CGP DERIVED SEMINAR

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1. January 16, 2018: Byunghee An, bar and cobar

Today I am going to talk about bar and cobar constructions again, between categories of algebras and coalgebras.

I think that one of the goals is to explain this diagram. You have a category Alg of algebras and a category Alg_∞ of ∞ algebras, and a category Cog of coalgebras, and a full subcategory of fibrant and cofibrant objects. Among these four categories we can think of algebras as a non-full subcategory of A_∞ algebras. We can think of coalgebras which are fibrant and cofibrant as a full subcategory of coalgebras. We want to define functors between these:

$$\begin{tikzcd} \mathrm{Alg} \arrow[r] \arrow[d, shift left, "B"] & \mathrm{Alg}_\infty \arrow[d, "B_\infty"] \\ \mathrm{Cog} \arrow[u, shift left, "\Omega"] & \mathrm{Cog}_{cf} \arrow[l] \end{tikzcd}$$

and B_∞ will be an equivalence and all of these will induce equivalences on the homotopy categories.

I will define everything but this is my goal.

Let's start with algebras Alg. This is the category of unital augmented dg algebras over $\mathbf{k}$ a field, objects are (A, ϵ) where A is a unital algebra and ϵ is an algebra map $A \to \mathbf{k}$, and this has a model category structure where the weak equivalences are the quasi-isomorphisms, the fibrations are the degreewise surjections, and the cofibrations are the maps with the left-lifting property against trivial fibrations.

It is known

Theorem 1.1. *This data defines a model structure on* Alg.

Now I want to define the category of coalgebras, so let me denote Cog' the category of coaugmented dg coalgebras, an object consists of a complex C with differential d, a coproduct Δ, a counit η and a coaugmentation ϵ. We require that d is a coderivation against Δ, so that $(d\otimes 1 + 1\otimes d)\Delta = \Delta d$, that $\eta\epsilon = 1_{\mathbf{k}}$.

For example, let (V, d) be a complex. Then $T^c(V)$, the "tensor coalgebra" on V, is $\bigoplus_{n\ge 0} V^{\otimes n}$, and the coproduct is defined by the sum of all possible separations.

$$\Delta(v_1\otimes\cdots\otimes v_n) = \sum (v_1\otimes\cdots\otimes v_i)\boxtimes(v_{i+1}\otimes\cdots\otimes v_n).$$

So for example $\Delta(v) = 1\boxtimes v + v\boxtimes 1$.

There is a canonical projection $T^c(V)\to V$ taking the V summand. But $T^c(V)$ is not cofree on V. This does not have a universal property, that coalgebra maps to $T^c(V)$ are the same as maps to V. Suppose we have C a coalgebra and take a chain map $C \to V$. Then there need not be a lift to $T^c(V)$. The answer is no.

So I want a smaller (full) subcategory Cog whose objects are *cocomplete*, meaning that $C = \sqcup \ker(C \to C^{\otimes n} \to (C/\mathbf{k})^{\otimes n})$ where $C\to C^{\otimes n}$ is the iteration $\Delta^{(n)}$ of Δ. If you take the coproduct enough times it has the base field element in at least one factor. It cannot be decomposed in a trivial way any more.

We can easily see that the tensor coalgebra is cocomplete, any element is in the form, a finite sum of $v_1 \otimes \cdots v_n$, and if you take the coproduct to $n+1$ copies, there will be some 1 somewhere. Actually it is not only cocomplete, it is cofree in the category of cocomplete coalgebras. For any C in cocomplete coalgebras and any chain map from $\bar{C}$ to V we can always find a map from C to $T^c(V)$. These should be counital maps and send $\mathbf{k}$ to 0. So $T^c(V)$ is cofree on V in Cog.

So let's let C be a cocomplete coalgebra and A a unital augmented algebra. Then we want to consider $\mathrm{Hom}_{\mathbf{k}}(C,A)$, well, really, we want chain maps that are compatible with augmentations. We want a differential and algebra structure, and I'll put this in Alg, and the differential will be D and the multiplication $*$. The differential is $D(f) = d_A f - (-1)^{|f|} f d_c$. The product is $\mu_A(f \otimes g)\Delta_c$, the unit is $\eta_A \circ \eta_C$.

The nontrivial thing to check is that d is a derivation with respect to the product.

$$D(f * g) = Df * g + (-1)^{|f|} f * Dg.$$

The left hand side is

$$d_A \mu_A (f \otimes g)\Delta_C - (-1)^{|f|+|g|}\mu_A(f \otimes g)\Delta_C d_C,$$

and the Δ_C and d_C have compatibility and can be interchanged, and likewise d_A and μ_A, so we get

$$\begin{aligned}\mu_A(d_A \otimes 1 + 1 \otimes d_A)(f \otimes g)\Delta_C - (-1)^{|f|+|g|}\mu_A(f \otimes g)(d_C \otimes 1 + 1 \otimes d_C)\Delta_C \\ = \mu_A(d_A f \otimes g + (-1)^{|f|} f \otimes d_A g) - (-1)^{|f|+|g|}\mu_A((-1)^{|g|} f d_C \otimes g + f \otimes g d_C) \\ = \mu_A(Df \otimes g + (-1)^{|f|} f \otimes Dg)\Delta_C\end{aligned}$$

but this is the multiplication in A and the coproduct in C so this is

$$Df * g + (-1)^{|f|} f * Dg,$$

which is the right-hand side.

We call $\tau \in \mathrm{Hom}^1_{\mathbf{k}}(C,A)$ a *twisting cochain* if $D\tau + \tau * \tau = 0$ and $\epsilon \tau \epsilon = 0$. We define a set $\mathrm{Tw}(C,A)$ as the set of all twisting cochains.

For a given A you get a contravariant functor $C \mapsto \mathrm{Tw}(C,A)$. We need to check functoriality, that if you have a map $C' \to C$ that you get a map $Tw(C,A) \to Tw(C',A)$, by postcomposing.

We need to check that this is a twisting morphism. If $\tau \in \mathrm{Tw}(C,A)$ and $f : C' \to C$, we need to check that $D(\tau \circ f) + (\tau \circ f) * (\tau \circ f) = 0$.

But this is

$$\begin{aligned} d_A(\tau f) - (-1)(\tau f) d_{C'} + \mu_A(\tau f \otimes \tau f)\Delta_{C'} \\ = (d_A \tau - (-1)\tau d_C)f) + \mu_A(tau \otimes \tau)(f \otimes f)\Delta_{C'} \\ = D(\tau) f + \mu_A(tau \otimes \tau)\Delta_C f \\ = (D(\tau) + \tau * \tau) f = 0.\end{aligned}$$

This functor is nice. It's representable, and I want to give an explicit representation, which is the bar construction. We define BA as the tensor coalgebra $T^c(S\bar{A})$, where SA is the shift of the algebra A. Then the differential is $\sum 1^{\otimes -} d_A \otimes 1^{\otimes -}$ plus another term using the (shifted) algebra b_2 (which is $s^{-1}\mu s \otimes s$) which is $\sum 1^{\otimes -} \otimes b_2 \otimes 1^{\otimes -}$.

Then $BA \in \mathrm{Cog}$, and the canonical projection is to $S\bar{A}$, and by postcomposition you have a morphism to $\bar{A}$, which we denote τ_0. I want to show that τ_0 is a twisting cochain. The projection map is degree 0 and the other map is degree 1, so I want to check that $D(\tau_0) + \tau_0 * \tau_0 = 0$.

