CGP DERIVED SEMINAR

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1. March 14: Damien Lejay (Categories)

Thank you for being here, for supporting it by coming to the first lecture. I've named this the derived seminar and set up a webpage for the seminar, with pdfs that would be the skeleton of the seminar and I will put information like who is going to talk next and what it's going to be about. I'll upload the pdfs every week. If you want to recall anything then it's just for you.

I've been thinking about the roadmap and how to articulate the things on the wishlist. Today is the first day and I'll talk for two hours about category theory. There will be another session where we do category theory again. Then one session of introduction to the problematics of differential graded categories and why we are interested in these tools. One session of introduction, one session just on differential graded categories, then two sessions on model categories. Then we'll study differential graded categories again with those tools. This will take us through April. Then we'll have a session on triangulated categories. Already we will have seen these things. then we will have ∞-categories, stable ∞-categories, and comparison theorems. I won't plan more than that but by that time we will want to change the wishlist and can add and change things.

[Discussion of timing]

In the first hour I will give definitions, examples, and vocabulary, and in the second hour we'll do computations. Many many very important things will be said in the next week.

Category theory is a language, a theory that helps you to write down mathematics. It's like set theory. You don't do it for its own sake but rather do it to help you do mathematics. To define categories and work with them we will use the language of sets, and I'll start with some fun about set theory.

When you start with set theory, you talk about sets and it's great and then you try to take the set of all sets and it's not so great. So when you take the "set" of all sets, this is a "class." Classes are much bigger and that means you can't do quite the same operations on them.

What happens if I want to talk about the collection of all classes. You could could call this a "superclass," it's not inside the theory of sets. What can I can do with this? I don't know. What if I take a collection of collections? I can't really do anything here anyway. So we haven't really solved the problem. There's some boundary, but I want to be able to consider big objects. There's a nice solution to this, called Grothendieck universes.

Definition 1.1. A *universe* is a set $\mathbb{U}$ with some properties:

- if x is in $\mathbb{U}$ then the power set $P(x)$ is in $\mathbb{U}$.
- if x is in y and y is in $\mathbb{U}$ then x is in $\mathbb{U}$ (transitivity)

- if $\{x_i\}_{i\in I}$ have $I \in \mathbb{U}$ and $x_i \in U$ then $\bigcup x_i \in \mathbb{U}$.
- the natural numbers are in the universe. Let me say instead that I want the empty set and one infinite set.

These properties tell you that all the operations of set theory work in $\mathbb{U}$. You don't need to go elsewhere because all you do uses these sets.

The empty set is one example. The second one is natural numbers, where each natural number is the union of the numbers before. You never go outside of this universe. But these don't match the final axiom. Already when we said the natural numbers we use the axiom of set theory that there is an infinite set. In fact, if you just take the ZF axioms and the axiom of choice, you cannot prove that there is a universe bigger than $\mathbb{N}$, you don't know. So I should add an axiom that for every set X there exists a universe $\mathbb{U}$ with $X \in \mathbb{U}$. This is a harmless axiom. You can put it next to ZF and it will still be mathematics, but it will be helpful.

Something about universes, they are sets, so they have a cardinal, the cardinal of a universe, because of the stability under these operations, you get that they are strongly inaccessible. If $K < \mathrm{card}(\mathbb{U})$ then $2^K < \mathrm{card}(\mathbb{U})$. The existence of universes is equivalent to the existence of strongly inaccessible cardinals. This is completely harmless and transparent in the rest of your life in mathematics.

Now there's a bonus. Since $\mathbb{N}$ is a set, there is a universe $\mathbb{U}$ containing $\mathbb{N}$. But then there is a universe $\mathbb{V}$ containing $\mathbb{U}$. Then the set of all 'sets' lives in $\mathbb{U}$. If I want the set of all things like this, I move to $\mathbb{W}$. Every time I do something illegal in set theory I just jump up a level of universes.

With this said, I will stop set fun and start categorical fun. I'll give the definition of a category and then some examples. At some point we'll take a break before computations.

Definition 1.2. A *category* $\mathcal{C}$ is a set of objects $\mathrm{Ob}\,\mathcal{C}$ (I have a universe and a "set" is inside my universe). For x and y in $\mathrm{Ob}\,\mathcal{C}$ I have a new object $\mathrm{Hom}(x,y)$, the *arrows* or *homomorphisms* from x to y, and I'll picture these like $x \xrightarrow{f} y$. I have a special arrow for $x \in \mathrm{Ob}\,\mathcal{C}$, I have a special arrow $\mathrm{Id}_x \in \mathrm{Hom}(x,x)$, which goes $x \xrightarrow{\mathrm{Id}} x$, a special arrow. When I have arrows that I think of as functions, if I have $x \xrightarrow{f} y \xrightarrow{g} z$ I get an arrow $g \circ f \in Hom(x,z)$. I want

- $\mathrm{Id} \circ f = f$ and $f \circ \mathrm{Id} = f$.
- For three composable arrows I want $(f \circ g) \circ h = f \circ (g \circ h)$.

Next I have to to give an example. Instead I will give *the* example, the category of sets. I'll call it $\mathbb{U} - \mathrm{Set}$, and for this I need the set of objects. The objects of my category will be $\mathbb{U}$. If I have two sets x and y in $\mathbb{U}$, I need to know $\mathrm{Hom}_{\mathcal{U}-\mathrm{Set}}(x,y)$, this will be the set of functions from x to y. I have to give my identity element, which will be the identity function, and I have to give composition, and that's composition of functions.

We don't want to speak about universes all the time so I will talk about Set instead of $\mathcal{U} - \mathrm{Set}$. This is the most basic example, you'll have more complicated categories, and a lot of the time they will be built like this because they will have underlying sets.

I'll give some basic examples and classical notation for them. We're going to see.

There's a category called Ab which is the category of Abelian groups. The objects of Ab are the Abelian groups, but I cannot take "all" Abelian groups, that's too big, so I want the underlying set to be in $\mathbb{U}$. If I have two Abelian groups A and B, then $\mathrm{Hom}_{\mathrm{Ab}}(A, B)$ is the linear (additive) functions from A to B. I should be sure that composition of additive functions is an additive function and so on. No problem, so this is a category.

There's a category Rings of rings, whose objects are rings such that the underlying set is in $\mathbb{U}$. Then $\mathrm{Hom}_{\mathrm{Rings}}(R, S)$ is the set of ring homomorphisms from R to S.

So these are natural. Most concepts in mathematics can be described in this language. Let me give a category Ban of Banach spaces, so the objects are Banach spaces whose underlying sets are in $\mathbb{U}$. Then $\mathrm{Hom}_{\mathrm{Ban}}(V, W)$ are the continuous linear functions from V to W.

I could also take a category $\mathrm{Ban}_{\leq 1}$, which has the same objects, Banach spaces, but different arrows. Here $\mathrm{Hom}_{\mathrm{Ban}_{\leq 1}}(V, W)$ is the set of continuous linear maps $f : V \to W$ such that $\|f\| \leq 1$. If you compose two contracting morphisms you still get a contracting morphism, and the identity is a contracting morphism.

Let me give two other well-known categories. $\mathrm{Vect}_{\mathbb{R}}$ is the category of real vector spaces, here the objects are vector spaces with underlying set in $\mathbb{U}$. I'll end the list of known categories by Top, again the objects are topological spaces with underlying set in $\mathbb{U}$. The topology will then be inside $\mathbb{U}$. There is no problem here. The functions will be continuous functions.

Immediately from the definition is a principle that allows you to build twice as many categories, the duality principle. In categories, there is a difference between left and right, and what I can do is build a category by swapping the arrows. Let $\mathcal{C}$ be a category. Let $\mathcal{C}^{\mathrm{op}}$ be the category with the same objects and the arrows reversed, for $x, y \in \mathrm{Ob}\,\mathcal{C}$ I want $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x)$. So let's give an example. So say you have a category with only the following arrows

$$\begin{tikzcd} 0 \arrow[loop left, "\mathrm{id}_0"] \arrow[r] & 1 \arrow[loop right, "\mathrm{id}_1"] \end{tikzcd}$$

and you take the opposite category you get

$$\begin{tikzcd} 0 \arrow[loop left, "\mathrm{id}_0"] & 1 \arrow[l] \arrow[loop right, "\mathrm{id}_1"] \end{tikzcd}$$

If you take $\mathrm{Rings}^{\mathrm{op}}$ you get a useful and well-known category, the category of affine schemes. Taking the opposite category is useful because of things that depend on the side of the arrow. So if you have property P on $\mathbb{C}^{op}$, that means you have "co"-P on $\mathbb{C}$. If you change the direction of the arrow, things are swapped.

Since Descartes, we have tried to embed geometry in algebra, and here we see the principle that "geometry" is "algebra"$^{\mathrm{op}}$. I know how to do this computation on my algebra and I hope it will give me what I want on my geometry.

Now I'll give a definition specific to category theory. If I have two arrows

$$\begin{tikzcd} x \arrow[r, shift left, "f"] & y \arrow[l, shift left, "g"] \end{tikzcd}$$

whose compositions are equal to the identity, $g \circ f = \mathrm{Id}$ and $f \circ g = \mathrm{Id}$, then we call f an *isomorphism* and say that $x \cong y$, and I want to give a table of differences of how you think in set and in category theory.

Sets	Categories
set	object
$\in$	$x \to y$
$=$	$\cong$

You never want to talk about elements in category theory, you want to describe things in terms of arrows. It's forbidden. Equality is awkward, I'll say isomorphic or equivalent. There are plenty of examples but probably it's good to take a break right now. I'll see you in five minutes.

Definition 1.3. $\mathcal{D}$ is a *subcategory* of $\mathcal{C}$ if $\mathrm{Ob}(\mathcal{D}) \subset \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{D}}(x,y) \subset \mathrm{Hom}_{\mathcal{C}}(x,y)$ and identities and compositions are compatible with the inclusion.

Examples include $\mathrm{Ban}_{\leq 1} \subset \mathrm{Ban}$, or finite sets inside sets. I could take the objects of $\mathcal{U}$-sets with surjective morphisms. The identity is surjective and compositions of surjective morphisms are surjective. I could take torsion Abelian groups.

Definition 1.4. $\mathcal{D} \subset \mathcal{C}$ is *full* if for $x, y \in \mathrm{Ob}\,\mathcal{D}$, we have $\mathrm{Hom}_{\mathcal{D}}(x,y) = \mathrm{Hom}_{\mathcal{C}}(x,y)$. I will say $\mathcal{D} \subset \mathcal{C}$ is *wide* if $\mathrm{Ob}(\mathcal{D}) = \mathrm{Ob}(\mathcal{C})$.

So the Banach and surjection examples are wide and the finite set and torsion examples are full.

Definition 1.5. A category is a *groupoid* if every morphism is an isomorphism.

Take G a group, then we can build a category BG with one object and then $\mathrm{Hom}_{BG}(*,*) = G$. It's a point with G-many arrows, with composition given by composition in the group. Because it's a group you always can invert arrows.

If you have an action of G on X, then you can build a category a bit like G, where the objects of the category are the elements of X and $\mathrm{Hom}(x,y) = \{g \in G : y = g.x\}$. An arrow means there is an action of an element of G that goes from x to y. I can always go back since G is a group.

If $\mathcal{C}$ is a category, we have the interior groupoid of $\mathcal{C}$, the objects are the objects of $\mathcal{C}$, and the morphisms are the isomorphisms of $\mathcal{C}$.

