hom}_{\mathcal{C}}(W,X)$ to $\mathrm{Hom}_{\mathcal{C}}(W,Y)$.

This is a subset of the domain, $\phi \in \mathrm{Hom}_{\mathcal{C}}(W,X)$ such that $f \circ \phi = 0$.

We want this to be representable, so when this functor is representable, its representing object we call the *kernel* of f and denote it ker f. Unlike the kernel of linear homomorphisms. It's not a subset of X necessarily. It's another object in the abstract category, and at the moment $\ker f \subset X$ does not make sense.

This is the object level, I want to state the characterizing property. By definition, we have the following isomorphisms:

$$\mathrm{Hom}_{\mathcal{C}}(W, \ker f) \cong \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(W,f)).$$

Similarly, the cokernel is defined by considering another functor

$$(-) \mapsto \mathrm{Coker}(\mathrm{Hom}_{\mathcal{C}}(f,-)).$$

Let me state several properties of kernels. Suppose $f : X \to Y$, in an additive category the kernel may or may not exist. Suppose $f : X \to Y$ has a kernel. Then we have a natural morphism (natural transformation) which I'll denote $\beta : \mathrm{Hom}_{\mathcal{C}}(-, \ker f) \to \mathrm{Hom}_{\mathcal{C}}(-,X)$. How is it defined? Let Z be an object of $\mathcal{C}$ and consider $\beta_Z : \mathrm{Hom}_{\mathcal{C}}(Z, \ker f) \to \mathrm{Hom}_{\mathcal{C}}(Z,X)$. Let

$$\alpha \in \mathrm{Hom}_{\mathcal{C}}(Z, \ker f) \cong \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(Z,f)),$$

which is a subset of the domain $\mathrm{Hom}_{\mathcal{C}}(Z,X)$. Therefore we have this map β_Z. The property of morphisms you can check. In particular, let's apply this to $Z = \ker f$, then we have the identity element of $\mathrm{Hom}_{\mathcal{C}}(\ker f, \ker f)$, then β of this identity element defines a map ker $f \to X$, which we denote α. In this way, we cannot think of it as a subset but at least there's a natural map, and the universal property of kernels, the kernel of f has the universal property that for any morphism $e : W \to X$, whenever you look at $f : X \to Y$, whenever $f \circ e = 0$, there is a lift

$$\begin{tikzcd} W \arrow[r, "e"] \arrow[dr, dashed] & X \arrow[r, "f"] & Y. \\ & \ker\ f \arrow[u, "\alpha"] & \end{tikzcd}$$

Now I want to denote images or coimages.

Definition 3.2. Assume $\alpha : \ker f \to X$ has cokernel, then we denote $\mathrm{coker}(\alpha)$ as the coimage of f. Similarly, if the associated canonical map for the cokernel $\gamma : Y \to \mathrm{coker}\, f$ has a kernel, then we denote it by the image of f.

One important lemma is the following.

Lemma 3.1. *Assuming all these exist, there is a natural map from the coimage to the image.*

I'll give the proof of this.

Proof. We have

$$\ker f \xrightarrow{\alpha} X \xrightarrow{f} Y.$$

We have a natural map from X to the cokernel $\operatorname{coker}(\alpha : \ker\ f \to X)$ By definition, $f \circ \alpha = 0$ so there is a map δ from this cokernel to Y. Then we consider a similar diagram, we have

$$\operatorname{coker}(\alpha) \xrightarrow{\delta} Y \xrightarrow{\gamma} \operatorname{coker} f$$

and we have an inclusion of $\ker \gamma$ into Y. We check that $\gamma \circ \delta = 0$, so by universal properties there is a map from $\operatorname{coker}(\alpha)$ to $\ker(\gamma)$. □

In general this map may not be an isomorphism. Here comes the definition of an Abelian category.

Definition 3.3. An additive category $\mathcal{C}$ is called an *Abelian* category if

(1) any morphism has a kernel and cokernel, and
(2) the natural map from the coimage of f to the image of f is an isomorphism for any f.

You may wonder what is going on. Here are some examples of additive categories which are not Abelian.

(1) First you have topological categories. If you have Banach spaces, sometimes you need to do things that aren't algebraic, like closure. In Banach spaces (normed complete vector spaces with morphisms the bounded linear maps), the kernel of f is canonically defined, but the cokernel is not. For linear things, it's usually the quotient of the target by the image. So there is no canonical way of defining a Banach space on the quotient. The cokernel of f is $Y/\overline{\operatorname{im} f}$. You can check that this satisfies the defining property of a cokernel. But the problem is, there exists a homomorphism f such that both $\ker f = 0$ and $\operatorname{coker} f = 0$ but f is not an isomorphism. If you densely define an operator.
(2) this is more algebraic, this is the category of filtered modules over a filtered ring, maybe this is more relevant to the Fukaya category.

Let me take a five minute break.

Now I want to talk about triangulated categories, and the motivation will be from example, with the homotopy category of complexes, once we talk about this, the homotopy category of complexes over an additive category, this will eventually give us triangulated categories.

So let's talk about the homotopy category of complexes. Let's say $\mathcal{C}$ is an additive category, then a complex in $\mathcal{C}$ is a sequence $X = (X^n, d_X^n)$ for $n \in \mathbb{Z}$ where X^n is an object and d_X^n is a map from X^n to X^{n+1} satisfying that consecutive compositions are zero in $\operatorname{Hom}_{\mathcal{C}}(X^n, X^{n+2})$.

The morphisms between two complexes X and Y are the $\{f^n\}$, a sequence of maps $f^n : X^n \to Y^n$ that makes, that commute, so that the diagram commutes

$$\begin{tikzcd} X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[d, "f^{n+1}"] \\ Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \end{tikzcd}$$

Let $C(\mathcal{C})$ be the category of complexes. This is again an additive category, let me not talk about this but you do most things termwise.

Definition 3.4. The *shifted complex*, for k an integer and X a complex in $C(\mathcal{C})$, a new complex $X[k]$, is defined by

$$X[k]^n = X^{n+k}$$
$$d^n_{X[k]} = (-1)^k d^{n+k}_X.$$

For a given morphism $f : X \to Y$, we denote by $f[k] : X[k] \to Y[k]$ the map $f[k]^n = f^{n+k}$.

Given a category, you can quotient by a collection of morphism, we'll define the homotopy category as the quotient by some collections.

Definition 3.5. A morphism $f : X \to Y$ in $C(\mathcal{C})$ is called *homotopic to zero* if there exist morphisms $s^n : X^n \to Y^{n-1}$ such that

$$f^n = s^{n+1} d^n_X + d^{n-1}_Y s_n.$$

Lemma 3.2. *The composition, we want to quotient this category by a collection of morphisms, we want this to be multiplicative, so the composition*

$$\mathrm{Hom}_{C(\mathcal{C})}(X,Y) \times \mathrm{Hom}_{C(\mathcal{C})}(Y,Z) \to \mathrm{Hom}_{C(\mathcal{C})}(X,Z)$$

maps

$$\mathrm{Ht}(X,Y) \times \mathrm{Hom}_{C(\mathcal{C})}(Y,Z) \sqcup \mathrm{Hom}_{C(\mathcal{C})}(X,Y) \times \mathrm{Ht}(Y,Z)$$

to $\mathrm{Ht}(X,Z)$*. So this allows us to define*

Definition 3.6. The category $K(\mathcal{C})$ is defined by, at the object level it's the category of complexes $C(\mathcal{C})$, and the morphisms from X to Y are the maps in $C(\mathcal{C})$ from X to Y quotiented by $\mathrm{Ht}(X,Y)$, i.e., the set of homotopy classes of chain maps. There are many ways of changing the morphisms, you can try to define these as some kind of homology.

The category of complexes will be an Abelian category of $\mathcal{C}$ is Abelian, but $K(\mathcal{C})$ may not be Abelian even if $\mathcal{C}$ is Abelian. So you need to go beyond in general.

Definition 3.7. Assume that $\mathcal{C}$ is Abelian. We denote by $Z^k(X)$ the kernel of d^k_X and by $B^k(X)$ the image of d^{k-1}_X and $H^k(X)$ the cokernel of the canonical map $B^k(X) \to Z^k(X)$, in other words $Z^k(X)/B^k(X)$, the kth cohomology group of X.