But this is, well, $\tau_0 = S^{-1}\pi$. Then it's the same as saying that

$$d_A(S^{-1}\pi) - (-1)(S^{-1}\pi)d_{BA} + \mu_A(S^{-1}\pi \otimes S^{-1}\pi)\Delta_{BA} = 0.$$

If we put $v_1 \otimes \cdots \otimes v_n$ where $n \geq 3$ then the projection right away gives 0. We need to check or $v_1 \otimes v_2$. If you take the latter, then you get, by taking the definition, you get $S^{-1}b_2(v_1, v_2)$, and the differential d_{BA} is some d_A terms and a $b_2(v_1, v_2)$ term. One of these vanishes because of the projection; the other gives the A_∞ relation. The case with just v is easier.

So what I proved here is that τ_0 is a twisting cochain, it's contained in the set $\mathrm{Tw}(BA, A)$. Now I want to prove that $\mathrm{Tw}(C, A)$ is bijective with $\mathrm{Hom}_{\mathrm{Cog}}(C, BA)$. I wnat to show this representation statement. So a map $\tau : C \to A$ gives a map $C \to BA$ by the universal property. To prove bijectivity, we need to check that $\tilde{\tau} \circ \tau_0 \in \mathrm{Tw}(C, A)$.

But this is not hard. The equation for $D(\tau_0 \circ \tilde{\tau}) + (\tau_0 \circ \tilde{\tau}) * (\tau_0 + \tilde{\tau})$ can be rewritten (as previously $(D\tau_0 +$
$tau_0 * \tau_0) \circ \tilde{\tau} = 0$.

Dually, this construction, we started with a fixed A and get a contravariant functorp Dually if we fix a coalgebra then we get a covariant functor by assigning the same set. It's corepresentable, and the elements represented by it are "cobar." Let's have a break.

Damien asked why we consider twisting cochains. I said I don't know why.

[Christophe: They are the first nontrivial examples of Maurer–Cartan elements. These are very simple elements on which we can express the calculus on A_∞ categories. Knowing these is enough to reconstruct your A_∞ category. You can reduce to calculating these. This corresponds in the Fukaya category, say, to a very precise calculus.]

You can use a twisting cochain to deform the A_∞ structure. So I want to define a functor $\Omega : \mathrm{Cog} \to \mathrm{Alg}$ and will show that Ω and B are adjoint to each other.

Now I fix a coalgebra C, and whenever we have an algebra A we can define a set of twisting cochains $\mathrm{Tw}(C, A)$, and this is functorial, $A \mapsto \mathrm{Tw}(C, A)$, covariantly. So we should prove that a morphism $A \to A'$, by postcomposing you get a map $\mathrm{Tw}(C, A) \to \mathrm{Tw}(C, A')$. We should check that $D(f\tau) + (f\tau) * (f\tau) = 0$, and this is the same as $f(D\tau + \tau * \tau) = 0$, so that this functor is well-defined. Moreover it is actually representable by an element "Cobar," ΩC, which is nothing but the tensor algebra $T(S^{-1}\bar{C})$, this is the tensor algebra. So we want to regard this as an algebra, so we need a differential, and $d = \sum 1^{\otimes -} \otimes d_C \otimes 1^{\otimes -} + \sum 1^{\otimes -} \otimes S^{-1}\Delta \otimes 1^{\otimes -}$. Here $S^{-1}\Delta$ is something like $(S^{-1} \otimes S^{-1})\Delta S$. We need to check that d is actually a derivation of the tensor product. I don't want to check the details.

There's a canonical map, something like $C \to S^{-1}C \to \Omega C$, this is a degree 1 map, and we can denote this by, well, I want to show that this is in $\mathrm{Tw}(C, \Omega C)$. So we need to check that $D(iS^{-1}) + (iS^{-1}) * (iS^{-1}) = 0$ but this is

$$d_{\Omega C}(iS^{-1}) - (-1)iS^{-1}d_C + \mu_{\Omega C}(iS^{-1} \otimes iS^{-1})\Delta_C$$

but this is just

$$d_{\Omega C} is^{-1} - (-1)id_{S^{-1}C}S^{-1} + \mu_{\Omega C}S^{-1}\Delta.$$

but this is the definition of $d_{\Omega C}$, and so this is very complicated but conceptually this is nothing but the definition of the differential on ΩC. So this $\tilde{\iota}$ is a twisting cochain, and the functor from algebras to sets $A \mapsto \mathrm{Tw}(C, A)$ is represented by ΩA, it's $\mathrm{Hom}_{\mathrm{Alg}}(\Omega C, A)$. To see this we have the kind of situation $C \to A$, and the canonical inclusion $C \to \Omega(C)$, but this tensor algebra is a free object in the category of algebras, so we can always find a morphism from $\Omega(C)$ to A so we only need to prove the other way, that the composition of $\tilde{\tau} \circ \tilde{i}$ is a twisting morphism. But this is the same as like $\tilde{\tau}(D\tilde{\iota} + \tilde{\iota} \star \tilde{\iota})$, and we showed that $\tilde{\iota}$ is a twisting cochain, so this is zero.

So what we've shown is that there are two bijections

$$\mathrm{Hom}_{\mathrm{Cog}}(C, BA) \cong \mathrm{Tw}(C, A) \cong \mathrm{Hom}_{\mathrm{Alg}}(\Omega C, A),$$

there are such bijections, and we have these two functors, and these are adjoint to each other. So Ω is left adjoint. This is bar and cobar.

I wnat to mention a model structure on the category of coalgebras. A map is a weak equivalence in Cog if and only if $\Omega(f)$ is a weak equivalence in algebras. The cofibrations in Cog are the degreewise injective morphisms. The fibrations are the morphisms which have the right lifting property against trivial cofibrations.

Theorem 1.2. *(Lefèvre–Hasegawa)*

- *This data gives a model structure on* Cog *and* Ω *preserves cofibrations and trivial cofibrations and* B *preserves fibrations and trivial fibrations, so* (Ω, B) *are a Quillen adjunction. Actually* Ω *and* B *are Quillen equivalences. In other words they induce an equivalence of homotopy categories.*
- *All objects in* Alg *are fibrant; all objects in* Cog *are cofibrant. An algebra* A *is cofibrant if and only if it is a retract of* ΩC *for some* C *in* Cog *and a coalgebra* C *is cofibrant if and only if it is isomorphic as an underlying graded coalgebra to* $T^c V$ *for some* V*.*
- *If* A *and* A' *are fibrant and cofibrant in* Alg*, then* $f \sim g$ *as maps* $A \to A'$ *if and only if there exists* $h : A \to A'$ *of degree* -1 *with* $h\mu_A = \mu_{A'}(f \otimes h + h \otimes g)$ *and* $f - g = d_{A'}h + hd_A$*. There is a dual statement for coalgebras.*

So we have

$$\begin{tikzcd} \mathrm{Alg} \arrow[d, shift left, "B"] \\ \mathrm{Cog} \arrow[u, shift left, "\Omega"] & \mathrm{Cog}_{cf} \arrow[l] \end{tikzcd}$$

and to complete the corner, we should pass to Alg_∞ the category of augmented strongly unital A_∞ algebras, which is equivalent to considering non-unital A_∞ algebras. This has this sequence of maps (A, b_n), where $b_n : A^{\otimes n} \to A$, with all maps of degree 1. The unital means there is a unit element and it should be zero unless $n = 2$.

As before we want to define something like $\mathrm{Hom}^\bullet_{\mathbf{k}}(C, A)$ for $C \in \mathrm{Cog}$ and $A \in \mathrm{Alg}_\infty$. We want to equip this with, this has an A_∞ structure and there is a unit and stuff like that.