Now I want to spend the last forty-five minutes making computations of one of the major things in category theory, called *limits* and *colimits*. This is a key big thing in category theory.

So far I don't have a lot of things in the structure. I have objects and arrows. Maybe I have something like this:

$$\begin{tikzcd} * \arrow[loop left] \arrow[r] & * \\ & * \arrow[ul, shift left] \arrow[ul, shift right] \end{tikzcd}$$

In a category you have a notion of approximation of a diagram by a single object of your category, and you have a problem of what it means to approximate. Close means arrows, and arrows have a side. I can approximate on the left or on the right.

If I have an approximation on the left, I call this the *limit* and denote it $\varprojlim \mathcal{D}$ and on the right I call it the *colimit* and write $\varinjlim \mathcal{D}$. Let me simplify my diagram. I want an arrow from my limit A to every element of my category and I want the compositions to be commutative:

$$\begin{tikzcd} & * \\ A \arrow[ur] \arrow[r] \arrow[dr] & * \arrow[u] \arrow[d] \\ & * \end{tikzcd}$$

I will say I have a *limit* if I have a universal thing like this. If A is my limit, it's closer to my diagram than any other one. If I have another approximation B then I should get an arrow $B \to A$ which is unique.

For colimits it's the same picture. An approximation on the right is something like this Z:

$$\begin{tikzcd} * \arrow[dr] & \\ * \arrow[u] \arrow[d] \arrow[r] & Z \\ * \arrow[ur] & \end{tikzcd}$$

The "best" approximation is one Z so that whenever you have an approximation on the right, W, you get a map $Z \to W$, unique.

This is called the colimit because it's on the right. If I swap directions, in the opposite category limits become colimits and vice versa.

Let X be an object of $\mathcal{C}$. The best approximation on both the left and right is X, $\varprojlim \mathcal{D} = X$ and $\varinjlim \mathcal{D} = X$. What if $\mathcal{D}$ is a single arrow, $X \xrightarrow{f} Y$. To make an approximation on the left, Z, I need a map $\varphi : Z \to X$ and a map $\psi : Z \to Y$. Then $\psi = f \circ \varphi$ by compatibility. I want an approximation by only one object, and now I want just a map $Z \xrightarrow{\varphi} X$, and so my limit is X. So $\varprojlim \mathcal{D} = X$.

What about my approximation on the right? This will be an object Z, with maps $\psi : X \to Z$ and $\varphi : Y \to Z$. Compatibility says that $\psi = \varphi \circ f$ so ψ is useless. Then I want an approximation by one object and Y looks like a good candidate. So the colimit is Y.

When you have an arrow $X \to Y$ then the approximation on the left is X and on the right is Y.

If I have two arrows $X \xrightarrow{f} Y \xrightarrow{g} Z$, what are the best approximations on the left and on the right? On the left it's X with the identity map because the maps from the limit to Y and Z are useless. On the right it's Z with the identity.

What about the empty diagram? Can you have a best approximation on the left and on the right? On the left it's called a *final object* $*$, every object in the category has a morphism to this object, a unique one, to $*$. An initial object is a final object in the opposite category. An *initial object* is one, notated $\varnothing$, an object so that $\mathrm{Hom}(\varnothing, c)$ is always of cardinality one.

In the category of sets, the empty set is initial, there is a unique map from the empty set to any set, and a set with only one element is final. In the category of Abelian groups, the Abelian group 0 is initial, and it's also final because you always have a map from any Abelian group to 0.

I want to give you more types of diagrams now. What about two objects and no arrows. What is the limit or colimit if you have just two objects A and B. You have some approximation Z, maybe, with arrows $f : Z \to A$ and $g : Z \to B$. If this exists we call it $A \times B$. In sets this is the product.

So for the colimit, you need two maps, one from $A \to W$ and one from $B \to W$. In sets it's called $A \sqcup B$, what about in Ab, there you have $A \oplus B$. In the category Ab, the coproduct of A and B is $A \oplus B$ also. Then you can get the product and the coproduct are the same.

Now I want to give you the definition of the coproduct in rings, but maybe, I have two ring maps from R and S to W, then I want this to be $R \sqcup S \to W$, and the answer is $R \otimes_{\mathbb{Z}} S$, this is ring theory. How do you build this map? You take the product of the two things. You take $R \otimes_{\mathbb{Z}} S \to W \otimes_{\mathbb{Z}} W \to W$. In rings this is the tensor product, while for Abelian groups it's sum and for sets it's disjoint union. So these things behave in the same way. If you can prove something abstractly about coproducts then it will apply in all these contexts at once, so tensor products of rings is like disjoint unions of sets is like sum of Abelian groups.

Another type of small diagrams,

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

So what is the limit, if it exists? I get $Z \xrightarrow{\varphi} X$ and $Z \xrightarrow{\psi}$ with two conditions $\psi = f \circ \varphi$ and $\psi = g \circ \varphi$. So I have just the data φ and the condition $f \circ \varphi = g \circ \varphi$. I'll say that I factorize through the *equalizer* $\mathrm{Eq}(f, g)$ if this is in sets, that's $\{x \in X, f(x) = g(x)\}$ and then that includes in X. Any time you have an arrow like this and compose you get this condition. And any time you have an approximation it lands inside this subset.

What about for colimits? I want an approximation W with maps $\varphi : Y \to W$ and $\psi : X \to W$. The conditions are $\varphi \circ f = \psi$ and $\varphi \circ g = \psi$. So ψ is useless and we just need $\varphi \circ f = \varphi \circ g$. Sometimes there is a colimit, and here in set theory the colimit is called the *coequalizer*, this is the equalizer in the opposite category. Can you guess what is the coequalizer? It's going to be a quotient, it will be $Y/\sim$, the equivalence relation generated by $y \sim z$ if there exists $x \in X$ with $f(x) = y$ and $g(x) = z$. We know how to compute this.

Let me give this in a context that is much more visible. Let A and B be Abelian groups, and take the two maps 0 and g:

$$A \underset{g}{\overset{0}{\rightrightarrows}} B$$

and then the limit or equalizer is the kernel of g and the colimit or coequalizer is the cokernel of g.

Now I want to go to fiber products. A fiber product is a different kind of diagram, you have

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

The colimit of this is Z with the identity map, this is not very interesting. In the other direction it will be the *fiber product* $X \times_Z Y$, which may or may not exist. In sets it will exist, and there it will be pairs $\{(x, y) : f(x) = g(y)\}$. You can check that this is the correct limit in the category of sets.

Another tye of diagram, I swap the arrows and get

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$$

If this diagram has a limit it's called a *pushout* and is written $X \sqcup_Z Y$, and may not exist. In sets it exists, and is modeled by $X \sqcup Y / \sim$ where $x \sim y$ if there is z in Z with $x = f(z)$ and $y = g(z)$.

In rings, the pushout is the tensor product

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, dotted] \\ C & B \otimes_A C \end{tikzcd}$$

So far I have written down limits and colimits for finite diagrams but we have them for infinite diagrams. The simplest ones will have uninteresting compositions. Consider $X_0 \subset X_1 \subset X_2 \subset \cdots$, say I have an infinite tower of sets like this. What is the limit of this diagram? It's X_0. The colimit will be the union of these. It receives every one and anything that receives every one accepts a map from the union. OWhat if I have $\cdot \subset \cdots \subset X_1 \subset X_0$. The colimit is X_0 and the colimit is the intersection.

These are *filtered*, a *filtered poset* is one where you can always compare two elements with the help of a third element, for any x and y there is z with arrows $x \to z$ and $y \to z$. Field extensions of $\mathbb{Q}$ are something like that, you can add $\sqrt{2}$ or i or both. You can extend from either of the smaller ones to the big one. Filtered diagrams are good.

What is good about filtered posets is that it is easy to compute their colimits in sets. If I have a filtered poset with at most one arrow $x_i \xrightarrow{f_{ij}} x_j$. The formula for the colimit is that you take the disjoint union of the X_i modulo the equivalence relation that $x_i \in X_i$ and $x_j \in X_j$ are equivalent if there exists k and f_{ik} and f_{jk} so that $f_{ik}(x_i) = f_{jk}(x_j)$.

I have to give a basic idea for constructing any limit or colimit. Let me give a definition.

Definition 1.6. A category $\mathcal{C}$ is *complete* if it has all limits. This means that every time I give you a diagram you can find the limit. I mean a diagram of a reasonable size, relative to the universe $\mathbb{U}$. I'll say it's *cocomplete* if it has all colimits.

Theorem 1.1. $\mathbb{U}-\mathrm{Set}$ *is complete and cocomplete.*

There are several different recipes. One is to compute infinite products and equalizers. This is a way to compute limits. In the other direction you need infinite coproducts and coequalizers. On the other hand if you have pushouts and filtered colimits you can define all colimits, a different recipe.

2. March 21: Gabriel Drummond-Cole (Functors and adjunctions)

I do not take notes on my own talks.

3. March 21: Chang-Yeon Chough (Adjoint functor theorem)

(1) statement of the theorem
(2) adjoint functors preserve limits /colimits
(3) accessible categories
(4) presentable categories
(5) compact objects and Freyd's theorem

Adjoint functors are everywhere. If you pick up a book you'll see adjoint functors. These are useful in your real life. I'll give you some criterion for when you can find adjoint functors. My game plan, let me say one more thing, I'll say everything in terms of 1-categories, but if you replace every category with ∞-categories and sets with spaces, some model of spaces, then everything will be the same except on a set of measure zero.

I'll state the theorem without any explanation.

Theorem 3.1 (Adjoint functor theorem)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between* presentable categories *(I'll explain this later).*

(1) *F admits a right adjoint if and only if F preserves all colimits*
(2) *F admits a left adjoint if F is accessible and preserves all limits*

Let me give you a quick application. Gabriel talked about the free group functor. You can construct a free Abelian group on a set, but you can think of the existence of the functor as a consequence of this theorem. The forgetful functor will preserve all limits, and then by the adjoint functor theorem it will admit a left adjoint. If a left adjoint exists, it's essentially unique by the Yoneda lemma. Theoretically, you have a free Abelian functor. Concretely if you'd like to construct the free Abelian group, you can write down a formula. That concrete description is never useful. The definition in your first year as a grad student, the construction is, you have a word, juxtaposition, blah blah blah, you only ever use the universal property of a free group, you never use the construction.

If you're an algebraic geometer, given a Grothendieck topology you can describe a sheaf, and there's a forgetful functor from sheaves to presheaves, that satisfies this theorem and admits a left adjoint called sheafification.

That was the statement of the theorem. Second, this is one of the most important consequences of having adjoint functors. As always, one direction of this theorem is immediate, which is that if F admits a right adjoint, it will preserve all colimits, and if F admits a left adjoint, it will preserve all limits.