So H^k is an additive functor from $C(\mathcal{C})$ to $\mathcal{C}$, and $H^k(X)$ satisfies $H^0(X[k])$. We have canonical short exact sequences that hold in cohomology groups in the category of complexes in Abelian groups. For example, let me state $X^{k-1} \to Z^x(X) \to H^k(X) \to 0$. You have to check things because you don't have subgroups and so on. I want to state one important proposition.

Proposition 3.1. *You can define the notion of exact sequence, and for a short exact sequence* $0 \to X \to Y \to Z \to 0$ *in* $C(\mathcal{C})$ *you have a long exact sequence*

$$\cdots \to H^n(X) \to H^n(Y) \to H^n(Z) \to H^{n+1}(X) \to \cdots$$

The only thing you might want to check is the existence of the connecting morphism, but I want to skip it. So many of the basic homological algebra tools work in Abelian categories. Mitchell's theorem says that any Abelian category is embedded into a category of modules, so Abelian categories are in many senses equivalent to the category of modules.

So now I want to talk about mapping cones. This works for any additive category. Suppose $f : X \to Y$ are morphisms in the category of complexes $C(\mathcal{C})$.

Definition 3.8. The mapping cone $M(f)$ of $f : X \to Y$ is the object in $C(\mathcal{C})$ defined by, $M(f)^n = X^{n+1} \oplus Y^n$ with differential the lower triangular matrix

$$\begin{pmatrix} d^n_{X[1]} & 0 \\ f^n & d_Y \end{pmatrix}$$

There is a natural map $\alpha(f) : Y \to M(f)$ and $\beta(f) \to X[1]$ defined by $\alpha(f)^n = \begin{pmatrix} 0 \\ \mathrm{id}_{Y^n} \end{pmatrix}$ and $\beta(f)^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \end{pmatrix}$.

Here is an important lemma satisfied by the mapping cone. For any morphism $f : X \to Y$, there is a map $\phi : X[1] \to M(\alpha(f))$ such that

(1) ϕ is an isomorphism in $K(\mathcal{C})$ the homotopy category and
(2) the diagram

$$\begin{tikzcd} Y \arrow[r, "\alpha(f)"] \arrow[d] & M(f) \arrow[r, "\beta(f)"] \arrow[d] & X[1] \arrow[r, "{[1]}"] \arrow[d] & Y[1] \arrow[d] \\ F \arrow[r] & M(f) \arrow[r] & M(\alpha(f)) \arrow[r, "\beta(\alpha(f))"] & Y[1] \end{tikzcd}$$

commutes.

Maybe I will write down what this ϕ is, and its homotopy inverse.

Proof.

$$M(\alpha(f))^n = Y^{n+1} \oplus M(f)^n = Y^{n+1} \oplus X^{n+1} \oplus Y^n$$

and we define $\phi^n : X[1]^n \to M(\alpha(f))^n$ as

$$\begin{pmatrix} -f^{n+1} \\ \mathrm{id}_{X^{n+1}} \\ 0 \end{pmatrix}$$

and $\psi^n = (0, \mathrm{id}_{X^{n+1}}, 0)$ is a homotopy inverse of ϕ. You can check that $\psi \circ \phi = \mathrm{id}_{X[1]}$ and the other direction is homotopic to the identity. Also $\psi \circ \alpha(\alpha(f)) = \beta(f)$ and $\beta(\alpha(f)) \circ \phi \cong -f[1]$. □

Definition 3.9. A *triangle* $X \to Y \to Z \to X[1]$ in $K(\mathcal{C})$ is *distinguished* if it is isomorphic to the basic triangle $X \to Y \to M(f) \to X[1]$ for some f.

So a triangle is called a distinguished triangle if the triangle is isomorphic to the basic triangle. Now let me state the basic properties will be the same as the definition of a triangulated category if you replace the homotopy category by any general additive category.

Theorem 3.1. *The collection of distinguished triangles above satisfy the following.*

TR0 Any triangle isomorphic to a distinguished triangle is distinguished.

TR1 (existence of enough distinguished triangles) if X is in $\mathrm{Ob}(K(\mathcal{C}))$*, then $X \to X \to 0 \to X[1]$ is distinguished.*

TR2 Any $f : X \to Y$ in $K(\mathcal{C})$ can be embedded into a distinguished triangle, meaning that $X \xrightarrow{f} Y \to Z \to X[1]$ is for some Z and maps.

TR3 (rotation) If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$, then the rotation

$$Y \to Z \to X[1] \to Y[1]$$

is distinguished.

TR4 *Given two distinguished triangles $X \to Y \to Z \to X[1]$, then $X' \to Y' \to Z' \to X'[1]$ and maps $u : X \to X'$ and $v : Y \to Y'$ making the diagram commute, this can be extended to a morphism of triangles.*

TR5 *(the most mysterious axiom, the octahedron axiom)—then I'll stop. Suppose given two triangles $X \xrightarrow{f} Y \to Z' \to X[1]$ and $Y \xrightarrow{g} Z \to X' \to Y[1]$, and there is a distinguished triangle $X \xrightarrow{g \circ f} Z \to Y' \to X$, then there exist triangle $Z' \to Y' \to X' \to Z'[1]$:*

$$\begin{tikzcd}
X \arrow[rr] \arrow[rrrr, bend left] & & Y \arrow[rr] \arrow[dl] & & Z \arrow[dl] \arrow[dll, bend left] \\
& Z' \arrow[ul] \arrow[dr] & & X' \arrow[ul] \arrow[ll] & \\
& & Y' \arrow[ur] \arrow[ull, bend left] & &
\end{tikzcd}$$

and various things commute and are distinguished.

For the homotopy category all these are satisfied.

Now here is the definition, you replace the homotopy category with an additive category which satisfies these five axioms, and the prototype is the homtopy category of an Abelian category.

4. Feb 12: Taesu Kim: t-structures on triangulated categories

So actually I have to say, I changed my original plan, apologies for that. I'm going to use a little bit different lanugage for talking about triangulated categories and T-structures. Suppose we have a category $\mathcal{C}$ with a 0 object, meaning initial and terminal, and suppose we have homotopy pushouts. Let me not elaborate on the precise definition of this, but you can think of it as a usual pushout, where the diagram only commutes up to some homotopy, and the uniqueness for the universal property is replaced by uniqueness up to some contractible choice in some sense, and that's the kind of rough definition of a homotopy pushout. We'll consider a category which admits all possible homotopy pushouts, and the diagram we'll think of will be of this type:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & Z
\end{tikzcd}$$

and we'll call this a distinguished triangle. I'll denoe this by Δ from now on. If we look at this diagram, for instance,

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \arrow[d] \\
0 \arrow[r] & 0
\end{tikzcd}$$

this is a homotopy pushout so this is always a distinguished triangle. Another example is the suspension

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X[1]
\end{tikzcd}$$

and then we see a rotation:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\
 & 0 \arrow[r] & Y[1]
\end{tikzcd}$$

and all of these are distinguished.

So another is

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[rr, bend left] & Y \arrow[d] \arrow[rr, bend left] & X' \arrow[r] \arrow[d] & Y' \arrow[d] \\
0 \arrow[r] & Z \arrow[rr, bend right] & 0 \arrow[r] & Z'
\end{tikzcd}$$

but this doesn't have to be unique.

One last example, suppose we are given

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
0 \arrow[r] & Z'
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r] \arrow[d] & Z \arrow[d] \\
0 \arrow[r] & X'
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d] & Z \\
0 \arrow[r] & Y'
\end{tikzcd}$$

then we get

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
0 \arrow[r] & Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\
 & 0 \arrow[r] & X'
\end{tikzcd}$$

and we get that

$$\begin{tikzcd}
Z' \arrow[r] \arrow[d] & Y' \arrow[d] \\
0 \arrow[r] & X'
\end{tikzcd}$$

is a homotopy pushout.

Last time Yong-Geun talked about $K(A)$, where you had complexes and chain maps. Today I'll talk about the derived category $D(A)$, where this is given by definition as follows, we take $C(A)$ and localize by the quasi-isomorphisms, and we have the following universal property, if $C(A)$ to D' takes quasi-isomorphisms to isomorphisms then there is a unique functor from $D(A) = C(A)[W^{-1}]$ to D' making the triangle commute.