If f is a map, and we want to define $b_1(f) = b_1^A f - (-1)^{|f|} f d_C$. For $n \geq 2$, we have

$$b_n(f_1, \ldots, f_n) = b_n^A(f_1 \otimes \cdots \otimes f_n)\Delta^{(n)},$$

and we need to check that $\mathrm{Hom}(C,A)$ with these multiplications is an A_∞ algebra. It's not hard to check this but I won't prove it all. I'll show one simple thing. If $n=2$, then $b_2(1\otimes b_1)+b_2(b_1\otimes 1)+b_1b_2$ should be zero, and this is

$$(-1)^{|f_1|}b_2^A(f_1\otimes(b_1^Af_2-(-1)^{|f_2|}f_2d_C))\Delta+b_2^A((b_1^Af_1-(-1)^{|f_1|}f_1d_C)\otimes f_2)\Delta \\ +b_1^A(b_2^A(f_1\otimes f_2)\Delta)-(-1)^{|f_1|+|f_2|+1}b_2^A(f_1\otimes f_2)\Delta d_C$$

and [some cancellation]. So one can check A_∞ relations, so this defines an A_∞ algebra structure on the Hom set.

The equation is something like

$$\sum_{n\geq 1} b_n(\tau\otimes\cdots\otimes\tau)=0,$$

and now $\tau\in\mathrm{Hom}^0_{\mathbf{k}}(C,A)$.

There is a canonical way to consider an algebra by setting all higher multiplications to be zero, so this equation is the same as $b_1(\tau)+b_2(\tau\otimes\tau)=0$. So then in this case by carefully considering the sign, this is nothing but the equation $D\tau+\tau*\tau=0$.

So we want to define the set $\mathrm{Tw}_\infty(C,A)$ as the solutions to this equation.

Then this comment $\mathrm{Alg}\subset\mathrm{Alg}_\infty$, if A is an algebra, then this set is the same as $\mathrm{Tw}(C,A)$.

As before one can regard this as a functor $\mathrm{Cog}\to\mathrm{Set}$ which sends the coalgebra C to $\mathrm{Tw}_\infty(C,A)$. I'll skip the proof of functoriality (this is actually very easy, you just pull $f:C'\to C$ out to the left, this is a standard argument we've used many times).

This functor is representable. You define $B_\infty A$ for an A_∞ algebra as $T^c(A)$, the reduced version, then we want to define a differential, the differential is the sum of $1^{\otimes -}\otimes b_i\otimes 1^{\otimes -}$, and one can check that d is a coderivation with respect to the coproduct. But I want to skip. So it's almost the end. So I want to show that representability $\mathrm{Tw}(C,A)\cong\mathrm{Hom}_{\mathrm{Cog}}(C,B_\infty(A))$, I want to prove this, and before I prove that, let's consider the canonical projection $B_\infty A\to A$, this is degree 0, and I want to show that this is a twisting cochain in $\mathrm{Tw}_\infty(B_\infty,A)$, and then by universal properties, for any twisting cochain $\tau:C\to A$ this lifts to a coalgebra map $C\to B_\infty A$ which pulls the canonical twisting cochain on $B_\infty A$ to the given one on C.

Let me mention some facts. If V is a graded vector space then the set of A_∞-structures on V is in one to one correspondence with coalgebra differentials on TSV. If A and A' are A_∞ algebras, then $\mathrm{Hom}_{\mathrm{Alg}_\infty}(A,A')$ is in one to one correspondence with $\mathrm{Hom}_{\mathrm{Cog}}(B_\infty A,B_\infty A')$. This means that B_∞ is a functor from A_∞ algebras to Cog, it's actually fully faithful.

Theorem 1.3. *If C is a coalgebra, then C is fibrant cofibrant if and only if $C\cong B_\infty A$ for some A_∞ algebra.*

This implies that the functor is essentially (quasi-)surjective. Then this is very close to an equivalence of categories, it's a (quasi-)equivalence. Moreover, C in Cog has a *minimal model* where $I\in\mathrm{Cog}_{cf}$ and $I\xrightarrow{\sim}C$ and there exists $f^{-1}\in\mathrm{Cog}$ if and only if there is an inverse in $\mathrm{Ho}(\mathrm{Cog})$. The theorem is that any cocomplete coalgebra has a minimal model, and on the other hand, if $A_{\min}$ is a minimal model for A, then this minimal model, this is a kind of A_∞ algebra with $b_1=0$. Of course, this is quasi-isomorphic to A.

Then the functor B_∞ makes a bridge between the two minimal models. B_∞ of a minimal model of A. is isomorphic to the minimal model for $B_\infty(A)$.

Now I can draw my diagram.

$$\begin{tikzcd} \mathrm{Alg} \arrow[r] \arrow[d, shift left, "B"] & \mathrm{Alg}_\infty \arrow[d, "B_\infty"] \\ \mathrm{Cog} \arrow[u, shift left, "\Omega"] & \mathrm{Cog}_{cf} \arrow[l] \end{tikzcd}.$$

And if you take homotopy categories everything is an equivalence of categories. So in some sense we have four different descriptions of one algebra, but homotopically they are all the same, there are no new homotopic descriptions. In particular, the A_∞ description is homotopically not new but gives several types, that's the story I wanted to tell. I will stop here.

2. Feb 6: Christophe Wacheux: A_∞-structure on Fukaya categories II

I have way too much stuff. Last time I showed the formula of the derivative that we will, the differential of the Floer complex.

$$\partial(p) = \sum_{\substack{q \in L_0 \pitchfork L_1 \\ [u]:\mu([u])=1}} \#\mathcal{M}(p, q; [u], J) T^{\omega[u]} q.$$

In order to make it so that this is a zero dimensional manifold, I want to count only Maslov index 1 holomorphic disks. Because I have an orientation, in the good case, a spin structure on the Lagrangian, I can do this and get an orientation on the moduli space and get signs so that I can now count with signs, and that's the count I put here. This $\omega[u]$ is $\int_D u^*\omega$. There is lots of reason for this not to be well-defined. This is a compactness issue which is taken care of using the Gromov compactness theorem.

This, as we saw earlier, belongs to Λ^R, the Novikov ring.

The plan today is to try and define some of the things, I want you all to see the actual formula for the k-ary operation, and then after that I discuss as many details as possible.

Just to mention, the theorem was to show that

Theorem 2.1 (Floer). *If $k = \mathbb{Z}_2$ and $[\omega].\pi_2(M, L_i) = 0$, then*

(1) *∂ is well-defined,*
(2) *$\partial^2 = 0$,*
(3) *$HF(L, L) \cong H_*(L, \mathbb{Z}_2)$, and*
(4) *$HF(L_0, L_1)$ doesn't depend on the choice of J, of isotopy class of L_i*

This result helps prove the Arnold conjecture, at least in this case. Then it was extended to another very nice setting. It was extended to the case which is called monotone, also a very important case, Yong-Geun did it, that $\int_D u^*\omega = \lambda\mu([u])$ for u representing a class in $\pi_2(M, L)$.

Now I'll make a huge jump to define, to give any sense to this formula, this is assuming everything works fine. I should write "AEWF," and I'll try to give sense to what this acronym means. Now the product that we will call m_2, Now you have a pair of pants [sic]. It's the disk with three marked points, each of which is sent to one intersection (between L_0, L_1, and L_2 pairwise). So z_0 goes to q in the intersection of L_0 and L_2 and z_i to p_i in the intersection of L_{i-1} and L_i. Then we have

$$m_2 : CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$$

given by

$$m_2(p_2 \otimes p_1) = \sum_{\substack{q \in L_1 \pitchfork L_2 \\ [u] | \mu[u]=0}} \#\mathcal{M}(p_1, p_2, q; [u], J) T^{\omega[u]} q.$$

This is the formula. And I guess now we start to understand that we have some pattern going on. The whole general operation $m_k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_k)$ is given by

$$m_k(p_k \otimes \cdots \otimes p_1) = \sum_{\substack{q \in L_0 \pitchfork L_k \\ [u] | \mu([u])=2-k}} \mathcal{M}(p_1, \ldots, p_k, q; [u], J) T^{\omega[u]} q.$$

You can guess, these verify A_∞ relations. Maybe this is not worth taking time to verify.