Proposition 3.1. *Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction. There are many different notations for adjunctions, this one means that L is left adjoint to R and L goes from $\mathcal{C}$ to $\mathcal{D}$ and R goes from $\mathcal{D}$ to $\mathcal{C}$. Then L preserves all colimits which exist in $\mathcal{C}$ and R preserves all limits that exist in $\mathcal{D}$.*

The proof, what does this mean, you have a family of objects indexed by some category (c_i), with morphisms and compatibilities and so on, so a functor p from I to $\mathcal{C}$, I assume nothing on the index category. Then we can talk about the colimit of p, and that's an object in $\mathcal{C}$. I can apply L and get an object in $\mathcal{D}$. So when I say that L preserves colimits, I can instead take L to my diagram and then have a diagram in $\mathcal{D}$ and I can talk about the colimit of $L \circ p$. So we're comparing $L(\operatorname{colim} c_i)$ and $\operatorname{colim}(Lc_i)$. You have a map from the latter to the former and I claim this is an isomorphism in $\mathcal{D}$. I'll give a formal proof that I like, if you don't like this argument, close your eyes for thirty seconds. I'll implicitly use the Yoneda lemma. To say that these are isomorphic, it's enough to show that maps into an arbitrary guy d are isomorphic. Choose any object D and think about $\mathrm{Hom}_{\mathcal{D}}(L(\operatorname{colim} c_i), D)$ and by adjunction

$$\mathrm{Hom}_{\mathcal{D}}(L(\operatorname{colim} c_i), D) \cong \mathrm{Hom}_{\mathcal{C}}(\operatorname{colim} c_i, RD)$$

which is the same thing as $\lim \mathrm{Hom}_{\mathcal{C}}(c_i, RD)$. Apply adjunction again, by naturality this is

$$\lim \mathrm{Hom}_{\mathcal{C}}(c_i, RD) \cong \lim \mathrm{Hom}_{\mathcal{D}}(Lc_i, D).$$

and then what happens, this is by the universal property of colimits, isomorphic to $\mathrm{Hom}_{\mathcal{D}}(\operatorname{colim} Lc_i, D)$. This is true for every D and so this means they are isomorphic.

What does it mean to have $\operatorname{colim} Lc_i$? You have a map $L(c_i) \to L(c_j)$, all compatible with the colimit. If you have any object compatible with the maps on the top, there could be many morphisms, then you get a filler.

$$\begin{tikzcd}
L(c_i) \arrow[rr] \arrow[dr] \arrow[ddr] & & L(c_j) \arrow[dl] \arrow[ddl] \\
& \operatorname{colim} L(c_i) \arrow[d, dotted] & \\
& D &
\end{tikzcd}$$

now you forget the middle thing and think about adjunction. That amounts by adjunction, the outer guy, to exactly this data:

$$\begin{tikzcd}
c_i \arrow[rr] \arrow[dr] \arrow[ddr] & & c_j \arrow[dl] \arrow[ddl] \\
& \operatorname{colim} c_i \arrow[d, dotted] & \\
& RD &
\end{tikzcd}$$

and when you apply the adjunction to this the middle guy is $L(\operatorname{colim} c_i)$ which has the unique lift. So then you have the same thing, they satisfy the same universal property. That's the easiest proof, not rigorous. For example, one application, with forgetful functors, let's say from vector spaces, R-modules, to sets. That guy satisfies this condition so it admits a left adjoint and is a right adjoint. Think about the Hom tensor adjunction, if you think about the Hom functor that admits a left adjoint, which is the tensor product of modules. That tensor product is a left adjoint, and therefore it preserves all colimits. If you go back to first year grad studies, when you study commutative algebra, if you take direct limit or colimit of M_i and tensor with N, that's the colimit of $(M_i \otimes N)$. You do this by hand for direct limits, but this is a formal property now because tensoring with N is a left adjoint.

Let me skip accessibility. An accessible category, this arises naturally in mathematics. Sets, Abelian groups, modules, are accessible, there are examples of interest. In real life, like for Abelian groups, you may face set theoretical issues. Then accessiblity allows you to avoid set theoretical difficulties.

Let me give you an idea for how this goes, I want to assume I preserve all colimits and show that F admits a right adjoint, so that $\mathrm{Hom}_{\mathcal{D}}(Fc,d) \cong \mathrm{Hom}_{\mathcal{C}}(c,Gd)$ for some G. Having a right adjoint means I want to have such an object, if F admits this adjoint, I should have an object Gd satisfying this condition. So at this point we don't know, but we want to fill in the blank and define Gd. What does that mean, that means, consider a functor $H : \mathcal{C}^{op} \to \mathrm{Set}$ which assigns for each $c \in \mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{D}}(Fc,d)$. If this functor is representable by a certain object, by something in $\mathcal{D}$. I'll focus now on representability of this guy. Then what else can you do? Think about $\tilde{\mathcal{C}}$ which is a category with objects pairs (c,η) where $\eta \in \mathrm{Hom}(Fc,d)$. If you have this new category $\tilde{\mathcal{C}}$ and you have (c,η), you can map this to $\mathcal{C}$ by forgetting η, to make this representable, if this functor were representable, then you'd want to take the colimit of this diagram. You have a diagram in $\mathcal{C}$ indexed by $\tilde{\mathcal{C}}$ and you'd take the colimit of this guy. If you go back and forth between these two constructions, you get the representing object. If you are an algebraic geometer, you see something like this as a prestack and stack. So being representable will mean having a colimit of this diagram. But there's a set theoretical issue because $\tilde{\mathcal{C}}$ is very big. So presentability/accessibility is a set theoretic thing that lets this colimit exist. Accessibility says that $\mathcal{C}$ has a small subcategory $\mathcal{C}_0$, and instead of taking the colimit of $\tilde{C} \to \mathcal{C}$ we take the colimit over the pullback $\mathcal{C}_0 \times_{\mathcal{C}} \tilde{\mathcal{C}}$ which will be small. This is not the end of the world, we have to compare the two needed colimits, and that corresponds to adding the additional condition that turns accessible into presentable. I won't define what an accesible category is, but it is in some sense small because it is controlled by a small category having certain properties. The colimit, accessibility tells you, is large enough, even though it's small.

Any set is a filtered colimit of finite sets, any Abelian group is a filtered colimit of finitely presented Abelian groups, but topological spaces are not accessible.

In five minutes, let me say that accessibility saves your life because it lets you compare colimits. A nice theorem is:

Theorem 3.2. *Let $\mathcal{C}$ be a presentable category. Then a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is representable (this is what we want to get right adjoints) if and only if F preserves all small limits.*

So if you follow the theorem, you only need to check that this functor preserves small limits. This lets you construct an adjoint. The accessibility condition lets you avoid the set theoretical issues, and the construction is completed by the presentability assumption. This is not Freyd's original version. He proved the general and special adjoint functor theorems, in your real life this is going to be the useful one, your categories will be presentable so you will just be able to check these and not something more complicated.

4. March 28: Seongjin Choi: Motivation

[Seminar dinner leaves at 6:10.]

Thank you for coming today. I will talk about loclization of categories with respect to S. The situation is as follows. Let $\mathcal{C}$ be a category and S a subset of the morphisms. Then what we want is a "localization category functor" $\mathcal{C} \to S^{-1}\mathcal{C}$ such that for any category $\mathcal{D}$ and functor $F : \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for $s \in S$, there is a unique functor $S^{-1}\mathcal{C} \to \mathcal{D}$ so that the diagram commutes.

The claim is that there exists such a functor unique up to isomorphism. Uniqueness is not hard so I'll focus on existence. Before constructing the localization category, I'll define notation.

Definition 4.1. A *quiver* is a quadruple (E, V, s, t) where E is a set of edges, V is a set of vertices, and s and t are source and target maps from E to V.

A morphism between $Q = (E, V, s, t)$ and $Q' = (E', V', s', t')$ is a pair $f = (f_e, f_v)$ where f_e is a map $E \to E'$ and f_v a map $V \to V'$ such that the diagrams commute

$$\begin{tikzcd} E \arrow[r, "f_e"] \arrow[d, "s"] & E' \arrow[d, "s'"] \\ V \arrow[r, "f_v"'] & V' \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "f_e"] \arrow[d, "t"] & E' \arrow[d, "t'"] \\ V \arrow[r, "f_v"'] & V' \end{tikzcd}$$

We can think of this, roughly, as a category without identities morphisms or a composition structure. I'll denote Cat as the category of all categories, using Grothendieck universes, and there is a forgetful functor from Cat to Quiv.

There is a functor in the other direction called the free or path functor.

Definition 4.2. Let $Q = (E, V, s, t)$ be a quiver. Then I define the $\mathrm{Pa}\, Q$, the *path category* of Q to have objects V and morphisms between a and b finite paths of directed edges in Q, that is, $(a_n = b, f_n, \ldots, a_1, f_1, a_0 = a)$ with $s(f_i) = a_{i-1}$ and $t(f_i) = a_i$. The composition is concatenation; the identity is the path (a), so

$$\begin{aligned}(a_n, f_n, \ldots, f_1, a_0) &\circ (b_m, g_m, \ldots, b_1, g_1, b_0) \\ &= (a_n, f_n, \ldots, f_1, a_0 = b_m, g_m, \ldots, b_1, g_1, b_0)\end{aligned}$$

when $a_0 = b_m$.

So given the quiver Q I define $\mathrm{Pa}\, Q$ as described, so using this path category I will describe the localization category.

Define a quiver $Q = (E, V, s, t)$ as follows. We are given $\mathcal{C}$ and the set S. Using $\mathcal{C}$, the vertices are the objects of $\mathcal{C}$. The edges are $\mathrm{Hom}\,\mathcal{C} \amalg S$ (call the inclusions i_1 and i_2) and so $s \circ i_1 = s_{\mathcal{C}}$ and $t \circ i_1 = t_{\mathcal{C}}$ whereas $s \circ i_2 = t_{\mathcal{C}}$ and $t \circ i_2 = s_{\mathcal{C}}$. We want to define $S^{-1}\mathcal{C}$ as $\mathrm{Pa}\, Q/\sim$, where I'll explain the equivalence relation $\sim$, the objects are the objects of $\mathrm{Pa}\, Q$ and the morphisms are equivalence classes where we have

- $i_1(v) \circ i_1(u) \sim i_1(v \circ u)$ when $v \circ u$ is defined,
- $i_1(\mathrm{id}_{\mathcal{C}} a) = \mathrm{Id}_{\mathrm{Pa}\, Q}(a)$
- $i_2\sigma \circ i_1\sigma = \mathrm{Id}_{\mathrm{Pa}\, Q}(s_{\mathcal{C}}(\sigma))$
- $i_1\sigma \circ i_2\sigma = \mathrm{Id}_{\mathrm{Pa}\, Q}(t_{\mathcal{C}}(\sigma))$

So for example, let $\mathcal{C}$ be the category $\bullet \xrightarrow{f} \bullet$ with two objects and one arrow f and $S = \{f\}$. Then first I construct a quiver

$$\begin{tikzcd} a \arrow[loop left] \arrow[r, shift left, "i_1 f"] & b \arrow[l, shift left, "i_2 f"] \arrow[loop right] \end{tikzcd}$$

Next I construct $\mathrm{Pa}\, Q$ modulo the equivalence relations. Then I get only one morphism from a to b and likewise from b to a.

Let me introduce some examples. We want to construct a localization category adding some invertible morphisms.