That's an example of a triangulated category, and we have to specify what the triangles are, and the answer is

$$\begin{tikzcd} X_* \arrow[r, "f"] \arrow[d] & Y_* \arrow[d] \\ 0 \arrow[r] & C(f) \end{tikzcd}$$

and the suspension is degree shift by 1. The homology functor from $D(A)$ to A is the cohomology, and this is well-defined because quasi-isomorphism does not affect homology.

Now I consider $\tau_0 D(A)$ consisting of X the objects in $D(A)$ so that $H^i(X_*) = 0$ if $i \neq 0$. We consider all those objects, and these objects, we denote $\tau_0 D(A)$. We can show that this is equivalent to A.

We get an Abelian category inside the derived category, and the motivation for t-structures is to invert this. So we let D be an arbitrary triangulated category. We are going to identify some Abelian category inside of this, which will contain some important information. We consider two full subcategories $D_{\geq 0}$ and $D_{\leq 0}$, this is a *t-structure* if it satisfies the following axioms. Let me make the notation $D_{\geq n} = D_{\geq 0}[n]$ and similarly for $D_{\leq n}$

(1) $D_{\geq 1} \subset D_{\geq 0}$ and $D_{\leq 1} \supset D_{\leq 0}$
(2) $\mathrm{Hom}_D(X, Y) = 0$ if $X \in D_{\leq 0}$ and $Y \in D_{\geq 1}$
(3) For any $X \in D$ there exists a distinguished triangle

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] & X'' \end{tikzcd}$$

with $X' \in D_{\leq 0}$ and $X'' \in D_{\geq 1}$.

The *core* or *heart* of D is defined to be $D_{\geq 0} \cap D_{\leq 0}$, and it turns out that this is an Abelian category.

Here is a digression, not for the t-structure story, suppose we have a distinguished triangle $X \to Y \to Z$, and we assume that $\mathcal{C}$ is additive, and we consider the following sequence of Abelian groups,

$$\to \mathrm{Hom}_D(-, X) \to \mathrm{Hom}_D(-, Y) \to \mathrm{Hom}_D(-, Z) \to$$

and

$$\to \mathrm{Hom}_D(Z, -) \to \mathrm{Hom}_D(Y, -) \to \mathrm{Hom}_D(X, -) \to$$

are long exact, this is a fact that will be useful soon.

I'll define $\tau_{\geq 0} : D \to D_{\geq 0}$ and $\tau_{\leq 0} D \to D_{\leq 0}$.

Suppose X is an object, and we have $X' \to X \to X''$ a triangle, with X' in $D_{\leq 0}$ and X'' in $D_{\geq 1}$. We define $\tau_{\geq 0}(X)$ to be X'. For another object Y, we choose a pushout diagram $Y' \to Y \to Y''$ a triangle, and if we have $u : X \to Y$, then take the composition $X' \to X \to Y$, and taking $W = X'$ we get

$$\mathrm{Hom}_D(X', Y') \to \mathrm{Hom}_D(X', Y) \to \mathrm{Hom}_D(X', Y'')$$

and by our axioms, $\mathrm{Hom}_D(X', Y'')$ vanishes, so $\mathrm{Hom}_D(X', Y') \to \mathrm{Hom}_D(X', Y)$ is an isomorphism, so there is a lift to give a value for $\tau_{\geq 0}(u)$.

So we get a functor $\tau_{\leq 0}$ and similarly $\tau_{\geq 1}$ and we can see that $\tau_{\leq 0}$ is right adjoint to inclusion $D_{\leq 0} \to D$ and similarly $\tau_{\geq 1}$ is left adjoint to $D_{\geq 1} \to D$.

By using this functor we get $\tau_{\leq n}$ and $\tau_{\geq n}$.

So now we have the two truncation functors $\tau_{\leq m}$ and $\tau_{\geq n}$ which commute in the following sense, one can have $\tau_{\leq n} X \to X \to \tau_{\geq n+1} X$ as a distinguished triangle, and then we have $\tau_{\leq m} X \to X \to \tau_{\geq n} X$ and we have a unique isomorphism making the following diagram commute

$$\begin{tikzcd}
\tau_{\leq m} X \arrow[r] \arrow[d] & X \arrow[r] & \tau_{\geq n} X \arrow[d] \\
\tau_{\geq n}\tau_{\leq m} X \arrow[rr, dashed] & & \tau_{\leq m}\tau_{\geq n} X
\end{tikzcd}$$

By the way, all proofs are in the next talk or omitted. Then $\tau_{[n,m]} = \tau_{\geq n}\tau_{\leq m}$ and for $m = n$ we define $\tau_{[n,n]}$, which is $H^n D$.

In particular, when n is zero, H^0 goes from D to the core of D.

Theorem 4.1. *The core of D is Abelian and H^0 is cohomological (i.e., a distinguished triangle maps to a long exact sequence).*

Suppose $F : D \to D'$ respects the t-structure in the sense that $F(D_{\geq 0}) \subset D'_{\geq 0}$ and $F(D_{\leq 0}) \subset D'_{\leq 0}$ then we call this t-exact. So F should also respect the triangulated structure.

Then F restricts to the core, $F|_{\mathrm{Core}(D)} : \mathrm{Core}(D) \to \mathrm{Core}(D)$.

In the last part of my talk I'm going to talk about examples. In the derived category of an Abelian category, let me give an example of a t-structure. So $D_{\geq 0}$ consists of objects with no cohomology for $i < 0$ and $D_{\leq 0}$ consists of objects with no cohomology for $i > 0$. So then the core consists of X such that $H^i(X) = 0$ for $i \neq 0$.

The functor here from $D(A)$ to A sends X to $H^n(X)$, usual cohomology.

The next example is perverse sheaves, let me not give full details, all these things are from the original, the famous work by these Beilinson–Bernstein–Deligne, in their famous paper "Faisceaux Pervers", which contains t-structures and t-exact functors.

The next example is, since Damien introduced spectra, I felt an obligation to mention, the homotopy category of spectra. Recall that a spectrum is a sequence of pointed topological spaces $\{X_n\}$ equipped with $\Sigma X_n \to X_{n+1}$. For Ω-spectra, we have a map $X_n \xrightarrow{\cong} \Omega X_{n+1}$, and a map of spectra is f_n between X_n and X_{n+1}, and we require a commuting condition.

The stable homotopy groups are

$$\pi_k^s(X) = \lim_{n\to\infty} \pi_{k+n}(X_n)$$

and this stabilizes so that this is a well-defined notion. We consider a map between spectra, it's a π_*^s-isomorphism if $f_* : \pi_*^s(X) \to \pi_*^s(Y)$ is an isomorphism for each index.

Then we localize spectra with respect to these stable isomorphisms, and get the homotopy category of spectra.

Here there is a triangulated category structure, we have homotopy pushouts, the cone of f, and then you have Σ as the shift. The t-structure you give to this category is something like this: $h\,\mathrm{Sp}_{\geq 0}$ are X such that $\pi_i^s(X) = 0$ if $i < 0$ and similarly for ≤ 0. For this t-structure the core can be shown to be equivalent to the category of Abelian groups.

I'll continue next time.

5. Feb 19: Damien Lejay, T-structures

We'll review the modern theory about T-structures. I won't give any proofs. They are either evident or require computations I don't want to make.

I'll try to use the diagrammatic way of thinking, rectangles more than triangles.

So I won't be precise, and I want somehow to settle a bit of vocabulary, we have been talking about triangulated categories, but what we really want is stable ∞-categories, which are gadgets with certain properties.

An ∞-category has objects, for every two objects it has a homotopy type of maps between them, so you can think of this as a topological space up to weak equivalence. You see the problem of saying that I have maps. I don't want to say how to compose maps because it should be homotopical.

This is a regular ∞-category, instead of having a set of maps, you have a space of maps, and from this you can always make a regular category by taking the same objects, and the maps are the connected components of the maps from x to y. This is called the homotopy category.