I really want to address a bunch of problems. If I want an honest A_∞ structure, then I need to define a grading. So now I want to talk about Maslov index and grading of CF. First we need an orientation on the Lagrangian if you want to work over something other than $\mathbb{Z}_2$, then you will need to be able to define a consistent grading. The way to do this is with the Maslov index. If you have your Lagrangian manifold living in some big ambient symplectic manifold, if you look at the tangent space, then T_pL is a Lagrangian subspace of T_pM. What are the Lagrangian subspaces of M? If I give you a distribution of Lagrangian subspaces, does it integrate to a Lagrangian submanifold? So if I say now, I look at, just, on T_pM it's just the same as $\mathbb{R}^{2n}$, I can locally trivialize, and I define $LG(n)$ as the set of Lagrangian vector spaces of $\mathbb{R}^{2n}$, the "Lagrangian Grassmanian." There is a result which tells you that this is isomorphic to $U(n)/O(n)$. You can look at $\det^2 : U(n)/O(n) \to S^1$, and this goes, for $\pi_1[\det^2] : \pi_1(U(n)/O(n)) \to \mathbb{Z}$, and that's an isomorphism.

Essentially this is going to define μ. I'll define two things, I'm going to define the Maslov index of a holomorphic strip, the Maslov class of the Lagrangian, and the degree of a point. Let me write, now given u a J-holomorphic strip, let me remind you how this looks [picture].

If I look at $u^*_{\mathbb{R}\times\{i\}} TL_i : [0,1] \to LG(n)$ and call that ℓ_i, then my path of Lagrangian subspaces, since L_0 and L_1 intersect transversally, then

$$\ell_0(0) \pitchfork \ell_1(0) \text{ and } \ell_0(1) \pitchfork \ell_1(1).$$

As you said, Gabriel, I want to identify $(\mathbb{R}^{2n}, \omega_0) \cong (\mathbb{C}, \omega)$ and I want to say that $\ell_0 \in LG(n)$, there exists an $A_0 \in GL_n(\mathbb{C})$ such that $A_0(\ell_0(0)) = \mathbb{R}^n$ and $A_0(\ell_1(0)) = i\mathbb{R}^n$.

Now I call this $\lambda(t) := A_0^{-1}(e^{i\frac{\pi}{2}t}\mathbb{R}^n)$. [pictures].

Now I identify what is the path going from the tangent to the tangent, between $\ell_0(0)$ and $\ell_1(0)$. I can do the same stuff with a different identification for $\lambda_1(t)$, between $\ell_0(1)$ and $\ell_1(1)$. Now I will define the Maslov index of a J-holomorphic strip.

Definition 2.1. Define $\gamma : [0,1] \to LG(n)$ by $\gamma = \ell_0 \bullet \lambda_1 \bullet \ell_1^{-1} \bullet \lambda_0^{-1}$ and then

$$\mu([u]) := \pi_1[\overset{2}{\det}][\gamma]$$

and that's the Maslov index of a strip.

We've defined the Maslov index of a strip. In order to start doing all this business I need a spin structure, which is a choice of a section in the double cover of $U(n)$, or $O(n)$.

Now to define a $\mathbb{Z}$-grading, I will need exactly to make sure that $\mu[u]$ depends only on $|p|-|q|$ but not on $[u]$ even if I didn't define it yet. To make it happen, one thing is to ask for $2c_1(TM)=0$, which, now you get a hint for why this works in $\mathbb{Z}_2$. Why do you need this? I'll define something more elaborate than a Lagrangian, something we called a [unintelligible]Lagrangian submanifold, taking a universal cover of the Lagrangian Grassmannian.

So $c_1(TM)$ tells me, take $\Theta\in\wedge^n T^*M\otimes\mathbb{C}$, then $\varphi(D)=\arg(\Theta|_D)\in S^1$. Now you define $\tilde{\varphi}(D)$, a choice of smooth lift of $\varphi(D)$, and since $\pi_1(LG(n))$ is $\mathbb{Z}$, you can think of the universal cover,

3. February 13: Taesu Kim, an example of an A_∞ category

I'll talk about the Fukaya category, given a symplectic manifold (M,ω) which is oriented, compact, and spin and its spin Lagrangian submanifolds

$F\to M$ is the frame bundle whose fiber is frames of the tangent space. A frame is, given a basis of T_pM, the fiber F_p is the set of ordered orthonormal bases of T_pM. So I have $\iota c_1(TM)=0$ and $\mu_L=0$ in $H^1(L,\mathbb{Z})$ (this is the Maslov class) which are to give us a $\mathbb{Z}$-grading. Then we want $[\omega]\pi_2(M,L)=0$ in order to avoid disk bubbling.

With this data we can define $F(M,\omega)$, the Fukaya category of (M,ω).

Let's talk about the Maslov class. Suppose that $2c_1=0$. This means that $(\wedge^n_{\mathbb{C}}T^*M)^{\otimes 2}\to M$ has a nonvanishing section, Θ. In local coordinates this looks like $v_1,\ldots,v_n\mapsto\Theta_p(v_1\wedge\cdots\wedge v_n\otimes v_1\wedge\cdots\wedge v_n)$, and then you can write

$$\frac{\Theta_p(\vec{v})\Theta_p(\vec{v})}{\Theta_p(\vec{v})\bar{\Theta}_p(\vec{v})}$$

which should be in S^1. So then we can define $\varphi_\Theta(p)\coloneqq\frac{\Theta^2(v)}{|\Theta^2(v)|}$. So for ℓ in $Gr(n)$, the Lagrangian plane, so inside T_pM we can find T_pL and for ℓ we assign this fraction above for arbitrarily chosen v. Then choose a lift of this map.

[long discussion]

So for each Lagrangian we have an index, and if they vanish, we can get some sort of loop by doing something in the cover at the intersection points. Then we get the Maslov index of a loop, and that's how we get the Maslov index for each intersection point.

[Long discussion]

4. February 20: Kyoung-Seog Lee: Hochschild homology of DG categories

Today I will talk about both homology and cohomology of something. First let me discuss Hochschild homology and cohomology of algebras.

Let $\mathbf{k}$ be a commutative ring and R be a $\mathbf{k}$-algebra and M be an R-R-bimodule. Here R can be a non-commutative $\mathbf{k}$-algebra.

In this setting I can associate a simplicial $\mathbf{k}$-module $M\otimes R^{\otimes *}$ with

$$[n]\mapsto M\otimes R^{\otimes n}$$

and for concreteness, $M \otimes R^{\otimes 0} = M$.

I will make a complex

$$0 \leftarrow M \xleftarrow{\delta_0 - \delta_1} M \otimes R \xleftarrow{d} M \otimes R \otimes R$$

with $d = \sum (-1)^i \partial_i$.

Here $\delta_i(m \otimes r_1 \otimes \cdots \otimes r_n)$ is

$$\begin{cases} mr_1 \otimes r_2 \otimes \cdots \otimes r_n & i = 0 \\ m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n & 0 < i < n \\ r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1} & i = n. \end{cases}$$

And $\sigma_i(m \otimes r_1 \otimes r_n) = m \otimes \cdots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n$.

So call $C(M \otimes R^{\otimes *})$ the chain complex above, and then

Definition 4.1. The Hochschild homology $H_n(R, M)$ is the homology $H_n C(M \otimes R^{\otimes *})$.