- For S a set of isomorphisms, then we have $S^{-1}\mathcal{C} = \mathcal{C}$ and $\ell = \mathrm{id}_{\mathcal{C}}$.
- For $S = \mathrm{Hom}\,\mathcal{C}$, then $S^{-1}\mathcal{C}$ is the groupoid completion of $\mathcal{C}$.
- For A an Abelian category (where the morphisms between two objects are an Abelian group and composition is bilinear, you have finite direct sums and products, and every morphism has a kernel and cokernel, and the natural map $\operatorname{coker}\ker f \to \ker\operatorname{coker}$ is an isomorphism), such as modules over R, but not Set. Given an Abelian category, define $C(A)$ as the category of complexes in A, that is, sets $\{X_n, d^n\}$ for $n \in \mathbb{Z}$ with $d^n : X^n \to X^{n+1}$ and $X^n \in \mathrm{Ob}(A)$ with $d^{n+1} \circ d^n = 0$, with morphisms between two complexes a map from X^n to Y^n for each n commuting with d^n and d^{n+1}. Then you can define a functor $H^n : C(A) \to A$ by $\{X^n, d^n\} \to \ker d^n / \mathfrak{I} d^{n+1}$. Define S as the set of morphisms in $C(A)$ which become isomorphisms under these functors for all n. These are called *quasi-isomorphisms.* Then I'll call the localization category at S the *derived category* of A.

5. March 28: Morimishi Kawasaki: Motivation

So I'll talk about bad behavior of the localization. We want to claim that the derived category is not an Abelian category. $C(R)$, the category of chain complexes of R-modules is an Abelian category, but $D(\mathbb{Z})$ is not an Abelian category. We should say that $D(\mathbb{Z})$ has some good structure, it's an *additive category.*

Definition 5.1. $\mathcal{C}$ is an *additive category* if it satisfies the same conditions as an Abelian category except the existence of kernel and cokernels. It satisfies additivity, so that morphisms are Abelian groups and composition is bilinear, it has a zero object, and existence of finite direct sums (and products).

Definition 5.2. Suppose that for any X and Y in the objects of $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is an Abelian group. Then there is a *zero map* and we call Z a *zero object* if $0_{ZZ} = \mathrm{id}_Z$.

Proposition 5.1. *$D(R)$ is an additive category.*

Definition 5.3. Let $S \subset \mathrm{Hom}_{\mathcal{C}}$, then S is a *multiplicative system* if for all X in $\mathrm{Ob}\,\mathcal{C}$,

(1) id_X is in S and
(2) S is closed under composition,
(3) and for any f in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ and any morphism in S from Y' to Y, then there exists a morphism g in $\mathrm{Hom}_{\mathcal{C}}(X',Y')$ and a t in $\mathrm{Hom}_{\mathcal{C}}(X',X)$ such that the diagram commutes

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "t"'] & Y' \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and similarly for any g and t there exists f and s
(4) and for any f and g in $\mathrm{Hom}_{\mathcal{C}}(X,Y)$¡ there exists $s \in S$ such that $s \circ f = s \circ g$ if and only if ther exists t in S such that $f \circ t = g \circ t$.

Proposition 5.2. *the set of quasi-isomorphisms is a multiplicative system.*

I won't prove this statement.

Lemma 5.1. *Let $\mathcal{C}$ be an additive category and S a multiplicative system. Then the localization is also an Abelian category.*

Then Proposition 5.2 and Lemma 5.1 directly prove Proposition 5.2.

I'll sketch a proof of the lemma, only explaining what is the Abelian structure of $\mathrm{Hom}_{s^{-1}\mathcal{C}}(X,Y)$. By definition of the localization, you can write a morphism as $f \circ s^{-1}$ and another as $g \circ t^{-1}$. Fix $\hat{f}$ and $\hat{g}$ in $\mathrm{Hom}_{S^{-1}\mathcal{C}}(X,Y)$, then represent these by $[(f,s)]$ and $[(g,t)]$ and s and t are in S and f and g in $\mathrm{Hom}_{\mathcal{C}}$. Then define $\hat{f}+\hat{g}$ as $[(f\circ h + g\circ h', u)]$ where u will be in S and h and h' are in $\mathrm{Hom}_{\mathcal{C}}$. By definition of the multiplicative system, write

$$\begin{tikzcd}
\widetilde{X} \arrow[r, dotted, "h'"] \arrow[d, dotted] & {}^{h}X_1 \arrow[d] \\
X_2 \arrow[r] & X
\end{tikzcd}$$

So now I want to argue that the derived category is not Abelian. First, recall that $[\mathbb{Z}/2, \mathbb{Z}/2[1]] \cong \mathrm{Ext}^1(\mathbb{Z}/2,\mathbb{Z}/2)$ but these are objects in $C(\mathbb{Z})$. Here $\mathbb{Z}/2[1]$ has $\mathbb{Z}/2$ in the -1 position. Then $[X,Y]$ is $\mathrm{Hom}_{D(R)}(X,Y)$. Take P, which is the complex $\mathbb{Z}\to\mathbb{Z}$, a projective resolution of $\mathbb{Z}/2$. Then $\mathrm{Ext}^1(\mathbb{Z}/2,\mathbb{Z}/2) = \{e, 0\}$ where $e(n) = [n] \pmod{()2}$. So we see that $[\mathbb{Z}/2,\mathbb{Z}/2[1]] \neq 0$, where this is $\mathbb{Z}/2 \leftarrow P \rightarrow \mathbb{Z}/2[1]$. This is an explict representation of $\hat{e}$.

Now we're ready for the proof. Assume $D(\mathbb{Z})$ is Abelian. Then, we'll show that $[\mathbb{Z}/2,\mathbb{Z}/2[1]] = 0$. Then this shows that $D(\mathbb{Z})$ is not Abelian. So let us show that if $D(\mathbb{Z})$ is Abelian then $[\mathbb{Z}/2,\mathbb{Z}/2[1]]$ is 0.

Take $f \in [\mathbb{Z}/2,\mathbb{Z}/2[1]]$ and assume $D(\mathbb{Z})$ is Abelian. There exists the kernel of $f = (X,\iota)$ with $\iota : X \to \mathbb{Z}/2$. Since $D(\mathbb{Z})$ is Abelian, $0 \to X \to \mathbb{Z}/2 \to \mathbb{Z}/2[1]$ is an exact sequence in $D(\mathbb{Z})$. Then using the definition of localization, we can write

$$\begin{tikzcd}
 & & & W \arrow[dl, "u"'] \arrow[dr, "I_W"] & & \\
 & & Z \arrow[dl, "t"'] \arrow[dr, "I_Z"] & & Y \arrow[dr, "s"] & I_Y \arrow[dl] \\
0 & X \arrow[rr, "\iota"'] & & \mathbb{Z}/2 \arrow[rr, "f"'] & {} & \mathbb{Z}/2
\end{tikzcd}$$

where the downward arrows to the right are in $\mathrm{Hom}_{C(\mathbb{Z})}$, the downward left arrows are quasi-isomorphisms, and the horizontal arrows are in the derived category.

The third axiom of multiplicative systems gives us this diagram. Then exactness of $0 \to X \to \mathbb{Z}/2 \to \mathbb{Z}/2[1]$ implies exactness of the diagram $0 \to W \to Y \to \mathbb{Z}/2[1]$ in $C(\mathbb{Z})$. Then by definition of quasi-isomorphism, there is an isomorphism $H^i(W) \cong H^i(X)$ and $H^i(Y) \cong H^i(\mathbb{Z}/2)$ and so then we get an exact sequence at the level of homology $H^i(X) \to H^i(\mathbb{Z}/2) \to H^{i+1}(\mathbb{Z}/2)$, so I_W^* is surjective. I want to argue that I_W^* is an isomorphism so that ι is an isomorphism in $D(\mathbb{Z})$. Then by exactness, you'd say that f is zero.

So to prove that $X \to \mathbb{Z}/2$ is a quasi-isomorphism, you use the fact that $H_n(C)$ is $[Z, C[n]]$, and now you have $0 \to X \to \mathbb{Z}/2 \to \mathbb{Z}/2[1]$, and you can shift to get

$X[n] \to \mathbb{Z}/2[n] \to \mathbb{Z}/2[n+1]$, and then we'll have

$$[\mathbb{Z}, 0] \to [\mathbb{Z}, X[n]] \to [\mathbb{Z}, \mathbb{Z}/2[n]] \to [\mathbb{Z}, \mathbb{Z}/2[n+1]]$$

which is

$$0 \to H_n(X) \to H_n(\mathbb{Z}/2) \to H_{n+1}(\mathbb{Z}/2)$$

and the last map here is 0 and so $H_n(X) \to H_n(\mathbb{Z}/2)$ is an isomorphism, so $X \to \mathbb{Z}/2$ is a quasi-isomorphism, so f is a zero map, a contradiction.

6. April 4: Yoosik Kim: DG categories

Last time we saw the definition of the derived category of an Abelian category, first passing to the homotopy category and then inverting the quasi-isomorphisms. This is not an Abelian category in general, but it's triangulated, it has a shift functor and some distinguished triangles to generate long exact sequences in homology. This triangulated category is not the best structure, so what we're going to do is to enhance this category using the notion of dg categories. My mission is to define dg categories and give some typical examples to get familiar with that.

So let $\mathbf{k}$ be a commutative ring, you can think of it as the ring of integers or a field if you prefer linear algebra.

Definition 6.1. A $\mathbf{k}$-linear category $\mathcal{A}$ is called a *dg (differential graded) category* if the morphism spaces are dg $\mathbf{k}$-modules and the compositions and units are morphisms of dg $\mathbf{k}$-modules.

I probably need to explain some terminology here. Recall that a category $\mathcal{A}$ is said to be $\mathbf{k}$-*linear* if the morphism spaces are $\mathbf{k}$-modules, that is, $\mathrm{Hom}_{\mathcal{A}}(X,Y)$ is a $\mathbf{k}$-module and the composition $\mathrm{Hom}_A(Y,Z) \otimes \mathrm{Hom}_A(X,Y) \to \mathrm{Hom}_A(X,Z)$ is a $\mathbf{k}$-module homomorphism. A *dg* $\mathbf{k}$-*module* V is

(1) $V = \bigoplus V^p$
(2) d_V a differential, $d_V(V^p) \subset V^{p+1}$.

A *morphism* $f : V \to W$ of dg $\mathbf{k}$-modules of degree n has $f(V^p) \subset W^{p+n}$ and $d_W \circ f = (-1)^n f \circ d_V$

The tensor product $V \otimes W$ of graded $\mathbf{k}$-modules has

(1)
$$(V \otimes W)^p = \bigoplus_{i+j=p} (V^i \otimes W^j)$$
(2) $d_{V\otimes W} = d_W \otimes \mathrm{id}_W + \mathrm{id}_V \otimes d_W$.

The tensor product of morphisms of graded $\mathbf{k}$-modules, $f : V \to V'$ and $g : W \to W'$, this is defined by the Koszul sign rule, $(f \otimes g)(v \otimes w) \coloneqq (-1)^{|g||v|} f(v) \otimes g(w)$.

So a first example of a dg category is a dg algebra, a graded algebra over $\mathbf{k}$, so it has a grading, a multiplication, this is a dg category with one object, $d : A \to A$ satisfies the graded Leibniz rule $d(a \cdot a') = da \cdot a' + (-1)^{|a|} a da'$.

Any dg algebra is a dg category with a single object. Conversely, a dg category with a single object can be viewed as a dg algebra. Take $\{*\}$ as the objects and $\mathrm{Hom}_{\mathcal{A}}(*,*) = A$, with composition from multiplication.