Here I'm being very loose because once the language is set up I can just use the language. Then I can kill the homotopy information and just get the category. Now I want it to be stable, so I add a 0 object, coproducts, which are the same as products (finite ones). Because of my zero object, I have a map from $X \coprod Y \to X \prod Y$ which is an equivalence. I also want to have pushouts and pullbacks. So for instance the pushout of $X \xrightarrow{f} Y$ along $X \to 0$ is $\mathrm{Cof}(f)$ and the pullback of f along $0 \to Y$ is $\mathrm{Fib}(f)$. I use the special notation ΣX for the pushout $0 \leftarrow X \to 0$ and ΩX for the pullback $0 \to X \leftarrow 0$, and I want stability, Σ is the inverse of Ω.

These axioms are very strong, and there are many ways to see them, in different settings. An equivalent description, is that a square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & T \end{tikzcd}$$

is a pushout if and only if it is a pullback.

So for $Y = Z = 0$, that says that $\Omega(\Sigma X) \cong X$. I drew the pushout but I can do the other order.

The key thing is that when $\mathcal{C}$ is stable, the homotopy category is canonically triangulated. I think it's not true that every triangulated category is the homotopy category of a stable ∞-category but they are artificial.

Let me remind you of the octahedral axiom that is easy to get back in this context. What people call a distinguished triangle is a cofiber

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & \mathrm{Cof}(f) \end{tikzcd}$$

and then the thing you do is compose two maps.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \\
0 \arrow[r] \arrow[d] & \mathrm{Cof}(f) \arrow[r] \arrow[d] & \mathrm{Cof}(g\circ f) \arrow[d] \\
0 \arrow[r] & 0 \arrow[r] & \mathrm{Cof}(g)
\end{tikzcd}$$

By abstract nonsense, the bottom right triangle is a pushout, so it's a distinguished triangle, and this is the octahedral axiom.

Here I'll try not to say triangulated but stable.

I recall examples of stable categories, spectra, the derived category of a nice enough Abelian category. Once you have some building blocks, the traditional category theory language lets you do this, you can for instance take categories of sheaves, sheaves of chain complexes, of spectra, all the things you could do in the Abelian setting you can do in the stable setting.

The only problem with triangulated and stable categories is that there is an *anti-theorem.*

Theorem 5.1. *If $\mathcal{C}$ is stable and $X \to Y$ is a monomorphism (think of a subobject), then $X \cong Y$.*

You don't have monomorphisms. The same is true for epimorphisms. So you can't factorize through images. Somehow all your tools, your gadgets are broken, so you need another way to factorize your maps, and this is where you introduce T-structures.

Normally this is given in your category, so you need to add data in how you factor or truncate. You need to rewire your mind because your intuition changes things. The T-structure replaces the intuition.

If you have $\mathcal{C}$ stable you can define a T-structure either on $\mathcal{C}$ or on $h\mathcal{C}$, you give yourself full subcategories $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{0\geq}$. The axioms are:

(1) You want $\mathcal{C}_{\geq 0}[1] \subset \mathcal{C}_{\geq 0}$,
(2) you want $\mathrm{Hom}(x,y) = 0$ if $x \in \mathcal{C}_{\geq 0}$ and $y \in \mathcal{C}_{-1\geq}$, and
(3) you want a factorization, a fiber/cofiber sequence $\tau_{\geq 0}X \to X \to \tau_{-1\geq}X$.

Now you can resplit your things.

One comment I made last week is that there is too much data here. One thing is that you only need to know $\mathcal{C}_{\geq 0}$, because you can get back the other one, $\mathcal{C}_{0\geq}$ is the objects x such that $\mathrm{Hom}(x,y) = 0$ for $y \in \mathcal{C}_{\geq 1}$.

These two are very nice. For example, the category $\mathcal{C}_{0\geq}$ is a localization, and $\mathcal{C}_{\geq 0}$ is a colocalization, a coreflective subcategory.

Remember that the heart is the intersection of $\mathcal{C}_{0\geq}$ and $\mathcal{C}_{\geq 0}$ and this is always an Abelian category. If you compose truncations, you have truncations that go to the heart. In this you can call this π_0 or H_0, if you use a shift, this lets you define π_n for $n \in \mathbb{Z}$. You have the fact that the heart is quivalent to $\mathcal{C}_{n\geq} \cap \mathcal{C}_{\geq n}$, and this is how you get your H_n or π_n. If you don't give yourself a T-structure, then you can't get those things. Then the important thing is the long exact sequences. If you give yourself a fiber sequence, then by computing these homotopy groups you get a long exact sequence, and that's absolutely impossible to do unless you have a T-structure.

So what I wanted to do is show you how you can forget about one of the two subcategories.

We said that $\mathcal{C}_{0\geq}$ is a localization of $\mathcal{C}$, so when you have such a thing, a reflective subcategory, you always have the class S of maps in $\mathcal{C}$ such that $L(S)$ becomes an isomorphism. You look at maps that become isomorphisms after localization, and this is equivalent, so that $\mathcal{C}_{0\geq}$ is the same as $\mathcal{C}[S^{-1}]$. When you have a T-structure, you have in particular a localization, I'll now say that this is a special class of maps that has properties:

- every isomorphism is in S,
- if $h = g \circ f$, I have the two out of three property, meaning if any two maps are in S then the third one is in S. This is trivial, since it's something true of isomorphisms.
- The third condition (this is not typical) is that S should be pushout stable. The pushout of an S-map is in S. That's the property that this class S satisfies.

Call a class satisfying these three conditions (*quasi-saturated*) then the following things are

Now I'll say how you generate a T-structure.

Proposition 5.1. *Let $\mathcal{C}$ be stable, equipped with a localization, and suppose the localization is by a class of maps S that are quasi-saturated. Then the following are equivalent:*

(1) There exist a class of maps $f : 0 \to X$ which generate S (S is the smallest quasi-saturated class containing those maps),
(2) $\mathcal{C}_{ge0} = \{A | LA = 0\}$ and $\mathcal{C}_{-1\geq} = \{A | LA = A\}$

So you can write $\mathcal{C}^+$ as

$$\bigcup_n \mathcal{C}_{n\geq}$$

which is the subcategory of left-bounded objects. In particular if $\mathcal{C}$ is $\mathcal{C}^+$ then $\mathcal{C}$ is left-bounded. Similarly you can make a subcategory of right-bounded objects $\mathcal{C}^-$ and say that $\mathcal{C}$ is right-bounded if $\mathcal{C} = \mathcal{C}^-$.

When you have a stable category with a T-structure you want to figure out the long exact sequence. But you want a recognition principle that tells me something about going back from the information of the π_n. You do something by induction, proving n by n that something is an isomorphism or something is zero. So what you want is to be able to recover your full object from its truncations.

You say that $\mathcal{C}$ is *left t-complete* if

$$\mathcal{C} \to \lim_n \mathcal{C}_{n\geq}$$

is an equivalence. You say that $\mathcal{C}$ is *right t-complete* if

$$\mathcal{C} \to \lim_n \mathcal{C}_{\geq n}$$

is an equivalence.

In any case you call $\lim_n \mathcal{C}_{n\geq}$ is $\hat{\mathcal{C}}$. All the examples you know are already right t-complete and in nature the question is whether it's left t-complete. So $\hat{\mathcal{C}}$ is a stable ∞-category and always left complete.

Let me give an example, the category of spectra is both left and right t-complete.

You get a map from $\mathcal{C}^+$ to $\mathcal{C}$ and this gives a map $\widehat{\mathcal{C}^+} \to \hat{\mathcal{C}}$, which is always an equivalence, and you have an equivalence between the left bounded categories and the left t-complete categories, via $\mathcal{C} \mapsto \mathcal{C}^+$ and $\mathcal{D} \mapsto \hat{\mathcal{D}}$.

So I want $\mathcal{C}_{\geq 0}$ to be stable under countable products, and if you have this, and the intersection of $\mathcal{C}_{\geq n}$ is zero, then $\mathcal{C}$ is left complete.

So a non-example, $\mathcal{D}(A)$ is maybe not left t-complete. You take $\mathcal{A}$ to be $\mathbb{G}_a$-representations over $\mathbb{F}_p$.

You can take $\prod_{\geq 1} A[n]$, each component has nothing in degree zero, and H_0 of that can be non-zero.

If A is Grothendieck then $\mathcal{D}(A)$ is right-complete. This satisfies AB5 and so most derived categories are right complete. Then the question is about left completeness. Let's have a break and then I'll talk more.