When we look at $M \otimes R \to M$, the differential takes $m \otimes r$ to $mr - rm$, so $H_0(R, M) \cong M/[M, R]$. In the same setting I can define a cosimplicial $\mathbf{k}$-module, where n goes to $\mathrm{Hom}_{\mathbf{k}}(R^{\otimes n}, M)$, this is $\mathbf{k}$-linear maps from $R^{\otimes n}$ to M. We can again define a cochain complex

$$0 \to M \to \mathrm{Hom}_{\mathbf{k}}(R, M) \to \mathrm{Hom}_{\mathbf{k}}(R \otimes R, M) \to \cdots$$

and let me call this $C\,\mathrm{Hom}_{\mathbf{k}}(R^{\otimes *}, M)$ and d is defined the same way, $d = \sum (-1)^i \partial^i$ and let me define ∂^i as follows. This is a $\mathbf{k}$-module of functions, so $(\partial^i f)(r_0, \ldots, r_n)$ is

$$\begin{cases} r_0 f(r_1, \ldots, r_n) & i = 0 \\ f(r_0, \ldots, r_i r_{i+1}, \ldots, r_n) & 0 < i < n \\ f(r_0, \ldots, r_{n-1}) r_n & i = n. \end{cases}$$

We can define $\sigma^i f(r_1, \ldots, r_n) = f(r_1 ldots, r_i, 1, r_{i+1}, \ldots, r_n)$.

Definition 4.2. The Hochschild cohomology $H^*(R, M)$ is the $\mathbf{k}$-module which is the cohomology of the cochain complex $H^n(C\,\mathrm{Hom}_{\mathbf{k}}(R^{\otimes *}, M))$.

You have $0 \to M \to \mathrm{Hom}(R, M)$. If I have m, this goes to $\partial^0(m) - \partial^1(m)$, this is a function, which when you apply it to r, by definition, this is $rm - mr$. So $H^0(R, M)$ consists of the m in M such that $rm = mr$.

Let us compute $H^1(R, M)$. This is very closely related to derivations. I'll write

$$0 \to M \xrightarrow{d} \mathrm{Hom}(R, M) \xrightarrow{d} \mathrm{Hom}(R \otimes R, M)$$

and if I take f in $\mathrm{Hom}(R, M)$ it goes to $\partial^0 f - \partial^1 f + \partial^2 f$ and

$$(\partial^0 f - \partial^1 f + \partial^2 f)(r_0 \otimes r_1) = r_0 f(r_1) - f(r_0 r_1) + f(r_0) r_1$$

which means that $f(r_0 r_1) = r_0 f(r_1) + f(r_0) r_1$.

So the kernel of d is nothing but the set of $\mathbf{k}$-linear maps $f : R \to M$ satisfying this condition, which we call the $\mathbf{k}$-*derivation* condition. So $\mathrm{Der}_{\mathbf{k}}(R, M)$.

I should mod this out by the image of M, so M goes to $\mathrm{Hom}(R, M)$, so m goes to f_m which is $r \mapsto rm - mr$, and you can check that $f_m(r_0 r_1)$ is a derivation:

$$\begin{aligned} f_m(r_0 r_1) &= r_0 r_1 m - m r_0 r_1 \\ &= r_0(r_1 m - m r_1) + (r_0 m - m r_0) r_1 \\ &= r_0 f_m(r_1) + f_m(r_0) r_1. \end{aligned}$$

So we call the principal derivations

$$\mathrm{PDer}_{\mathbf{k}}(R, M) = \langle f_m \rangle.$$

So

$$H^1(R, M) \cong \mathrm{Der}_{\mathbf{k}}(R, M) / \mathrm{PDer}_{\mathbf{k}}(R, M).$$

Definition 4.3. Let R be a commutative $\mathbf{k}$-algebra. We can define the *Kähler differential* of R over $\mathbf{k}$ is

$$\Omega_{R/\mathbf{k}} = R\langle dr | d\alpha = 0 : \alpha \in \mathbf{k} \rangle.$$

So if $R = \mathbb{C}[x_1, \ldots, x_n]$ then $\Omega_{R/\mathbf{k}} = R\langle dx_1, \ldots, dx_n \rangle$.
This is an example,

Proposition 4.1. *Let R be a commutative* $\mathbf{k}$*-algebra and M be an R-R-bimodule, $rm = mr$. Then $H_0(R, M) \cong M$ and $H_1(R, M) \cong M \otimes_R \Omega_{R/\mathbf{k}}$.*

This is dual to derivation, this is dual to 1-forms.

Hochschild cohomology is related to derivations; homology is related to 1-forms in R.

When R is a polynomial ring, then $H_1(R, R) \cong \Omega^1_{R/\mathbf{k}}$ and $H^1(R, R) \cong T^1_{R/\mathbf{k}}$.

Let $R = \mathbb{C}[x]$, and $\mathbf{k} = \mathbb{C}$, and let us compute $\mathrm{Der}_{\mathbf{k}}(R, R)$. This is, by definition, $\mathbf{k}$-linear homomorphisms $R \to R$ such that $f(r_0 r_1) = r_0 f(r_1) + f(r_0) r_1$. In this case, this is a function, a $\mathbf{k}$-linear map. $f(x) = 1 f(x) + f(1) x$. This implies that $f(1) = 0$. Then $f(x^2) = 2x f(x)$.

I want to claim that $\mathrm{Der}_{\mathbf{k}}(R, R) \cong R\langle \frac{\partial}{\partial x}$.

So then for $\mathbb{C}[x]$ the principal derivations are 0 so $H^1(R, R) \neq 0$.

Exercise 4.1. Let $R = \mathbf{k}[x]/(x^{n+1} = 0)$. Then if $\frac{1}{n+1} \in R$, we hav $H_i(R, R) \cong H^i(R, R) \cong R/(x^n R)$ for all $i \geq 1$.

When R is $\mathbb{C}[x_1, \ldots, x_n]$, $M = R$, and $\mathbf{k} = \mathbb{C}$, then $H_0(R, R) \cong R$ and $H_1(R, R) \cong R dx_1 \oplus \cdots \oplus R dx_n \neq 0$ and $H^1(R) \cong R\frac{\partial}{\partial x_1} \oplus \cdots \oplus R\frac{\partial}{\partial x_n}$ then this is nonzero too.

On the other hand for $R = \mathbb{C}$, you get $H_i(R, R) \cong H^i(R, R) \cong 0$.

Let me show you one more example. This first homology is related to Kähler differentials. Let me give you one more, related to H^2. As I told you, H^2 is related to deformation. Let me show you. So a *square zero extension* of R by M is a $\mathbf{k}$-algebra E with $E \xrightarrow{\epsilon} R$ a projection such that $\ker \epsilon$ is an ideal of square zero and $M \cong \ker \epsilon$ as R-modules. So $0 \to M \to E \to R \to 0$ is short exact. This is called a Hochschild extension if $0 \to M \to E \to R \to 0$ is $\mathbf{k}$-split. This is an algebra, so as a $\mathbf{k}$-module, it's isomorphic to $R \oplus M$. As an algebra, I have a multiplication, I have $(r_1, m_1)(r_2 m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2 + f(r_1, r_2))$.