From this convention, you view this as a *dg* algebra, and the graded Leibniz rule crmes from our convention,

$$d(a \cdot a') = \mu\left[(d \otimes \mathrm{id}_A + \mathrm{id}_A \otimes d_A)(a \otimes a')\right] = da \cdot a' + (-1)^{|a|} a \cdot da'$$

The most important example, let $\mathcal{A}$ be a $\mathbf{k}$-linear Abelian category. Then $\mathcal{C}(A)$ the category of chain complexes in $\mathcal{A}$. Then this can be made into a dg category as follows. The objects and composition are the same as in $\mathcal{C}(A)$ and the morphisms for chain complexes C and D, the gradings are $\mathrm{Hom}_{\mathcal{C}dg}(C,D) = \bigoplus_p \mathrm{Hom}_{\mathcal{C}dg}(C,D)^p$ where $\mathrm{Hom}_{\mathcal{C}dg}(C,D)^p$ is $\prod_i \mathrm{Hom}_{\mathcal{A}}(C^i, D^{i+p})$.

Now I have to define the differential, so d^p takes f^i to $d_D^{i+p} \circ f^i + (-1)^{p+1} f^{i+1} \circ d_C^i$.

So if I draw one piece this looks like

$$\begin{tikzcd} C^i \arrow[r, "d_C"] \arrow[d, "f^i"] & C^{i+1} \arrow[d, "f^{i+1}"] \\ D^i \arrow[r, "d_D"'] & D^{i+1} \end{tikzcd}$$

so that d measures the failure of commutativity of this diagram.

That's the primary example of a dg category. You can take Z^0, objects are the same as in $\mathcal{C}dg(\mathcal{A})$. The morphisms are the kernel, they're the kernel of d^0. Then $H^0(\mathcal{C}dg(\mathcal{A}))$ has the same objects, and the morphisms are $\ker d^0/\mathrm{im} d^0$, which are chain maps modulo homotopy.

For a later purpose you want the opposite category, if $\mathcal{A}$ is a dg category with d and $\circ$ then the opposite category $\mathcal{A}^{\mathrm{op}}$ consists of the following data, the objects are the same as in $\mathcal{A}$ and the morphisms are opposite; the differential $d^{\mathrm{op}}(X,Y) = d(Y,X)$ and the composition $g \circ^{\mathrm{op}} f$ is defined as $(-1)^{|g||f|} f \circ g$.

Now let me introduce A_∞ categories, which is a cousin of a dg category.

Definition 6.2. A (unital) A_∞ category $\mathcal{A}$ consists of objects, and morphisms $\mathrm{Hom}_A(X,Y)$ a $\mathbb{Z}$ or $\mathbb{Z}/2$-graded $\mathbf{k}$module and it comes with a composition, structure maps m^d which goes

$$\mathrm{Hom}_{\mathcal{A}}(X_0, X_1) \otimes \cdots \otimes Hom_{\mathcal{A}}(X_{d-1}, X_d) \to \mathrm{Hom}_{\mathcal{A}}(X_0 \otimes X_d)$$

which is a multi-$\mathbf{k}$-linear map of degree $2-d$ satisfying the relations

$$\sum_{k_1+k_2=k+1} \sum (-1)^{|x_1|+\cdots+|x_{i-1}|+i-1} m_{k_1}(x_1, \ldots, mk_2(x_i, \ldots, x_{i+k-1}), \cdots, x_k) = 0.$$

How do these start? The first few are

$$m^1 \circ m^1 = 0$$

and

$$m^1(m_2(x_0, x_1)) + m^2(m^1(x_0), x_1) + (-1)^{|x_0|+1} m^2(x_0, m^1(x_1)) = 0.$$

Then there is the unit, which is that $m^k(\cdots, e_x, \cdots, 0)$ is zero, and that

$$m^2(e,x) = x = (-1)^{|x|} m^2(x,e).$$

Some remarks, in general this is not a category. If $m^k = 0$ for $k \geq 3$ this is a dg category.

Let me finish my talk by saing why these come into play in the ∞ category setting. We want to measure some A_∞ algebra from the algebro-geometric side.

To that purpose, we should enlarge the A_∞ category. We have structures in algebraic geometry, $\oplus, [1], \otimes$, *cone*.

So what do we do? We think about the Yoneda embedding.

7. April 4: Tae-Su Kim: morphisms of DG categories

I'm going to talk about morphisms of dg categories. These are topics in Bertrand Toën's lecture notes. I'll also talk about the homotopy category and about quasi-equivalences and the homotopy category of dg categories.

I'll give the definitions and examples and no theorems. So k will be a commutative unital ring.

Definition 7.1. A *morphism* of dg categories, or maybe I should call it a dg *functor* between dg categories T and T' is a functor in the usual sense such that f, the map between morphism spaces, $T(x,y) \to T'(f(x), f(y))$, is required to be a chain map.

Let me give an example. Fix an object x of T and define a functor from $T \to \mathrm{Ch}(k)$, the chain complexes of $\mathbf{k}$-modules. On objects it, takes y to $T(x,y)$. For morphisms it should take $T(y,z) \to \mathrm{Hom}_{\mathrm{Ch}(k)}(T(x,y), T(x,z))$ and here the definition is obvious, it takes α to ϕ_α which takes β to $\beta \circ \alpha$.

We can check that $d\alpha$ goes to $\phi_{d\alpha}$ which maps β to $\beta \circ d\alpha$. As Yoosik said, $d\phi_\alpha(\beta) = \pm\phi_\alpha d\beta \pm d(\phi_\alpha)\beta$ and these are the same by the compatibility condition for the differential with composition.

Let me give another example. Let R asd S be $\mathbf{k}$-algebras, unital associative. Let f be a $\mathbf{k}$-algebra morphisms from R to S.

Then we can define two functors $f^* : C(R) \to C(S)$ which are $(S \otimes_R --)$ and f_* from $C(S) \to C(R)$, extension and restriction of scalars.

This might be a digression but let me tell you about the product between two dg categories, the tensor product. Before doing that let me mention dgCat, which has objects dg categories and morphisms dg functors. This forms a category.

Now let me take the tensor product of T and T', this will be a dg category. The objects are pairs of objects, one in T and one in T'. The morphisms $\mathrm{Hom}_{T\otimes T'}(x \otimes x', y \otimes y')$ is $T(x,x') \otimes T'(y,y')$.

My last example is a functor $T \otimes T^{\mathrm{op}}$ to $\mathrm{Ch}(\mathbf{k})$, and the objects (x,y) goes to $\mathrm{Hom}_T(y,x)$ and $\mathrm{Hom}_{T\otimes T^{\mathrm{op}}}((x,y),(x',y'))$ goes from $\mathrm{Ch}(\mathbf{k}(T(y,x), T(y',x')))$ and this takes γ to $\beta \circ \gamma \circ \alpha$.

Let me move onto the second topic, the homotopy category of a dg category. We can define a linear category $[T]$ and Yoosik gave a definiiton, the objects are the same and the morphisms are $H^0(T(X,Y))$. This is the "*homotopy category of a dg category*" and this is a functor in the usual sense from $\mathrm{dg\,Cat} \to \mathrm{Cat}$ which takes $(T \to T') \mapsto ([T] \to [T'])$.

One issue in the homotopy category is composition $H^0(T(x,y)) \times H^0(T(y,z)) \to H^0(T(y,z))$. [some discussion].

For C a linear category, then the homotopy category $[C]$ concentreted on the zero level and is isomorphic to C itself.

Our goal is to study localization but to do this we need to specify the subset of morphisms spaces at which to localize, $S \subset \mathrm{Hom\,dg\,Cat})$. The answer is quasi-equivalences?

What is a quasi-equivalence?

Definition 7.2. A *quasi-equivalence* is a functor $T \to T$" such that

(1) $T(x,y) \to T'(f(x), f(y))$ is a quasi-isomorphism
(2) $[f] : [T] \to [T']$ is essentially surjective.

So on the homology level it's an equivalence of categories.

Definition 7.3. The *homotopy category* of dg categories, $\mathrm{Ho}\,\mathrm{dg}\,\mathrm{Cat}$, is the localization of $\mathrm{dg}\,\mathrm{Cat}$ at quasi-equivalences. The homotopy category $\mathrm{Ho}\,\mathrm{Cat}$ is the localization of Cat along equivalences.

As I said, $[\quad]$ is from $\mathrm{dg}\,\mathrm{Cat}$ to Cat and sends quasi-equivalence to equivalence. As we, in our talk, last week, we talked about localization, and there's a universality property, and by it you have a unique functor $\mathrm{Ho}\,\mathrm{dg}\,\mathrm{Cat} \to \mathrm{Ho}\,\mathrm{Cat}$.

Let me give an example you can see in the lecture notes. Consider a dg category T which satisfies $H^i(T(x,y)) = 0$ unless $i = 0$. We claim that T and $[T]$ are isomorphic in $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat})$. There is a "roof" T' above both of these. What is T'? This is a dg category whose objects are the objects of T. The morphisms from x to y is

$$\begin{cases} Z^0T(x,y) & i = 0 \\ T^i(x,y) & i < 0 \\ 0 & i > 0. \end{cases}$$

What are the quasi-equivalences? so $f_1 : T_{\leq 0} \to T$ and $f_2 : T_{\leq 0} \to [T]$. So f_1 is the identity on objects and inclusion on morphisms. Then f_2 is the identity on objects and projection on morphisms. The maps on morphisms are then quasi-isomorphisms, so that these are quasi-equivalences. Essential surjectivity is free.

8. April 11: Wanmin Liu: Model categories I

In Chinese philosophy, there are four levels of understanding. The first level is that we don't know that we don't know something. Level three is that we know that we don't know something. Before this seminar, I didn't know the word "model category." Maybe after some hard work, we don't know we know something, this is level two, and then we know we know something after fully studying it.

So today we want to talk about model category theory, introduced by Quillen many years ago, in the late 1960s, and this one provides a general setting to study the homotopy category, construct the basic machinery of homotopy theory. The motivation for myself is this so-called fundamental result, which I'll write here. This is given by:

Theorem 8.1. *(Tabuada 2005) Let k be a commutative unital ring. Then the category of dg categories over k (let's recall this: the objects are dg categories, and morphisms are dg functors) admits the structure of a cofibrantly generated model category where the weak equivalences are quasi-equivalences.*

So my goal in this talk is to give the definition of model categories via weak factorization systems. I'll also give some very basic properties. In the next hour we'll have many examples.

So today I will have many definitions, but they're not so hard.

8.1. Weak factorization systems (WFS).

Fix a category (a small category) $\mathcal{C}$.

Definition 8.1. Let $\iota : A \to X$ and $\pi : E \to B$ be morphisms. Suppose we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\iota"'] & E \arrow[d, "\pi"] \\ X \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

and there is a lift from $X \to E$ such that the two triangles commute. Then we say that ι is a left lifting with respect to π and that π is a right lifting with respect to ι, and denote this $\iota \boxslash \pi$.

Then we also define the collection, let $\mathcal{L}$ and $\mathcal{R}$ be two classes of morphisms in $\mathcal{C}$, we have two classes. We can define $\mathcal{L}^{\boxslash}$ is the collection of all π such that $\iota \boxslash \pi$ for all ι in $\mathcal{L}$. Similarly, ${}^{\boxslash}\mathcal{R}$ is the collection of all ι such that $\iota \boxslash \pi$ for all π in $\mathcal{R}$.