If I take $\mathcal{D}(A)^-$, the bounded (below) derived category of an Abelian category, then this is always left complete. If A is nice enough, and I take $\mathcal{C}$ a left-complete stable infinity category with a t-structure, and I have an exact functor from A to the heart of $\mathcal{C}$. Then you can extend to a map $\mathcal{D}(A)^- \to \mathcal{C}$ which is t-exact.

What does t-exact mean? A functor between categories with t-structures is t-exact if it sends $\mathcal{C}_{\geq 0}$ to $\mathcal{D}_{\geq 0}$ and $\mathcal{C}_{0\geq}$ o $\mathcal{D}_{0\geq}$.

This is the universal property of this $\mathcal{D}(A)^-$. There is a substatement where things are right exact you can look up. This is something that makes you wish for completeness of a category, you get a map from $\mathcal{D}(\mathcal{C}^\heartsuit)^-$ to $\mathcal{C}$, and usually you have enough injectives, so

$$\mathcal{D}(\mathcal{C}^\heartsuit)^+ \to \mathcal{C}$$

where $\mathcal{C}$ is right complete and $\mathcal{C}^\heartsuit$ has enough injectives.

You take a stable category with a t-structure. You have a comparison map.

So now if I have F a right exact functor from A to B, and what people usually do is take the derived functors, So now if you take $(\mathcal{D}(A)^-)^\heartsuit$ that's A, and so this is already a functor between hearts, and this gives a functor $\mathcal{D}^-(A) \to \mathcal{D}^-(B)$, and this is the derived functor $\mathbb{L}F$ of F (the *left*-derived functor). This is only right t-exact even though it preserves all small limits and colimits.

So as an example, derive the tensor product, you take the category of Abelian groups and tensor by $\mathbb{Z}_2$. You get a derived functor $\mathcal{D}(\mathrm{Ab})^- \to \mathcal{D}(\mathrm{Ab})^-$ which is $\otimes^{\mathbb{L}}_{\mathbb{Z}_2}$, and this will be exact and right t-exact. You can just reverse everything and you can reverse everything. So if you want to derive the global section functor of sheaves, this is left but not right exact, so you get something similar.

Let me make a remark for people who might recognize something. The completion issues, for derived functors it's well-known how to make derived functors.

If you do a bit of derived algebraic geometry, you're going to consider not just varieties but schemes and derived schemes and derived stacks, and something that people want to define are quasi-coherent sheaves over X. Sometimes this is not the object you want. Sometimes people will derive "ind-coherent" sheaves. The quasi-coherent sheaves is left t-complete and ind-coherent sheaves are not, and this map is just the completion. If X is a smooth variety then ind-coherent sheaves of X is already left complete, so these coincide. Most things are right complete, and that left completeness is the key difference.

If you remember correctly, we want to study motives, general cohomology theories on algebraic varieties. If you take the topological version of that, it's spectra, and the key property is that this is stable and has a t-structure. When people look for something motivic, they're looking for triangulated structures, and when we talk about pure motives, that's the heart of the t-structure, and this is the modern approach to all of that. So this is what people have in mind, at least as a goal.

A last comment for five minutes, stable ∞-categories plus t-structures is the input data to be able to speak of spectral sequences. If you've ever heard of spectral sequences, where can I compute these? a stable ∞-category and a t-structure. That's the typical place to do this. The goal here is to compute the π_n of a directed colimit of X_p, and then there's a recipe using the π_n of all the cofibers, and this has a modern interpretation in stable ∞-categories with a t-structure.

Every time you have a spectral sequence, usually you can rephrase it in something like this. The takehome is that a stable ∞-category without its t-structure is useless, and the t-structure is something very categorical that has a lot of implications, and in examples there are t-structures that are not obvious. It's not, yeah, I see my t-structures, they're obvious, but you have some subtle issues. Next week we have the pleasure of hearing Rune talking about a cohomology theory that people can compute on varieties.

6. Rune Haugseng: February 26: Yet another introduction to algebraic K-theory

I'm supposed to tell you something about algebraic K-theory. We heard a lot about motives and algebraic K-theory last time so I thought I would focus on the part I actually understand which is how you define K-theory.

Let me start with the warm-up, which is the Grothendieck group. Suppose $\mathcal{C}$ is a category with some notion of weak equivalences and a 0 object and some notion of direct sum (coproducts), and some sort of notion of a short exact sequence.

If you have this data (I'll be precise later) then you can define an Abelian group $K^0(\mathcal{C})$ generated by the objects of the category with the following relations:

(1) If c and c' are weakly equivalent then $[c] = [c']$, i.e., $c \sim c'$ in the equivalence relation generated by weak equivalences. In the simplest examples this will be isomorphisms.
(2) We want the direct sum to agree with addition in the group, $[c \oplus c'] = [c] + [c']$
(3) Given a short exact sequence

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \arrow[d] \\ 0 \arrow[r] & c \end{tikzcd}$$

you want $[b] = [a] + [c]$.

Example 6.1.
- (Grothendieck, late fifties) Let $\mathcal{C}$ be the category of algebraic vector bundles on a nice variety X. Here isomorphisms are the weak equivalences. This gives $K_0(X)$.
- As a special case, think of vector bundles on the affine scheme of R, and these are finitely generated projective modules over R. This is usually $K_0(R)$. Since we restrict to projective modules, all short exact sequences split and we can forget the third relation. So for R a field, every projective

module is free, and so you get $\mathbb{Z}$, one for each integer and then you get the negatives.
- If K is a number field you get the ideal class group of K and a copy of $\mathbb{Z}$, which I guess is just a consequence of how projective modules over $\mathcal{O}(K)$ look.
- If X is a topological space, you can look at $K_0(\mathbb{Z}[\pi_1 X])$ and this connects to geometric topology, containing the Wall finiteness obstruction, which measures whether you're a CW complex.

The number field gives something that number theorists are interested in, this gives something interesting to geometric topologists, if you plug in vector bundles on a scheme it might be interesting to algebraic geometers.

- You could plug in a small triangulated category with the short exact sequences the distinguished triangles. This is a non-example because this contains enough information to define K_0 but not the higher K-groups we'll come to later.
- So far all the examples have the weak equivalences be the isomorphisms. Let me give an example where they're not. If you want to define $K_0(X)$ for an arbitrary scheme X, then you need nice chain complexes of sheaves of $\mathcal{O}_X$-modules, the "perfect complexes with globally finite Tor-amplitude". The point is that in this case you use quasi-isomorphism.
- If you consider the category of pointed finite sets, and short exact sequences are pushouts of injective maps, and you get that K_0 of finite sets is $\mathbb{Z}$.

You can do more fancy things, cell complexes over and under a fixed space X, which gives Waldhausen's version and so on.

In the early 60s you get $K_1(R)$ and $K_2(R)$ explicitly with what look like parts of long exact sequences. Bass also constructed negative groups.

Quillen defined (in the 70s) higher K groups $K_n(\mathcal{C}) = \pi_n K(\mathcal{C})$ for a space $K(\mathcal{C})$ and eventually for some of the other examples you need a construction written up by Waldhausen in the early 80s, the $S.$ construction, although the story is that this was due to Graeme Segal but never appeared in print.

What I'll try to do in this talk is try to explain the Waldhausen $S.$ construction of K-theory. We can define K_0 as taking place via three steps.

(1) First, if we have some notion of weak equivalence, we can identify the weakly equivalent objects in the monoid of objects of $\mathcal{C}$ under direct sum
(2) You add negatives for the direct sum, something you can do for any monoid, called the group completion, in this case a commutative monoid
(3) Add the relations that split short exact sequences.

The goal is to explain how the $S.$ construction is a kind of homotopical analogue of these three steps. I want to explain in turn a homotopical version of each one of them.

So we want some way of inverting morphisms. So for this I have to talk about simplicial objects. We write $\mathbf{\Delta}$ for the category whose objects are finite non-empty ordered sets and $[n]$ is the set $\{0 < \cdots < n\}$ and the morphisms are order-preserving maps between these.