So $f : R \otimes R \to M$, and because this is an associative algebra, we should have $(r_1, 0)(r_2, 0)(r_3, 0)$ gives some condition. So you get $(r_1 r_2, f(r_1, r_2))(r_3, 0) =$

$(r_1r_2r_3, f(r_1,r_2)r_3 + f(r_1r_2,r_3))$ and I can do the other side and add up, and you eventually get a condition that f is a *cycle*, that

$$r_0f(r_1,r_2) - f(r_0r_1,r_2) + f(r_0,r_1r_2) - f(r_0,r_1)r_2 = 0$$

and this is nothing but $df(r_0,r_1,r_2)$, which is just $\partial^0 - \partial^1 + \partial^2 - \partial^3$. And from this associative rule, this says that $f \in Z^2(\ )$ of our cochain complex. If I choose another section, I had to choose a section, and if I choose another σ' I get another f' and we can check that the difference is in $B^2(\ )$ of our cochain complex. I want to say that this kind of extension, the equivalence class of Hochschild extensions is in one to one correspondence with $H^2(R,M)$. If M and R are commutative, then I have some commutative version which corresponds to another version of Hochschild cohomology.

Why is this kind of thing interesting? When M is R, then this kind of diagram is something like this. If I have $\operatorname{Spec}\mathbf{k}[\epsilon] \to \operatorname{Spec}\mathbf{k}$ and have $\operatorname{Spec}R \to \operatorname{Spec}\mathbf{k}$, then this diagram, this algebra.

So what this means, if you look at $R = \mathbf{k}[x]/(x^2)$, you have this kind of sequence:

$$0 \to (x) \to R \to \mathbf{k} \to 0$$

and this is exactly that situation. As Damien said, when I have this kind of $0 \to R \to E \to R \to 0$, then it means that I have some kind of, you have the deformation space of Spec R, and here you have some kind of choice of direction, to deform the algebra. This has this kind of feeling.

[Is it true that $H^2(R,R)$ is the same as equivalence classes of flat algebras so that when I point at $\mathbf{k}$, it reduces to R?]

Yes. So H^2 measures deformations of a certain kind of structure. Here it's deformations of algebra. This is some feeling I have.

I believe you have some feeling of this now.

Let me just state some general feeling. Let me write some general theorems that I think are quite important.

Let me give another definition of Hochschild homology. Let me define R^e to be $R \otimes_{\mathbf{k}} R^{\mathrm{op}}$. This op means it's the $\mathbf{k}$-algebra with $r\dot{s} = sr \in R$. Then this is a $\mathbf{k}$-module, and then a right R-module M is the same as a left R^{op}-module. Then an R-R-bimodule is a left R^e-module, $(r \otimes s)m = rms$. In the same way, it's also a right R^e-module.

You can check that, using the bar resolution, if R is flat over $\mathbf{k}$ then $H_*(R,M) \cong \operatorname{Tor}_*^{R^e}(M,R)$. If R is projective over $\mathbf{k}$ then $H^*(R,M) \cong \operatorname{Ext}^*_{R^e}(R,M)$. Here R is an R-R-bimodule, and M is one, so you can make everything a left R^e-module.

Let X be a smooth (projective?) variety over $\mathbf{k}$, let $\mathbf{k} = \bar{\mathbf{k}}$ of characteristic zero. Then

$$HH_*(X) \coloneqq H^*(X \times X, \Delta_*\mathcal{O}_X \otimes^L \Delta_*\mathcal{O}_X))$$

and

$$HH^*(X) \coloneqq \operatorname{Hom}^*_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X).$$

Theorem 4.1 (Hochschild–Kostant–Rosenberg)**.** *X is a smooth projective variety of dimension n, then*

$$HH_i(X) \cong \bigoplus_{p=0}^{n} H^{i+p}(X, \Omega_X^p)$$

and

$$HH^i(X) \cong \bigoplus_{p=0}^{n} H^{i-p}(X, \wedge^p T_X).$$

Finally let me discuss the Hochschild homology and cohomology of a dg algebra or category. Let C^* be a dg algebra. Then C^* is a dg bimodule over C^*, and $HH_*(C) = C \otimes^{\mathbb{L}}_{C\otimes C^{\mathrm{op}}} C$ and $HH^*(C) = \mathbb{R}\operatorname{Hom}_{C\otimes C^{\mathrm{op}}}(C, C)$.

Whenever I have X a smooth projective variety, I can consider the derived category $D(X)$, and it is known that this has a so-called *strong generator*. Let $\mathcal{E}$ be a strong generator of $D(X)$. I can consider $C = \mathbb{R}\operatorname{Hom}(\mathcal{E}, \mathcal{E})$. I mean I have a category, and a generator, and I have the endomorphism algebra of the generator. Then I can consider the Hochschild homology and cohomology of this algebra to be the Hochschild homology and cohomology of this category.

For example, $D(\mathbb{P}^1)$, this is generated by $\langle \mathcal{O}, \mathcal{O}(1)\rangle$, the strong generator is $\mathcal{O} \oplus \mathcal{O}(1)$, and C is the Kroenecker quiver on $\bullet \rightrightarrows \bullet$. You can compute that $HH^2(C) = 0$ because $\mathbb{P}^1$ is a rigid variety. So the right hand side of HKR is easy to compute. Sometimes we can compute it.

5. March 6: Taesu Kim: Introduction to the Fukaya category IV (or III)?

Let (M, ω) be a symplectic manifold and let L an oriented spin closed Lagrangian submanifold. We put some conditions on these geometric objects, for instance

- that the first Chern class of M is 2-torsion,
- that μ_L which lives in $H^1(L, \mathbb{Z})$, called the Maslov class of L, vanishes, and
- that $[\omega]\pi_2(M, L) = 0$.

Call these conditions (*). The first two of these are to give us a $\mathbb{Z}$-grading on Floer cochain complexes. The final condition is to prevent so-called "disk bubbling." Let me explain what this means later, so that we have the $\partial^2 = 0$ condition.

The spin condition is needed to put an orientation on the moduli space of pseudoholomorphic disks. We need this to appropriately count the number of rigid elements so that we can define the differential of the chain complex.

This is our geometric setting. Here are, what we called the Fukaya category, this is an A_∞-category $F(M, \omega)$. Its objects are Lagrangian submanifolds satisfying (*). The Floer chain complex between L_1 and L_2 is the direct sum over intersection points of Λp, where Λ is the Novikov ring. We assume $L_1 \pitchfork L_2$ for this definition. Later we'll modify this in some way. We should consider the Hamiltonian isotopy $\phi^t_{H_{L_1,L_2}}$ associated to $H_{L_1,L_2} \in C^\infty([0,1]\times M; \mathbb{R})$ so that $L_1 \pitchfork \phi^1_{H_{L_1,L_2}}(L_2)$ and then define $CF(L_1, L_2)$ as $CF(L_1, \phi^1_{H_{L_1,L_2}})$ which a priori depends on H_{L_1,L_2} and do this in a way that makes these transversal. This is a Λ-module with $\mathbb{Z}$-grading.

This construction anyway includes the case $CF(L, L)$. What about composition rules. We consider a map u from the disk with fixed holomorphic structure to M with fixed compatible almost complex structure J. We mark points z_0 to z_d on the boundary of the disk. We are given $L_1, \ldots, L_d, L_0$, objects in $F(M, \omega)$ (i.e. Lagrangians satisfying (*) which transversally intersect).

[pictures]

The conditions are that $u(z_i) = p_i$, and u is J-holomorphic in that $\bar\partial_J(u) = 0$. The image of the arc between z_i and z_{i+1} should lie in L_i and $[u] = \beta$. Then

$$\widetilde{\mathcal{M}}(p_1, \ldots, p_d, p_0, \beta)$$

is the set of such maps. In nice cases this has a manifold structure and the expected dimension (assuming generic J) this is $\mathrm{ind}(D\bar{\partial}_J(u)) = d+1+\mathrm{ind}(U)$, where $\mathrm{ind}(U)$ is the Maslov index of the disk, and so this is $d+1-\sum_{i=1}^d |p_i| + |p_0|$. Then we can reduce by an equivalence relation and get

$$\mathcal{M}(p_1, \ldots, p_d, p_0, \beta) = \widetilde{\mathcal{M}}(\vec{p}, \beta).$$

The dimension of $PSL(2,\mathbb{R})$ is three, so the expected dimension is $d-2-\sum_1^d |p_i|+|p_0|$.