Definition 8.2. A weak factorization is a pair $(\mathcal{L}, \mathcal{R})$, such that

(1) (factorization) any morphism f can be written as a composition $\iota \cdot \pi = \pi \circ \iota$ for some ι in $\mathcal{L}$ and π in $\mathcal{R}$.
(2) (closure) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{L}^{\boxslash} = \mathcal{R}$.

Maybe I need one example for you. You can take $\mathcal{C}$ to be the category of sets. Then $\mathcal{L}$ could be injective functions and $\mathcal{R}$ could be all surjective functions.

A morphism f is a retract of a morphism g if we have the following diagram:

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & \cdot \arrow[r] \arrow[d, "g"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Lemma 8.1. *$\mathcal{L}^{\boxslash}$ and ${}^{\boxslash}\mathcal{R}$ are closed under taking retracts.*

Let's just prove one of these. suppose $\pi : E \to B$ is in $\mathcal{L}^{\boxtimes}$ and f is a retract of it. Then since ι is in $\mathcal{L}$ and π is in its orthogonal we get a lifting

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\iota"] & \cdot \arrow[r] \arrow[d, "f"] \arrow[rr, bend left] & \cdot \arrow[r] \arrow[d, "\pi"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r] \arrow[urr, dotted] & \cdot \arrow[r] \arrow[rr, bend right] & \cdot \arrow[r] & \cdot
\end{tikzcd}$$

Let me give a second definition, equivalent.

Definition 8.3. A pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ is a WFS if

(1) for any morphism f there is a factorization $f = \iota \cdot \pi$ with $\iota \in \mathcal{L}$ and $\pi \in \mathcal{R}$.
(2) $\mathcal{L} \boxslash \mathcal{R}$
(3) $\mathcal{L}$ and $\mathcal{R}$ are closed under retracts.

Maybe it's obvious that the other definition implies this one. But how do we see that $R \supset \mathcal{L}^{\boxslash}$? So let π be in $\mathcal{L}^{\boxslash}$. Then we can factorize $\pi = i \cdot p$. Then we have

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i"] & X \arrow[d, "\pi"] \\
Y \arrow[r, "p"'] \arrow[ur, dotted, "h"] & Z
\end{tikzcd}$$

So we can rewrite this as

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "\pi"] \arrow[rr, bend left, "\mathrm{id}"] & Y \arrow[r, "h"] \arrow[d, "p"] & X \arrow[d, "\pi"] \\
Z \arrow[r, "\mathrm{id}"] \arrow[rr, bend right, "\mathrm{id}"'] & Z \arrow[r, "\mathrm{id}"] & Z
\end{tikzcd}$$

and so since $p \in R$ and R is closed under retracts, we conclude that π is in R.

8.2. **Model categories.**

Definition 8.4. A model cotegory M is a category with

1. all small limits and colimits exist, Damien talked about these,
2. equipped with three classes of morphisms, (W, C, F), called weak equivalences and denoted $\sim$, *cofibrations* (denoted $\rightarrowtail$), and *fibrations* (denoted $\twoheadrightarrow$) such that
   (a) (two out of three) If $f : X \to Y$ and $g : Y \to Z$ then if two of $\{f, g, f \circ g\}$ is a weak equivalence, so is the third.
   (b) $(W \cap C, F)$ is a WFS.
   (c) $(C, W \cap F)$ is a WFS.

What is the meaning of this here? It's better to give an equivalent definition.

Proposition 8.1. *Let M be a category with*

1. *M has all small limits and colimits*
2. *M has three classes of maps (W, C, F) so that*
   (a) *W satisfies the two out of three property,*
   (b) *W, C, and F are closed under taking retracts,*
   (c) *Given a diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\iota"'] & E \arrow[d, "\pi"] \\ X \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

   the dotted arrow exists if ι is in C and π is in F and at least one is in W.
   (d) *There exist factorizations of every morphism f into a cofibration followed by a fibration $\iota \cdot \pi$ where either one can be chosen to be* acyclic *(in W).*

(From the very beginning, the lifting is not unique.)

8.3. **Basic properties.**

Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system. Then $\mathcal{L}$ and $\mathcal{R}$ are closed under composition. To prove this for $\mathcal{R}$, take f and g in $\mathcal{R}$. To show that $f \cdot g$ is also in $\mathcal{R}$, it's just a diagram chase,

$$\begin{tikzcd} A \arrow[r] \arrow[dd, "h"] & B \arrow[d, "f"] \\ & C \arrow[d, "g"] \\ E \arrow[r] & D \end{tikzcd}$$

with h in $\mathcal{L}$. But we have a lift

$$\begin{tikzcd} A \arrow[r] \arrow[d, "h"] & C \arrow[d, "g"] \\ E \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

and then we have a lift

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "h"] & B \arrow[d, "f"] \\
E \arrow[r] \arrow[ur, dotted] & C
\end{tikzcd}$$

Another lemma is that $\mathcal{L}$ is closed under taking pushouts and $\mathcal{R}$ is closed under taking pullbacks. This means that if you have a pushout diagram

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d] & & h \\
Y \arrow[r, "i"'] & Z
\end{tikzcd}$$

if g is in $\mathcal{L}$ then h is in $\mathcal{L}$. If this is a a pullback and h is in $\mathcal{R}$ then g is in $\mathcal{R}$.

Let me prove this for $\mathcal{R}$. We know that h is in $\mathcal{R}$. We want to show g is $\mathcal{R}$. We'll show that it has the left lifting property. We are given a diagram

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "j"] & W \arrow[d, "g"] \\
B \arrow[r] & Y
\end{tikzcd}$$

with g in $\mathcal{L}$; we want a lift $B \to W$. So we will use the properties of pullbacks. Because of this diagram, we could draw another diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & X \arrow[d, "h"] \\
B \arrow[r] \arrow[urr, dotted] & Y \arrow[r] & Z
\end{tikzcd}$$

and we have a lift ℓ from B to X as indicated, which gives us

$$\begin{tikzcd}
& X \arrow[d] \arrow[dr] & \\
B \arrow[ur] \arrow[r, dotted, "p"] \arrow[dr] & W \arrow[d] & Z \\
& Y \arrow[ur] &
\end{tikzcd}$$

by the property of the pushout. We need to see that $k = j \cdot p$. If I compose the diagram with j we get a diagram

$$\begin{tikzcd}
& & X \\
A \arrow[urr, "k\cdot f"] \arrow[r, "j"] \arrow[drr, "k\cdot g"'] & B \arrow[r] \arrow[ur] \arrow[dr] & W \arrow[u] \arrow[d] \\
& & Y
\end{tikzcd}$$

then k is $j \cdot p$ by uniqueness of the map to the pushout

Let me give the last lemma.

Lemma 8.2. *$\mathcal{L}$ is closed under colimits and $\mathcal{R}$ is closed under limits.*

Proof. Let's do it for $\mathcal{L}$. Consider the best approximation from the right $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_\omega$.

Fix a morphism $\pi : E \to B$. We want a lifting

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ X_\omega \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

But this means we want a map $X_n \to E$ with compatibilities. In the $n = 0$ level we have X_0 to E. Then by induction, suppose we have it for X_n. Then because f_n is in $\mathcal{L}$ and π in $\mathcal{R}$ we have a lift

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"] & E \arrow[d, "\pi"] \\ X_{n+1} \arrow[r] \arrow[ur, dotted] & B \end{tikzcd}$$

□

9. April 11: Cheolgyu Lee: Model categories I

So you have seen the definition, so now we will give one hour of just examples.

Let $\mathcal{C}$ be a category with objects non-negatively graded chain complexes on Mod_R, for R a ring with identity and morphisms chain maps. Then we can define weak equivalences to be quasi-isomorphisms, cofibrations to be chain maps which is injective in each degree with projective cokernels. Let fibrations be surjections in positive degrees. Then we can check that for any $0 \to M_*$, then there is a projective resolution $0 \xrightarrow{\iota} P \xrightarrow{\pi} M$, a projective resolution. We can easily see that π is in $F \cap W$, it's an acyclic fibration and ι is a cofibration.

Suppose we're given a chain map $N_* \to M_*$, then we have a complex $N_* \oplus P_*$, and a factorization $N \to N \oplus P \to M$. So now we have P sitting inside $N \oplus P$ and also P projecting to M. We can define a homotopy category $\mathrm{Ho}\,\mathcal{C}$, where we invert weak equivalences, but this constructs a quiver, arrows in $\mathcal{C}$ where the arrows in W are inverted, this has index set objects in $\mathcal{C}$ and we need a bigger universe to construct it. With the structure of a model category, we can define the homotopy relation in morphisms in $\mathcal{C}$. Let $\mathcal{C}_{cf}$ be the full subcategory of objects that are both cofibrant and fibrant.

I need to define fibrant and cofibrant. We call X *cofibrant* if the unique morphism from the initial object is a cofibration and *fibrant* if the unique morphism to the final object is a fibration. I want to assume the existence of two functors Q and R which are called cofibrant replacement and fibrant replacement functors. I might need some version of the axiom of choice to define it. This is a functor from $\mathcal{C}$ to itself so that QX is a cofibrant replacement and RX is a fibrant replacement of X.

I want to explain a first lemma, Ken Brown's lemma.

Lemma 9.1. *Let $\mathcal{C}$ be a model category with structure (W, C, F) and $\mathcal{D}$ a category, not necessarily a model category, with some class of weak equivalences satisfying the two out of three property.*

Suppose a functor G sends acyclic cofibrations between cofibrant objects to weak equivalences. Then G sends weak equivalences between cofibrant objects to weak equivalences.

There is also a dual version that I won't state.

Proof. Let $f : A \to B$ be a cofibration between cofibrant objects. Then I can factorize $A \sqcup B \to B$ into a cofibration b followed by an acyclic fibration a. Then the identity is a weak equivalence. Because A and B are cofibrant, then c_1 and c_2 are cofibrations, then because pushout preserves cofibrations, then the inclusions of A and B into $A \sqcup B$ are cofibrations. Then $b \circ c_1'$ and $b \circ c_2'$ are both acyclic cofibrations. Then $F(b \circ c_1')$ and $F(b \circ c_2')$ are weak equivalences. So $F(a)$ is in W' by the two out of three property. Then $F(a) \circ F(b \circ c_2')$ is in W'; that's the same as $F(f)$. $\square$

Let me give a definition. A *cylinder object* $\mathrm{Cyl}(X)$ is an object of C such that

$$X \sqcup X \xrightarrow{c \in C} \mathrm{Cyl}(X) \xrightarrow{w \in W} X$$

where the composition is $\mathrm{id} \sqcup \mathrm{id}$ and a *path object* is dual:

$$Y \xrightarrow{w \in W} \mathrm{Path}(Y) \xrightarrow{f \in F} Y \times Y$$

where the composition is the diagonal $\mathrm{id} \times \mathrm{id}$.

A *left homotopy* from f to g is a map $H : \mathrm{Cyl}(X) \to Y$ satisfying that $Hi_0 = f$ and $Hi_1 = g$ where $i_0 \sqcup i_1$ is the map from $X \sqcup X \to \mathrm{Cyl}(X)$.

If there is a left homotopy from f to g then we write $f \overset{\ell}{\sim} g$.

If objects are cofibrant and fibrant then left homotopy implies right homotopy and vice versa. We say that $f \sim g$ if f is both right and left homotopic to g. We say that f is a homotopy equivalence if there is a "homotopy inverse" so that both compositions are homotopic to the respective identities.