For example, we can define a map d_i from $[n-1]$ to $[n]$ which is the inclusion where you skip i (coface) or in the other direction we can do s_i by repeating i once (codegeneracy). Every morphism in the category can be written as composites of face and degeneracy maps, but not uniquely, there are some relations. I won't tell you the relations because I think they're quite useless. The point is that you can think of the simplicial as looking like this:

$$\begin{tikzcd}
{[0]} \arrow[r, shift left] \arrow[r, shift right] & {[1]} \arrow[l] \arrow[r, shift left, bend left=60] \arrow[r, shift right, bend right=60] & {[2]} \arrow[l, shift left] \arrow[l, shift right] & \cdots
\end{tikzcd}$$

now a simplicial set is a functor $\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{X} \mathrm{Set}$ so $\mathrm{Set}_{\Delta} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathrm{Set})$ so X is a collection of sets X_0, X_1, and so on, along with face maps $X_n \to X_{n-1}$ and degeneracies $X_n \to X_{n+1}$.

I'll tell you a way to get a space out of a simplicial set and then how to get a simplicial set out of a category so putting them together we'll get a topological space out of a category.

So let me define geometric simplices. There's a functor from Δ to topological spaces which takes $[n]$ to a geometric n-simplex

$$|\Delta^n| = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : 0 \le x_i; \sum x_i = 1\}.$$

You can define maps between these, if you have a map $\varphi : [n] \to [m]$ a map of ordered sets, you can define a map $|\Delta^n| \xrightarrow{\varphi} |\Delta^m|$ where

$$(x_0, \dots, x_n) \mapsto (y_0, \dots, y_m)$$
$$y_i = \sum_{j \in \varphi^{-1}(i)} x_j$$

and this gives the face and degeneracy maps. The relations I'm not telling you give the identities on faces of faces, and the degeneracies are some kind of projections.

So now having this geometric simplex functor, I formally get from that an adjunction relating simplicial sets to topological spaces, with left adjoint geometric realization and right adjoint the *singular simplicial set*. I start with a space and take $(\mathrm{Sing}\, T)_n$ to be $\mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, T)$.

The left adjoint geometric realization is given by a coend

$$\int_{\mathbf{\Delta}^{\mathrm{op}}} X \times |\Delta^*|$$

or a coequalizer of

$$\coprod_{\varphi:[m]\to[n]} X_n \times |\Delta^m| \rightrightarrows \coprod_n X_n \times |\Delta^n|.$$

You should think of this as giving you a topological space given by building up using X as a blueprint, by taking a copy of a geometric n-simplex for every $\sigma \in X_n$ and glue them in along their faces, according to the face maps and collapse degenerate faces.

Here's something else I can do, I can define a functor from $\mathbf{\Delta}$ into categories, by taking $[n]$ to the category $0 \to 1 \to \cdots \to n$, also called $[n]$. Again formally this gives rise to an adjunction between simplicial sets and categories, where the right adjoint is N, the nerve, and the left adjoint h, kind of the "homotopy category" described by a coend but colimits in categories are not very nice.

This nerve has a very explicit description. The n-simplices of the nerve are $\mathrm{Hom}_{\mathrm{Cat}}([n],\mathcal{C})$, which is the sequences of n composable morphisms in $\mathcal{C}$. In this explicit description, the face maps correspond to composing, the face maps are composition at x_i if i is not 0 or n, and by forgetting the first or last morphism at the ends. The degeneracies are given by inserting identity maps.

Let me tell you some facts about this functor N. In fact it's fully faithful, a morphism of simplicial sets between nerves is exactly the same thing as a functor between the categories. Then for a general simplicial set I can't say much about h, but if I apply h to $\mathrm{Sing}\, T$ then I get something equivalent to the fundamental groupoid $\pi_{\le 1}X$, which is given by taking objects the points and the morphisms from p to q the homotopy classes of paths from p to q. It tells you all the fundamental groups of X at the same time.

If we start from $\mathcal{C}$ we can build a topological space by first taking the nerve and then the geometric realization, $\|\mathcal{C}\| \coloneqq |N\mathcal{C}|$, often called the *classifying space of* $\mathcal{C}$, and formally there is a map from the nerve of $\mathcal{C}$ to $\mathrm{Sing}\,\|\mathcal{C}\|$.

I claim that $\|\mathcal{C}\|$ is a homotopical upgrade of inverting the morphisms in $\mathcal{C}$—if you start by inverting morphisms of $\mathcal{C}$ you get a groupoid.

I want to say something about what this looks like, $\|\mathcal{C}\|$ has a point for every object of $\mathcal{C}$ and then an edge relating those two points for each morphism $p \to q$, and then you add in two-cell for all composable pairs of morphisms, and keep going.

In particular, you're making the morphisms invertible because edges have inverses.

The set of components $\pi_0\|C\|$ is the quotient of the objects of $\mathcal{C}$ by the equivalence relation generated by $c \sim d$ if there exists a morphism from c to d.

You can also show that $\pi_{\le 1}\|\mathcal{C}\| \cong \mathcal{C}[\mathcal{C}^{-1}]$.

I can view this construction taking $\mathcal{C}$ ot $\mathcal{C}[\mathcal{C}^{-1}]$ as left adjoint to the inclusion of groupoids into categories, and this construction $|-| : \mathrm{Set}_\Delta \to \mathrm{Top}$, you can view as modelling the ∞-version of this, left adjoint to the inclusion of ∞-groupoids into ∞-categories. We're regarding a category as a kind of ∞-category, and then doing this left adjoint there.

I guess, well, this was mainly a remark to those who know something about ∞-categories already, so not very useful maybe.

Let me tell you about this, if I start with a monoid M in Set, then I can define a category $\mathbb{B}M$ which has one object $*$ and $\mathrm{Hom}_{\mathbb{B}M}(*,*)$ is M with composition multiplication in M.

I can apply this construction to $\mathbb{B}M$, and get $\|\mathbb{B}M\|$ which is BM, which is called the classifying space of M, and so if G is a group, then $\|\mathbb{B}G\|$ is exactly the usual space $BG \cong K(G,1)$, the universal space where π_1 is G.

Let me say one more sentence to finish the first section. For K-theory we had a category with weak equivalences, and we wanted to upgrade modding out equivalent objects, so we take $\|w\mathcal{C}\|$ for some subcategory $w\mathcal{C}$ spanned by weak equivalences.

7. Group completion

Right, so before the break we were talking about modding out by weak equivilances homotopically, now this is a homotopical version of group completion. If I had a monoid, I can define $BM = \|\mathbb{B}M\| = \|N\mathbb{B}M\|$ which is obtained by inverting morphisms in $\mathbb{B}M$, which is the same as adding negatives or inverses to M. Indeed, BM is the same as BM_{gp} and since $M_{\text{gp}} \cong \Omega BM$. So then ΩBM implements group completion of monoids in sets.

We want to do something similar for "monoids" in topological spaces. We have to be, you have to think what is the right notion of a monoid. For K-theory, we'd like $\|w\mathcal{C}\|$ to be a "monoid" via direct sum or coproduct in $\mathcal{C}$. This might not be literally a monoid in the strict sense. But this is likely not strictly associative, that $a \oplus (b \oplus c)$ and $(a \oplus b) \oplus c$, they're probably not strictly equal but canonically isomorphic.

The classical way to solve this is to replace the category $\mathcal{C}$ with a different one where the sum is associative. The modern thing to do, and you kind of have to do it in a sense, is a notion of monoid that's homotopy coherent. Let me start by saying this in the classical sense. If I have M a category with products (such as sets) then I can define an associative monoid as the data of a functor X from $\mathbf{\Delta}^{\text{op}}$ to M such that $X_n \to X_1^{\times n}$ (coming from the inclusions of $[1]$ into $[n]$) is an isomorphism.

Let me try and justify that, X_0 says that this is isomorphic to a point, and so we have the degeneracy map $* \cong X_0 \to X_1$, which is a point, which tells you the unit of the monoid. If I look at X_2, the condition tells me that it's isomorphic to $X_1 \times X_1$, and the face map tells me that I have a binary product, which tells me the multiplication. Then if I look in X_3, there's an isomorphism to $X_1^{\times 3}$ and looking at the face maps I see that this is associative.

$$\begin{tikzcd} X_1^{\times 3} \arrow[r, "(m,\mathrm{id})"] \arrow[d, "(\mathrm{id},m)"] & X_1^{\times 2} \\ X_1^{\times 2} & X_2 \end{tikzcd}$$

and you can show that the other data tells you nothing else.