Then $\overline{\mathcal{M}}(\vec{p},\beta)$ can be defined, a compactification so that its boundary consists of maps with image nodally glued disks. Since $[\omega]\pi_2(M, L_j) = 0$, the area of this disk should be zero and so this cannot happen.

So now we can define the composition between these morphisms

$$\mu^d : CF(L_{d-1}, L_d) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_d)$$

by

$$\mu^d(p_d, \ldots, p_1) = \sum_{q \in L_d \cap L_0} \#\mathcal{M}((\vec{p}, q), \beta) T^{\omega(beta)} q$$

where the sum is over $q \in L_d \cap L_0$, where $\sum |p_i| - |p_0| = d-2$, and β.

We have to check that the operators μ^d satisfy the A_∞ relations. For $d-1 = \sum |p_i| - |p_0|$, then it's a compact 1-manifold, so a disjoint union of intervals and circles. So the signed count of a boundary of this 1-dimensional moduli space is zero.

This says that the signed count of nodal configurations of the appropriate dimension with one node are zero. Then this can be parameterized by gluing parameters somewhere, and near the limit there is a one-to-one correspondence between the nodal and glued smoothed configurations. The glued configurations, counting them is about the composition of two copies of μ^j for some smaller j. Then the sum being zero says that this sum of compositions is zero. Let me put the sign as a $\pm$ and that's how we get the A_∞ relation. Hence the Fukaya category is an A_∞ category.

One important point is that it's cohomologically unital, so that $HF(M,\omega)$ is unital.

The unit is in $HF^0(L,L)$, which is isomorphic to $H^0(L,[\Lambda])$, and this is the Poincaré dual of $[L]$.

6. Calin Lazariou: A_∞ structures on categories of matrix factorizations

Everything in the mathematics literature here is both trivial and trivially wrong. Not so much is known about this either in mathematics or in physics.

Why is the obvious idea trivial? Let A be a dg category. For any two objects, the space of morphisms $\mathrm{Hom}_A(a,b)$ is R-module. Then R is a unital commutative ring. This is already an A_∞ category of a very particular type. There's nothing to do.

So what would you do? You'd consider a minimal model. So the first (and failed) attempt. Any A_∞ algebra has an anti-canonical (dg) and canonical (minimal) model, which is finite dimensional if the homology is finite dimensional. So assume that A is (cohomologically) hom-finite, compact, or proper (these are all the same thing, please ask Kontsevich why he changed the terminology three times in the past ten years). I will taken $\mathrm{Hom}_A(a,b)$ to be $\mathbb{Z}/2\mathbb{Z}$-graded, and I'll denote this

$\operatorname{Hom}^k(a,b)$. That is, $\bigoplus H_d^\alpha \operatorname{Hom}^k(a,b)$ is finite dimensional over $\mathbf{k}$. Here $\mathbf{k}$ is a field in R and R is a $\mathbf{k}$-algebra, and $\mathbf{k}\cdot 1 = \mathbf{k}$.

If you have this, then you have a minimal model, which is realized on the total cohomology category $H(A)$. It's the category which has the same objects as A and the homs are the graded R-modules of cohomology. This is completely trivial by the minimal model theorem.

In the case of a proper dg category, this has the pleasing property that it's a finite dimensional model.

The anti-minimal model has the unpleasing property that the underlying space is infinite dimensional.

This is trivial! All you have done is the Kadeishvili minimal model theorem with more than one object. It's also *wrong*, misdirected, wrong in the philosophical sense of Kant. It's the wrong question to ask, it's the wrong way to think about this.

The question is not to find the minimal model. You haven't done anything. There is a traditional line Laudal, various people, Manetti, that says there are these Massey products controlling the deformation theory and the nicest way to arrange these is with a minimal model. Say I want the moduli stack of an object a, you can build it by representing a deformation functor $\operatorname{Def}_A(a)$. You can represent local Artinian rings, and this can be written as the deformation functor of the commutator L_∞ algebra induced on $\operatorname{End}_A(a) = \operatorname{Hom}_A(a,a)$ by the minimal model of $\operatorname{End}_A(a,a)$.

If you're interested in deformations you can do this, build a moduli stack $\mathcal{M}_a$, an ∞-stack in general. This is again trivial in the sense that it was well-known before, you just put objects in what was known before, and again, not so interesting, because what you really want is to understand the *structure* of $\mathcal{M}_a$. There are physical reasons to expect this to be a non-commutative Calabi–Yau scheme.

So you can find the literature on this but this is the wrong problem. So what's the right way to think about it? The right way to think about the problem is via string field theory. This really works correctly if you have some sort of "Calabi–Yau"-ness. Let me explain what I mean. Your dg category, as I said, I pick some base field and can consider it as a dg category over $\mathbf{k}$, it's $\mathbb{Z}/2\mathbb{Z}$-graded, and I'm fixing μ in $\mathbb{Z}/2\mathbb{Z}$. I say A is *μ-Calabi–Yau* if there exist cyclic *homologically non-degenerate* linear maps

$$\operatorname{tr}_a : \operatorname{End}_A(a) \to \mathbf{k}$$

of degree μ for every a in A. By non-degenerate I mean that the bilinear pairing defined by taking $\operatorname{Hom}_A(a,b)\times\operatorname{Hom}_A(b,a)\to\mathbf{k}\to\mathbf{k}[\mu]$, this is a dg map with zero in the target, cyclic so that $(u,v) = \operatorname{tr}_a(v\otimes u) = \operatorname{tr}_b(u\otimes v)$ and this defines a non-degenerate bilinear form on the cohomology. I required my space to be hom-finite; otherwise I'd need to topologize and require perfectness. You can never have a non-degenerate bilinear form on two vector spaces of infinite dimension. You want this to be invariant up to sign up to the obvious permutation. You want it to be compatible with the differentials here, so that the trace of a boundary is zero, and it should induce a nondegenerate pairing on the cohomology.

I can write down the other properties explicitly:

$$
\begin{aligned}
tr_a(v \circ u) &= (-1)^{|u||v|} \operatorname{tr}_b(u \circ v) \\
\operatorname{tr}_a((dv) \circ u + (-1)^{|v|} v \circ (du)) &= 0 \\
\operatorname{tr}_a(v \circ u) &= 0 && \text{unless } |v| + |u| = \mu \\
\overline{\operatorname{tr}_a} : H(\operatorname{End}_A(a)) &\to \mathbf{k}[\mu] && \text{is nondegerate.}
\end{aligned}
$$

A $\mathbb{Z}/2\mathbb{Z}$-graded category with these maps, such a category, with degree μ non-degenerate traces, is usually called a *Calabi–Yau* category, and this is the extension to the dg world, except that you only require the non-degeneracy at the homological level.

I will tell you the interesting problem. What does this have to do with matrix factorizations.

Theorem 6.1. *Let X be a smooth Stein manifold which is holomorphically Calabi–Yau in the sense that its canonical line bundle is trivial. Let W be a holomorphic function on X such that the critical set is compact (in this case finite). Then the $\mathbb{Z}/2\mathbb{Z}$-graded dg category of matrix factorizations $PF(X,W)$, of projective analytic factorizations of W is proper and μ-Calabi–Yau with $\mu \equiv d \pmod 2$ where d is the dimension of X as a complex manifold.*

One of the nicest types of Landau–Ginzburg pairs is (X,W) where X is Stein and W is holomorphic. I insist on this compactness to get a proper category.