From now on I will construct an equivalence relation on the space of morphisms between two objects. We can check the following

Lemma 9.2. *$f \overset{\ell}{\sim} g$ implies that $h \circ f \overset{\ell}{\sim} h \circ g$ and $f \overset{r}{g}$ implies that $f \circ h \overset{r}{\sim} g \circ h$.*

Lemma 9.3. *If Y is fibrant then $f \overset{\ell}{g}$ implies $f \circ h \overset{\ell}{\sim} g \circ h$. Dually if X is cofibrant and $f \overset{r}{\sim} g$ then $h \circ f \overset{r}{\sim} h \circ g$.*

Let me give a partial proof. So you have $X \sqcup X \to \mathrm{Cyl}(X) \to X$ and $H : \mathrm{Cyl}(X) \to Y$. We can assume that w is an acyclic fibration because Y is fibrant, you can form a lift

$$\begin{tikzcd}
\mathrm{Cyl}(X) \arrow[rr] \arrow[dd] \arrow[dr, dotted] & & Y \arrow[dd] \\
& \mathrm{Cyl}'(X) \arrow[ur, dotted] \arrow[dl, dotted] & \\
X \arrow[rr] & & *
\end{tikzcd}$$

Then we can use this to lift.

I didn't prove that this is an equivalence relation. I didn't prove transitivity.

Lemma 9.4. *If X is cofibrant then $\overset{\ell}{\sim}$ is an equivalence relation.*

It suffices to show transitivity. So assume $f \overset{\ell}{\sim} g$ and $g \overset{\ell}{\sim} h$. So we have $\mathrm{Cyl}(X)_0 \to Y$ and $\mathrm{Cyl}(X)_1 \to Y$. Then we can take a colimit of $\mathrm{Cyl}(X)_0 \sqcup \mathrm{Cyl}(X)_1$ over X with respect to i_0' and i_1. We have a map to X and both maps are weak equivalences. Since X is cofibrant, the maps i and i' are cofibrations, in fact acyclic cofibrations (by an easy diagram chase).

Then i_0 and i_1 are trivial cofibrations. We can't deduce that $X \sqcup X \to Z$ is a cofibration, but if we factorize this into a cofibration followed by a trivial fibration, then we get a cylinder obect K, as desired.

So we get a homotopy relation.

Lemma 9.5. *Suppose X is cofibrant and Y and Z are fibrant. Then a weak equivalence $Y \to Z$ induces a bijection $\mathrm{Hom}_{\mathcal{C}}(X,Y)/\overset{\ell}{\sim} \to \mathrm{Hom}_{\mathcal{C}}(X,Z)/\overset{\ell}{\sim}$*

I should also have assumed that the morphism spaces are sets.

Lemma 9.6. *If X is cofibrant then $f \overset{\ell}{\sim} g$ implies $f \overset{r}{\sim} g$.*

By these five lemmas we can deduce that

Theorem 9.1. *The category $\mathcal{C}_{cf}/\sim$ exists.*

But we didn't see how we can invert the weak equivalences. I will state it.

Theorem 9.2. *A map of $\mathcal{C}_{cf}$ is a weak equivalence if and only if it is a homotopy equivalence.*

So this is how we invert weak equivalences in that category. Actually,

Theorem 9.3. *Let $\delta : \mathcal{C} \to \mathcal{C}_{cf}/\sim$ and Q and R be cofibrant and fibrant replacement functors. Then δQR satisfies the universal property for the homotopy category $\mathrm{Ho}(\mathcal{C})$.*

Let me just give a proof. Let $\mathcal{D}$ be a category and F a functor from $\mathcal{C}$ to $\mathcal{D}$ sending weak equivalences to isomorphisms. Then there is a "unique" functor that I will construct $G : \mathcal{C}_{cf}/\sim \to \mathcal{D}$. Then $G(\delta QRX) = F(\delta QRX)$ and $[f] \in \mathrm{Hom}_{\mathcal{C}_{cf}}(\delta QRX, \delta QRY) \cong \mathrm{Hom}_{\mathcal{C}}(\delta QRX, \delta QRY)/\sim$, whith $G([f]) = F(f)$.

We should check athat it's well-defined. Suppose $f \sim g$, then there exists a cyclinder on δQRX with

$$\delta QRX \sqcup \delta QRX \to \mathrm{Cyl}(\delta QRX) \to \delta QRY$$

and because δQRX is cofibrant, then i_0 and i_1 are cofibrations and using the identity $wi_0 = \mathrm{id}_{\delta QRX} = wi_1$, and $F(w)$ is an isomorphism, so $F(i_0)$ and $F(i_1)$ have the same image. So $F(f) = F(Hi_0) = F(Hi_1) = F(g)$. Then we can check composition. Now I can say that I defined the homotopy category for a model category.

10. April 18: Damien Lejay: Model Categories II

[Damien told me not to take notes for this talk]

11. April 18: Mehdi Tavakol: The small object argument

I want to discuss some conditions for getting factorizations for free.

I'll start by recalling some definitions for small and compact objects. This is confusing in the literature. Let me first say that I want to define small objects. They behave like, they have a small amount of information. To say precisely, I should first say something about a filtered diagram. If I have a partially ordered set J which is an index set, and I assume it's filtered, by which I mean that if you have two objects i and j, there is something bigger than both of them, when you have an object X in the category, I say it's *small* if for any such diagram indexed by J, say $\{Y_j\}_{j\in J}$ with colimit Y, and I have an induced diagram which passes to $\mathrm{Hom}_{\mathcal{C}}(X, Y_j) \to \mathrm{Hom}(X, Y)$, we say X is small if for any such diagram, the map from the colimit of these hom sets to the hom set into the colimit is a bijection.

This, let me give some examples. If $\mathcal{C}$ is a set, you'll see that it should be a finite set. If $X = \mathbb{N}$, then you can write it as a union of $\mathbb{N}_n$ where $\mathbb{N}_n = \{1, \ldots, n\}$. Then let $Y_n = \mathbb{N}_n$. Then we think about the identity map from $\mathbb{N}$ to $\mathbb{N}$. Then we can't find a collection of maps from $\mathbb{N}$ to $\mathbb{N}_n$ for any n that leads to this, I can't factor through a finite set. So from this I can say that small objects in sets are finite sets.

You can use the same kind of idea to say that if you want to look at groups, then small objects are finitely presented groups. Then you can see that by the same idea, these are going to be small objects.

For algebraic structures you can think of them as objects with some kind of finiteness conditions. "Compact" is a little confusing because in topological spaces, compact spaces are not compact objects.

Let me make one remark, I'm ignoring a cardinal condition. You can define "κ-small objects" for κ a regular cardinal (here κ is $\aleph_0$), and, well, let me do the example later.

Now I'll define presentable categories. There are two conditions on the category $\mathcal{C}$

(1) $\mathcal{C}$ admits all $\mathcal{U}$-small colimits
(2) $\mathrm{Hom}(X, Y)$ is $\mathcal{U}$-small
(3) there is a set of $\aleph_0$-small objects which generate $\mathcal{C}$ under $\mathcal{U}$-small colimits.

So some examples. For $\mathcal{C}$ the category of sets or algebraic things like groups. If $\mathcal{C}$ is the category of Banach spaces, then it's not $\aleph_0$-presentable, but it's $\aleph_1$-presentable, which I'll leave as an exercise.

Another example that is not $\aleph_0$-presentable but $\aleph_1$, there are some examples in topological groups, complicated ones. So topological spaces are *not* presentable.

Let me say a little more about lifting problems. If I have a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "p"] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

then we say that p has the *left lifting property* with respect to q and q has the *right lifting property* with respect to p. Then for a collection of maps we can talk about lifting properties. If S is any collection of maps, then ${}^{\boxslash}(S^{\boxslash})$ has a stability condition (we can easily see that this contains S). Let me define another thing, a *weakly saturated class*, which is, I have small colimits, and my class of morphisms is weakly saturated if it's closed under pushouts:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ B \arrow[r, "f'"'] & Y \end{tikzcd}$$

where if this is a pushout and f is in S then f' is in S. It's closed under infinite composition, so if you have

$$\begin{tikzcd} & D_i \arrow[dd, "\phi_{ij}"] \\ C \arrow[ur, "\phi_i"] \arrow[dr, "\phi_j"'] & \\ & D_j \end{tikzcd}$$

and we have this for D_i for $i \in I$, and if $\mathcal{D}$ is the colimit of the diagram, then there is, for any j, there's a map $D_j \to \mathcal{D}$, and I have this map from $C \to D_j \to \mathcal{D}$, and this is in S.

It's also closed under retracts, which means that if you have the following diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "f"] & C' \arrow[r] \arrow[d, "g"] & C \arrow[d] \\ D \arrow[r] & D' \arrow[r] & D \end{tikzcd}$$

where the two horizontal compositions are the identity, and we know that g is in S, then f is in S.

Now I can state the proposition, the small object argument.

If $\mathcal{C}$ is a presentable category and I have a collection A_0 of maps ϕ_i which is indexed by $\mathcal{U}$-small I, and if you have $f : X \to Y$, then you can decompose f as $X \to Z \to Y$ with $f' : X \to Z$ where f' is in the smallest weakly saturated class of morphisms generated by A_0 and f'' has the right lifting property with respect to all elements in A_0.

Let me explain the construction, we want to construct this, I just want to know the statement, so to do the construction, I look at $C_i \to D_i$, and look at all collections of such maps, with $Z_0 = X$

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] & X \arrow[d] \\ D_i \arrow[r] & Y \end{tikzcd}$$

and I take the colimit for all guys in A_0, and then I get a colimit Z_1 which maps to Y. Then I can do something similar

$$\begin{tikzcd}
C_i \arrow[rr] \arrow[dd] & & Z_0 \arrow[dl] \\
& Z_1 \arrow[dr] & \\
D_i \arrow[rr] & & Y
\end{tikzcd}$$

and I play the same game. Then I take the colimit and get Z, and then it's easy to check that this is in the smallest weakly saturated class.

So what is this used for? This weakly saturated class is ${}^{\boxslash}(A_0^{\boxslash})$.

12. Byung Hee An: Quillen adjunctions and equivalences

First of all it was hard to prepare this talk because I'm very much a beginner at this category theory. If I say something wrong, then it's Damien's fault.

Let me say something about Quillen adjunctions and equivalences. Let's start with two categories $\mathcal{C}$ and $\mathcal{D}$, we have two functors $F : \mathcal{C} \to \mathcal{D}$ left adjoint to $G : \mathcal{D} \to \mathcal{C}$.

Suppose both $\mathcal{C}$ and $\mathcal{D}$ have model category structures. In other words, $\mathcal{C}$ has three classes of morphisms, weak equivalences and cofibrations and fibrations, and likewise for $\mathcal{D}$. How can we say that these two model structures are related by adjoint functors, by these two, right? So, to say about these two model categories, related to those two adjoint functors, I want to first state one lemma.

Lemma 12.1.
1. *Suppose that F preserves cofibrations, that's equivalent to G preserving trivial fibrations.*
2. *The functor F preserves trivial cofibrations if and only if G preserves fibrations.*

Let me give the idea of the proof. This is very easy. Let's consider $f : X \to Y$ a $\mathcal{C}$-cofibration. What does it mean to say that F preserves cofibrations? That means that $F(f)$ is in $C_{\mathcal{D}}$.