For a monoid M in Set the corresponding map $\mathbf{\Delta}^{\text{op}} \to \text{Set}$ is exactly $N\mathbb{B}M$.

Definition 7.1. An A_∞-*monoid* in Top (for the purpose of this talk) is a functor $\mathbf{\Delta}^{\text{op}} \to \text{Top}$ such that $X_n \to X_1^{\times n}$ is a weak homotopy equivalence.

A monoidal category corresponds to functors from $\mathbf{\Delta}^{\text{op}} \to \text{Cat}$ so that $X_n \to X_1^{\times n}$ is an equivalence of categories.

In particular, explicitly in the case of $\mathcal{C}$ a category with coproducts, we can define $\mathcal{C}^{\otimes} : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ by taking $\mathcal{C}^{\otimes}_n$ to be the category of diagrams

$$\begin{tikzcd}
& 0 \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & \\
X_1 \arrow[dr] & & X_2 \arrow[dl] & & \cdots & & X_n \\
& X_{12} \arrow[dr] & & & & & \\
& & \ddots \arrow[dr] & & {} \arrow[dl] & & \\
& & & X_{1n}. & & &
\end{tikzcd}$$

If I remember right Milnor showed that $|-| : \mathrm{Set}_{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ preserves homotopy equivalence. Using that you can show that if $\mathcal{C}$ is a monoidal category viewed as $\mathcal{C}^{\otimes} : \mathbf{\Delta} \to \mathrm{Cat}$ then $\|\mathcal{C}\|$ is an A_∞ monoid. For $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Top}$ we can define $|X|$ by some coequalizer formula, and then for X an A_∞-monoid in Top we have $BX_1 \coloneqq |X|$.

There is a canonical map $X_1 \to \Omega BX_1$ which comes about by adjoint to $\Sigma X_1 \to BX_1$.

Then this gives a morphism of A_∞ monoids, the loop space is always an A_∞ monoid, by concatenation of loops. This is even an A_∞ group, which is an A_∞ monoid such that $\pi_0(X_1)$, that's a monoid, and if that's actually a group, we say that this is an A_∞ group.

The fact is that ΩBX_1 is the universal A_∞ group with a map from X. In particular, if you take π_0 you get the group completion, $\pi_0 \Omega BX_1 \cong (\pi_0 X_1)_{\mathrm{gp}}$. You can think of this as the ∞ analog of what I started with in Set. You can think of this as the ∞-category with one object, and then we invert morphisms to have inverses, invert morphisms to get an ∞ groupoid, and then we recover a monoid as the endomorphisms of the base point.

How do we apply this to K-theory? We start with $\mathcal{C}$ and take $\|w\mathcal{C}\|$ and that's an A_∞-monoid, and we can form its group completion in this sense as $\Omega B\|w\mathcal{C}_1\|$, and if $\mathcal{C}$ was one of these examples where all short exact sequences split, then this is already the K-theory space of $\mathcal{C}$. So $K_n(\mathcal{C})$ is $\pi_n(\mathcal{C})$.

If I want to split short exact sequences then I need something more complicated.

I won't make precise the notion of a Waldhausen category, but I can build a new category called $S_n\mathcal{C}$ which consists of diagrams

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_01 & \longrightarrow & A_{02} & \longrightarrow & \cdots & \longrightarrow & A_{0n} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & 0 & \longrightarrow & A_{12} & \longrightarrow & \cdots & \longrightarrow & A_{1n} \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & 0 & & & & \vdots \\
 & & & & & & & & \downarrow \\
 & & & & & & \ddots & & \vdots \\
 & & & & & & & & \downarrow \\
 & & & & & & & & 0.
\end{array}$$

I should say a Waldhausen category is a category $\mathcal{C}$ of cofibrant objects with cofibrations, weak equivalences, 0 and $\coprod$ and pushouts of cofibrations exist and are cofibrations. So these diagrams, the morphisms that are horizontal are cofibrations and all the squares are pushouts.

These categories have a natural simplicial structure $S_\cdot\mathcal{C} : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Cat}$ where $wS_n\mathcal{C}$ is the subcategory where the morphisms are objectwise weak equivalences, I have this simplicial diagram of categories, and I can take $||wS_\cdot\mathcal{C}||$, then the loop space of that.

This is the definition of $K(\mathcal{C})$, so for $\mathcal{C}^{\amalg}$ the diagram I had before, then I can get a map $w\mathcal{C}^{\amalg}$ to $wS_\cdot\mathcal{C}$ by taking split exact sequences. If I look in level two, the data corresponds to a pair, [unintelligible], and so you get a map from $\Omega B||w\mathcal{C}|| \to \Omega\left|||wS_\cdot\mathcal{C}||\right|$, and you can think of this as quotienting out by splitting exact sequences.

So $S_0\mathcal{C}$ is a point, $S_1\mathcal{C}$ is $0 \to X \to 0$ which is just $\mathcal{C}$ itself. Then an object in $S_2\mathcal{C}$ is a short exact sequence, so this is the category of short exact sequences, so I add in a two simplex when I take geometric realization which has as its boundaries the entries, so that the middle entry (as a loop) is the sum of the other two entries.

I'll say one more word. Each category is a Waldhausen category, and $K(S_2\mathcal{C})$ has maps (two of them) to the K-theory of $\mathcal{C}$, and the additivity says that $K(S_2\mathcal{C})$ is equivalent to $K(\mathcal{C})\times K(\mathcal{C})$ and similarly for higher n, which shows that $K(S_\cdot\mathcal{C})$ is an A_∞-monoid (in fact an A_∞-group) and I can deloop it, and iterate that, applying the $S_\cdot$ construction as many times as I want, and doing that I get an Ω-spectrum. So it's not just a space, it's a spectrum.

I should definitely stop.

8. March 25: Kyoung-Seog Lee: Introduction to Voevodsky motives

Because there are few, sorry for the two audience members, but since there are only two overlaps, let me briefly recall what I did last time and then go to Voevodsky motives.

Last time I explained why Grothendieck introduced Chow motives. I could only give the definition and construction and couldn't go further than that. Recall the construction of Chow motives, I'll spend twenty minutes or so, Grothendieck's original motivation was to construct, his idea was that there should be a universal cohomology theory taking values in a $\mathbb{Q}$-linear category of motives $M(k)$, so that every smooth projective variety over k and every Weil cohomology to graded K-algebras, he wants to find some kind of category of motives $M(k)$ and a functor h from varieties to motives so that for every such cohomology there is a unique functor from $M(k)$ making the triangle commute.

Let me briefly recall how he constructed that kind of category. I start from the category of smooth projective varieties over k and then I define the so-called category of correspondences, the objects is the same as the smooth projective varieties. The morphisms $\mathrm{Mor}(X,Y)$ is the Chow group $\mathrm{CH}_{\mathbb{Q}}^{\dim X}(X\times Y)$. That's, $Z^r(S)$ for S a scheme or variety is the free Abelian group on the codimension r cycles on S modulo rational equivalence (please do not ask what this is, I explained this for 20 minutes last time). If I had another equivalence, numerical or homological equivalence, I could have another group. So we call this Chow motives because we're using rational equivalence. If we chose numerical equivalence we'd call this maybe numerical motives.

If we choose two varieties, you can't add morphisms generically. You can take a graph of f and graph of g in the correspondences and can add these in this cycle group. Then I choose a pseudo-Abelian closure $\widetilde{CV_k}$, inputting a kernel and cokernel for each projector. Call this M_k^+. Then I localize this and obtain $M_k = M_k^+[\mathbb{L}^{-1}]$, inverting the Lefschetz motive, and this is my Chow motives. This is a pseudo-Abelian category (it has kernels and cokernels of idempotents).

I will also tell you why I, Grothendieck's original idea was to prove the Riemann hyothesis for this thing. Let me recall Grothendieck's standard conjecture C. This says that for X a smooth projective variety over k, he thinks that $h(X)$ should be $h^0(X)\oplus\cdots\oplus h^{2\dim X}(X)$ and then by a realization functor we have $H^*(X) = H^0(X)\oplus\cdots\oplus H^{2\dim X}(X)$ and from this we have the Riemann hypothesis for finite fields.

I can give you two applications that I think are important. Last time we had a very hard time to construct this one, so I want to tell you why this thing is maybe useful. I want to explain what kind of thing we get from this, some kind of possible applications.