What is K_X? It's the top wedge product $\wedge^d T^*X$ (the holomorphic cotangent bundle) is trivial, isomorphic as a holomorphic line bundle to $\mathcal{O}_X$.

There is a particular example of Gromov's principle that says that the topological and holomorphic classifications coincide in this setting (Stein) so topologically trivial (first Chern class vanishes) implies holomorphically trivial.

If you try to do a non-Calabi–Yau version, then you get an anomaly in the $U(1)_X$ symmetry. So twisting with K_X like Pantov, Katzarkov, Pomerleano, Orlov, et cetera, have done, is physically wrong. Then you have to do something very weird on the other side to the Fukaya category. What is behind is that the correct data, you have to build an open-closed field theory.

Why did I mention compact? There's a version of this category, the so-called correct version, which doesn't require Stein, which is *not* the version they have proposed. There's something called DF which is again triangulated and $\mathbb{Z}/2\mathbb{Z}$-graded and makes sense for any X complex non-compact, and any W holomorphic with compact critical locus. There's a hypercohomology description, but this is a 2-periodic thing.

Of course any affine variety is a Stein analytic space, and in that case you can do an algebraic version of this category, but this is a much nicer statement, I think.

So what is $PF(X,W)$? They are pairs (P,D) where P is a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathcal{O}(X)$-module, degreewise projective and finitely generated. And D is an endomorphism of this module such that $D^2 = W$. The morphisms are the obvious ones, if I give you $a_1 = (P_1, D_1)$ and $a_2 = (P_2, D_2)$, then the hom space in PF is $\underline{\operatorname{Hom}}_{\mathcal{O}(X)}(P_1, P_2)$, with the defect differential

$$d_{a_1,a_2}(f) = D_2 \circ f - (-1)^{|f|} f \circ D_1.$$

There's a Serre–Swan theorem for Stein manifolds. Their condition is satisfied by the sheaf of holomorphic functions on a Stein manifold. [Something about

Cartan Theorem B.] This says that finitely generated projective $\mathcal{O}(X)$-modules are equivalent to holomorphic vector bundles.

There's a general result that if X is a non-compact complex manifold with $K_X \cong \mathcal{O}(X)$. Let W have compact critical locus. Then $DF(X,W)$ is proper and d (mod 2)-Calabi–Yau. This is the twisted Dolbeault category of the holomorphic factorizations. The objects are (E,D) where E is a $\mathbb{Z}/2\mathbb{Z}$-graded holomorphic vector bundle and $D \in \Gamma(X, \operatorname{End}(E))$ such that $D^2 = W\mathrm{id}$.

The morphisms between two is $\mathcal{A}^{0,*}(X, \operatorname{Hom}(E_1, E_2))$ equipped with the differential $\bar\partial + \partial_{a_1,a_2}$, where this latter on ω is $D_2 \circ \omega - (-1)^{|\omega|}\omega \circ D_1$.

The way I prove this with Dmitry is by combining Serre's original result with sophisticated spectral sequence arguments. This is a very general example.

I didn't introduce this notion of Calabi–Yau category, of course.

This is still not what you need. I will tell you in a moment how this is induced by a holomorphic volume form. But you need more, you need this notion of a Calabi–Yau structure, which is more than these traces.

Definition 6.1. Let A be proper, $\mathbf{k}$-linear (I'll assume $\mathbf{k}$ of characteristic zero, my interest is in $\mathbb{C}$) dg category. A cochain level Calabi–Yau structure (of degree μ) on A is a linear map from the cyclic complex $\theta : CC_*(A) \to \mathbf{k}[\mu]$, so

(1) $\theta \circ \delta = 0$
(2) $\theta_* : HC(A) \to \mathbf{k}[\mu]$ induces nondegenerate traces on $H(A)$ via precomposition with the natural map q from $H(A)$ to the Hochschild complex, and then this gives a natural map to to the cyclic homology. So this restriction is a homologically non-degenerate trace.

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So that's the Calabi–Yau structure. They only cared about the cohomology class, but this is a trivial extension, this was basically introduced by Kontsevich–Soibelman. To be precise, string field action is a *strict* cyclic structure, where the traces induecd by θ are nondegenerate at the cochain level.

So either this cyclic structure is established at the level of the minimal model or you topologize and require a perfect pairing. Everything you see here is defined for any A_∞ category. I can consider a minimal A_∞ category which is proper, and there require nondegeneracy off-shell.

The punchline, the point, there's a theorem, the particular case was proved by Sklyarov, that says the cohomologically non-degenerate traces of $DF(X,W)$ have a natural extension to a chain level μ-Calabi–Yau structure which is induced by a cubic open string field theory (in the sense of Witten). Cubic means that you have only, you have a dg model, but the trace is non-degenerate off-shell. This is something with compact supports. The trace is induced by the volume form. You do a gauge-fixing procedure, trying to find a quasi-isomorphic model by projecting on a small tubular neighborhood of your critical locus.

A *minimal Calabi–Yau* structure or strictly cyclic minimal A_∞-category is a minimal A_∞-category which is proper, the spaces are finite dimensional, and the traces are strictly cyclic with respect to the A_∞ structure, so

$$\langle f_0, m_n(f_1, \ldots, f_n)\rangle = (-1)^{\text{whatever}}\langle f_1, m_n(f_2, \ldots, f_n, f_0)\rangle.$$

In practice this was hard to construct, and Sklyarov gives you such a theory. You replace DF with a compactly supported version DF_c, which naturally includes in

DF, so the objects are the same, but the morphisms are *compactly supported* forms of type $0,*$ as before. If Ω is a volume form, a holomorphic section of $K_X \backslash \{0\}$, then you have for $\omega \in \mathrm{End}_{DF_c}(E,D)$ the following:

$$\mathrm{tr}_c(\omega) = \int \Omega \wedge \mathrm{str}(\omega)$$

and you have $\delta_W = \delta + \partial_{a_1+a_2}$, and this is a *perfect trace* if $DF_c(X,W)$ is topologized using the Fréchet topology. Then the cubic string field action is the functional S with

$$S(\phi) = \int_X \Omega \wedge [\mathrm{str}(\phi\delta_W\phi) + \frac{2}{3}\mathrm{str}(\phi^3)]$$

a ($\mathbb{Z}/2\mathbb{Z}$-graded twisted-by-W, categorified: $\phi \in \mathrm{End}(A) = \bigoplus_{a,b} \mathrm{Hom}(a,b)$) Chern–Simons type action).

But this only makes sense on the compactly supported one, and it uses smooth things, none of the algebraic geometers and few of the complex geometers would touch this.

Then what you do, the idea is the following, how does that object transfer into something defined on the other category. These are dg categories. You can prove that the map induced on cohomology is an isomorphism, so that HDF_c is a quasi-equivalence. If we know anything about quasi-equivalences, there should be a (non-unique) quasi-equivalence. This is not just an ordinary map, it's an A_∞ quasi-isomorphism. I'm sure you've seen this at least for algebras. It inverts i. It's an ordinary thing that commutes with differentials, but it has an inverse with many pieces.

You want to make a choice, getting rid of anything smooth, Fréchet, et cetera. You choose some tubular neighborhood of the (compact) critical locus and try to construct π_1 as a projector, I won't give the formula, and then π_n are given by some universal formula using π_1 and some property. This depends on the choice of infinitesimal neighborhood. You take some sort of inductive limit in which this neighborhood shrinks to Z_W and in that limit you use a residue theorem of [unintelligible]–Andersson (not Grothendieck, you need to upgrade this, a representation of Bochner–Martinelli type) so when you do this you find that all the θ_n of the corresponding Calabi–Yau structure have an expression in terms of these W–A residues.