Let X be a smooth projective variety over k, then H^* a Weil cohomology functor. By the axioms I have a cycle map $\gamma_i : Z^i(X)\to H^{2i}(X)$, and let me call $A^i(X)$ the image of γ_i.

For example for $k=\mathbb{C}$ and H^* usual singular cohomology, then

$$A^i(X)\subset H^{2i}(X,\mathbb{Q})\cap H^{i,i}(X,\mathbb{C}).$$

Or another example for char $k=p$, and H^* the ℓ-adic cohomology, then

$$\mathbb{Q}_\ell A^i(X)\subset (H_\ell^{2i}(X)(1)(i))^{G_k}.$$

The Hodge conjecture in the first case and Tate conjecture 1 in the second case says that these are the same. So now I can make these kinds of conjectures for motives. I can say M is an *effective motive* meaning it belongs to M_k^+ and $A^i(M)$ is $\mathrm{im}\gamma_i$. Then the Hodge conjecture for M is that $A^i(M) = H^{2i}(M,\mathbb{Q})\cap H^{i,i}(M,\mathbb{C})$ while the Tate conjecture 1 for M is that if k is finitely generated over its prime field then $\mathbb{Q}_\ell A^i(M) = H_\ell^{2i}(M)^{G_k}$.

Proposition 8.1. *For M and N in M_k, the Hodge conjecture for $M\oplus N$ implies the Hodge conjecture for M and N, and the same is true for the Tate conjecture 1.*

Theorem 8.1. *When X is $\mathcal{SU}_C(r, \mathcal{L})$ vector bundles of rank r with fixed degree with $\deg \mathcal{L}$ coprime to r for C a curve then I can give a nice such decomposition into the curve and its Jacobian, which lets me prove the Hodge conjecture is true.*

This kind of application is possible.

One more application of Manin, who learned from Grothendieck.

Theorem 8.2 (Manin)**.** *Let X be a smooth projective 3-fold, unirational over k a finite field. Then X satisfies the Riemann hypothesis.*

Maybe you think this was known by Deligne, but this is prior to Deligne. This is one approach. The idea of the proof, by unirational, there's a birational map $\mathbb{P}^3 \dashrightarrow X$.

This is characteristic p so I have Abhyankar's result, and I can get $\mathbb{P}^3 \leftarrow \tilde{X} \to X$, with a blowup of a smooth center in $\mathbb{P}^3$, so this gives $h(\tilde{X}) \cong h(\mathbb{P}^3) \oplus \bigoplus h(\mathrm{pt})^{\otimes} \oplus h(C_i)$ and these satisfy the Riemann hypothesis by previous results. Then $h(X)$ fits in $h(\tilde{X})$ so X satisfies the Riemann hypothesis.

Let me take a five minute break and then go to triangulated categories of motives.

I realized that this is maybe the right title. In the 1980s, there was a conjecture of Beilinson and Deligne independently.

Conjecture 8.1. When k is a field then there exists an Abelian tensor category MM_k of mixed motives containing Grothendieck's category of pure (homological) motives as the semi-simple object and some conditions or properties.

There should be some category containing this? Why? Deligne constructed mixed Hodge structures. When X is a smooth variety, we have

$$H^{2i}(X) = \bigoplus_{p+q=2i} H^{p,q}(X)$$

and if I have a Frobenius action on X then I have $H^{2i}(X, \mathbb{Q}_\ell)$ with eigenvalue q^i.

When X is non-projective or singular then you have no such thing but you have a weight filtration $W_j H^i(X)$ and the associated graded is pure. They think that, the motive is defined only for smooth projective varieties but even for singular varities you should have this one and the weight filtration should give the Hodge filtration.

People tried really hard to construct such categories. I don't know, maybe not really hard, but some people tried. They did not succeed. If you have time and energy, maybe you can try.

Remark 8.1.
1. This is still open and thought to be hard.
2. (suggestion of Deligne) It's hard to construct this but constructing the bounded derived category of these motives might be easier. Maybe one reason is that the construction of mixed Hodge structures used this so much.
3. There has been lots of progress. People really constructed tensor triangulated categories $DM(k)$ that have many expected properties of $D^b(MM_k)$. People really succeeded in this, but what one really wants is to construct a t-structure in $DM(k)$ to give MM_k.

There's something called non-commutative motives, constructed by Tabuada and Kontsevich. Today I only consider the original construction of Voevodsky.

Let me just say a few words about triangulated categories of motives. There are several constructions of this triangulated category of motives by Hanamura, Levine, and Voevodsky, who independently constructed tensor triangulated categories. They are expected to be equivalent. Indeed the Levine construction is equivalent to Voevodsky. People also expect Hanamura's to be equivalent. All three give the same $\mathbb{Q}$-motivic cohomology theory. Today I'll follow Voevodsky.

Indeed the construction is quite similar to the construction of Chow motives. Let k be a field and Sm/k is the category of smooth schemes over k, so one thing is good, because I erased the proper projective condition, admitting open varieties. It's not good that I'm in the smooth case but that will be removed. I'll construct a functor from Sm/k to $DM(k,\mathbb{Q})$ and later extend it to a functor from general schemes. Okay so let X and Y be smooth schemes over k. Then $c(X,Y)$ is the free Abelian group generated by integral closed subschemes W in $X\times Y$ such that $W\to X$ is finite and surjective over a connected component of X.

Then this is a really amazing fact, imposing this simple condition I can define composition.

Lemma 8.1. *If X_1, X_2, and X_3 are smooth schemes over k and $\phi\in c(X_1,X_2)$ and $\psi\in c(X_2,X_3)$, then you can really easily check that $\psi\circ\phi$, defined as $p_{13*}(p_{12}^*\phi\cap p_{23}^*\psi)$.*

Then I can define a new category of smooth correspondences over k with the same objects and morphisms $\mathrm{Hom}(X,Y)=c(X,Y)$ which is an additive category. Then I can define the so-called homotopy category $K^b(\mathrm{Sm\,Cor}/k)$ of $\mathrm{Sm\,Cor}/k$. I want to impose two conditions.

Let T be the class of complexes of the following two forms:

(1) For $X\in\mathrm{Sm}/k$, I have $[X\times\mathbb{A}^1]\to[X]$.
(2) For X in Sm/k and every $U\cup V=X$ open covering, I have
$$[U\cap V]\to[U]\oplus[V]\to[U]\cup[V]$$
and if these are in T then I define $\bar{T}$ as the minimal thick subcategory of K^b containing T.

Then whenever I have a thick subcategory of a triangulated subcategory, I can define the quotient by $\bar{T}$, the localization. Some people take this as the definition of the triangulated category of motives, but Voevodsky goes further and takes the pseudoAbelian closure. This is $\mathrm{DM}_{\mathrm{gm}}^{\mathrm{eff}}(k)$. It has a very nice property. From the beginning, because I've modded out by T, it means that I have a Mayer–Vietoris sequence and as Philsang said, one might ask that this be a tensor triangulated category, and indeed it is.

Also when k admits resolution of singularity, some people say this can be removed these days, but anyway, then this contains effective Chow motives fully faithfully.

Then these motives have nice properties, all the properties you might expect. Let me give a few remarks and finish. Voevodsky said it's easy to construct but hard to deal with these. So he gave a sheaf theoretic definition of these motives. I don't have time to say this. In here the ingredients are Nisnevich topology, which I never saw before Voevodsky, and algebraic K-theory, and also homotopy invariant sheaves.

Let me tell you properties. This $DM:\mathrm{Sm}/k\to DM(k)$ can be extended to schemes of finite type over k. Then I can say that

(1) Some Künneth formula holds: $M_{\mathrm{gm}}(X\times Y)\cong M_{\mathrm{gm}}(X)\otimes M_{\mathrm{gm}}(Y)$,
(2) homotopy invariance, (i.e. $M_{\mathrm{gm}}(X\times\mathbb{A}^1)\cong M_{\mathrm{gm}}(X)$ for all X)

(3) a Mayer–Vietoris sequence,
(4) a blowup formula
(5) a projective bundle formula
(6) compactly supported cohomology, which extends beyond the smooth projective case.

If you want to see another definition of $DM(k)$ maybe I can personally talk to you.