Consider a diagram like this:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"] & G(A) \arrow[d, "G(g)"] \\
Y \arrow[r] & G(B)
\end{tikzcd}$$

Then using the adjunction we have a diagram like this:

$$\begin{tikzcd}
F(X) \arrow[r] \arrow[d, "F(f)"] & A \arrow[d, "g"] \\
F(Y) \arrow[r] & B
\end{tikzcd}$$

So if F preserves cofibrations, there is a lift of the second diagram, and then its adjoint is a lift for the first diagram. So then $G(g)$ is in the right orthogonal of f, and so is in the trivial fibrations.

So among these four conditions, we pick two of these, like preserving cofibrations and trivial cofibrations. So there are four equivalent conditions.

(1) $F(C_{\mathcal{C}}) \subset C_{\mathcal{D}}$ $F(W_{\mathcal{C}} \cap C_{\mathcal{C}}) \subset W_{\mathcal{D}}$
(2) $\cdots$
(3) $\cdots$
(4) $\cdots$

Definition 12.1. We call (F, G) a *Quillen adjunction* if one of the four equivalent conditions is satisfied.

This doesn't literally map a model structure to the other but it maps enough of the structure of one to the other. So we call in this case F the *left Quillen functor* and G the *right Quillen functor*.

Then if you have a Quillen adjunction, one of the nice properties is that those functors induce functors at the level of the homotopy category. Before seeing that I want to mention one remark: a left Quillen functor preserves weak equivalences between cofibrant objects and a right Quillen functor preserves weak equivalences between fibrant objects.

This follows from "Ken Brown's lemma" which Cheolgyu already mentioned. We know that F preserves trivial cofibrations and then this is exactly the conclusion that we draw from that lemma.

So now let's consider a subcategory $\mathcal{C}_c$, the full subcategory whose objects are cofibrants. Then F preserves cofibrants and this maps to $\mathcal{D}_c$, the full subcategory of cofibrants. Now we take a localization to take a homotopy category, $\mathcal{D}_c \to \mathrm{Ho}(\mathcal{D}_c)$ but this is the same as $\mathrm{Ho}(\mathcal{D})$. Then the composition satisfies a universal property, that it maps all weak equivalences to isomorphisms, and this is the universal property of the homotopy category, so it must factor through the homotopy category of $\mathcal{C}_c$ which is isomorphic to the homotopy category of $\mathcal{C}$. So a left Quillen functor induces a functor between homotopy categories, this is unique up to unique natural transformation.

This induced functor we denote by $\mathbb{L}F$. Similarly we can think the induced functor from G, but instead of considering the cofibrants we think of the fibrants, we have $\mathcal{D}_f$, and since G preserves fibrations this maps to $\mathcal{C}_f$, and then this localizes to $\mathrm{Ho}(\mathcal{C})$, and this takes weak equivalences to isomorphisms so it factors through $\mathrm{Ho}(\mathcal{D}_f)$ which is $\mathrm{Ho}(\mathcal{D})$, called $\mathbb{R}G$.

We can consider $\mathcal{C}_{cf}$, the full subcategory of cofibrant fibrant objects, which sit inside $\mathcal{C}_c$ and $\mathcal{C}_f$, which sit inside $\mathcal{C}$. These may not be model categories. But these are categories with weak equivalences. These have a special class of morphisms with the two out of three condition. Then we can make a homotopy category. Indeed we have functors among all of these. The key point is that both satisfy the universal property.

Sometimes these derived functors may be equivalences of categories. There are several equivalent conditions for those derived functors to be equivalences of categories.

Lemma 12.2. *The following are equivalent:*

(1) *The left derived functor $\mathbb{L}F$ is an equivalence*
(2) *the right derived functor $\mathbb{R}G$ is an equivalence.*

(3) *for any cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, then $f \in \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$ is a weak equivalence if and only if its adjoint in $\mathrm{Hom}_{\mathcal{D}}(F(X), Y)$ is a weak equivalence*
(4) *For any cofibrant X, the composition $X \to GFX \to G(R(FX))$ (where R is a fibrant replacement) and for any fibrant Y, the composition $F(Q(GY)) \to FGY \to Y$ is a weak equivalence.*

These four conditions are equivalent.

Let me mention a weaker fact about the derived functors. So we have functors $\mathcal{L}F : \mathrm{Ho}(\mathcal{C}) \to \mathrm{Ho}(\mathcal{D})$ and $\mathbb{R}G : \mathrm{Ho}(\mathcal{D}) \to \mathrm{Ho}(\mathcal{C})$, and these two functors are adjoint to one another. The only thing to prove to see this is that F and G preserve some homotopy relation. That's basically about preserving cylinders or path objects. But a cylinder is a factorization $X \sqcup X \to C(X) \to X$. Anyway, you can do it this way or also with units and counits.

Let's go back to the lemma. The first two are equivalent because they are adjoint to one another. It's not too hard to prove the equivalences of the other statements, but it's not trivial. By the way, if you find a Quillen adjunctions and Quillen equivalences from google, then you can find a website, the nlab, which says that these last two are separate conditions that are equivalent, but that's wrong.

I want to say only one sketch of the proof of only one thing because I think I'm faster than I expected. I'm going to show one proof. Let's see that the third and fourth conditions are equivalent. The equivalence between the second and third is in higher topos theory but not this one. So consider $f : X \to GY$, here X is cofibrant and Y is fibrant, then by adjunction there is a corresponding morphism $f' : FX \to Y$. and we apply fibrant replacement and get a morphism $\tilde{f}' : RFX \to Y$. But these two objects are fibrant and so we take a functor G. We get

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & GFX \arrow[r] \arrow[d] & G(RFX) \arrow[dl] \\ & GY & \end{tikzcd}$$

So if we assume the fourth statement, then the composition along the top is a weak equivalence. So by the two out of three condition, oh, suppose f' is a weak equivalence. Since fibrant replacement is weak equivalence, so is $\tilde{f}'$. Then G preserves weak equivalence between fibrants so the arrow $G(\tilde{f}')$ is as well. Then the composition is, so f is as well. If we use the other diagram we get the other direction.

Definition 12.2. A pair (F, G) is a *Quillen equivalence* if one of the conditions of the lemma is satisfied.

I have these units and counits $1_{\mathrm{Ho}\,\mathcal{C}} \to \mathbb{R}G \circ \mathbb{L}F$, and being an equivalence means that this is an isomorphism. Let's assume that c is cofibrant, then what is $\mathbb{L}F(c)$? At the object level, the localization does nothing, so $\mathbb{L}F$ maps c to $\mathrm{Ho}(\mathcal{D})$, then the image is a cofibrant thing. So then $\mathbb{R}G$ does a fibrant replacement and takes G. So the composition is something like $C \mapsto F(C) \to D \to G(D)$ where D is fibrant. If we pick any weak equivalence $F(C) \to D$, the natural transformation defines an isomorphism; then the composition $C \to G(D)$ we want to have as a weak equivalence [missed something].

Let's see some examples. I need some model structures to show examples. I won't go into any detail.

Let's consider the category of topological spaces, with objects topological spaces and morphisms continuous maps. As in chain complexes there are two standard model structures, the "Quillen" model structure and the "Strøm" model structure. In the first one, the weak equivalences are weak homotopy equivalences, continuous maps that induce isomorphisms on all homotopy groups. In the second model, weak equivalences are homotopy equivalences. These two model structures cannot be "the same" in some sense. If we localize and see homotopy categories then these are different. The identity functor from $\mathrm{Top}_Q \to \mathrm{Top}_S$, this is not a Quillen equivalence. This is a Quillen adjunction even though it's not an equivalence. Quillen to Strøm is the left adjoint.

The second example is the category of simplicial sets. This has objects functors from Δ^{op} to the category of sets. This is a "simplicial set" which has some kind of face and degeneracy maps. Then this has a standard model structure. The second example is that Top_Q and sSet are Quillen equivalent via $|\cdot|$, the geometric realization, and the singular functor Sing, maps from the simplex. The important thing is that this pair is a Quillen equivalence. Their homotopy categories are equivalent. So if you only want to see homotopy types, in this category, then it's enough to consider simplicial sets. Damien said that the benefit of simplicial sets is that this is a presentable category.

In the Strøm category, cofibrations and fibrations are maps satisfying the homotopy extension and homotopy lifting properties, respectively. For the Quillen case, it's Serre fibrations, which satisfy a lifting property only with respect to some certain kind of maps $D^n \to D^n \times I$.

I didn't remark about cofibrant generation, where you have a small set of generating cofibrations, and then you can get all cofibrations from transfinite compositions, pushouts, and retracts. You can do some kind of small object argument and build some Serre cofibrations here.

A third example is chain complexes. We saw two model structures, but if $\mathcal{A}$ and $\mathcal{B}$ are Abelian categories (with enough injectives), then we can make $\mathrm{Ch}(\mathcal{A})$ and $\mathrm{Ch}(\mathcal{B})$, the chain complexes on them. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are adjoint, then there are induced functors on the chain complexes which are Quillen adjunctions.

The next example, I want to introduce one more model category, the category sMod_R, where this is simplicial R-modules. The definition is similar: functors $\Delta^{op} \to \mathrm{Mod}_R$. You have special maps, face and degeneracy. So the fourth example is the Dold–Kan correspondence. Consider $\mathrm{Ch}(R)$ and sMod_R, and there are equivalences of categories, N and Γ which I don't want to define. These are not only just Quillen equivalences but also equivalences of categories. So, I missed something. We have Quillen functors from Top to sSet and sMod_R to $\mathrm{Ch}(R)$, and we also have functors $\mathrm{sMod}_R \leftrightarrows \mathrm{sSet}$. Maybe you remember the forgetful and free functor between groups and sets. This is a forgetful functor, get rid of the module structure, and in the other way it's the free R-module. Then this pair is a Quillen adjunction.

So I want to write in this way.

$$\mathrm{Top}_Q \leftrightarrows \mathrm{sSet} \leftrightarrows \mathrm{sMod}_R \leftrightarrows \mathrm{Ch}(R)$$

So here we have as composition the singular chain complex on spaces, so you get homology at the level of homotopy categories.

I want to mention one more thing, this is not an example, say one more thing, about transfered model structures. So suppose we have adjoint functors $F : C \leftrightarrows$

$D : G$, and suppose that $\mathcal{C}$ has a model structure. Can we define a model structure on $\mathcal{D}$ using this adjunction? How can we hope to do this? The baby example is when F and G are equivalences. Then it's easy to get the model structure over to the other place. You let the image be the desired class of morphisms. You only have an adjoint pair. So what we want to do is to define the weak equivalences in $\mathcal{D}$ to be $G^{-1}(W_{\mathcal{C}})$, all morphisms whose image under G is a weak equivalence, and fibrations all the maps whose preimage are fibrations. When do you get this kind of model structure? I don't know a necessary and sufficient condition but I know a sufficient condition.

Proposition 12.1. *Assume that $\mathcal{C}$ is cofibrantly generated (I don't want to define this). Then $(W_{\mathcal{D}}, F_{\mathcal{D}})$ defines a model structure on $\mathcal{D}$ if it satisfies two conditions (quite technical, I think. I'll use stronger but easier conditions than the most technical ones I know).*

(1) *G preserves filtered colimits.*
(2) *$\mathcal{D}$ has a fibrant replacement functor and path objects which are functorial for fibrant objects.*

The first condition we all know, you send a filtered colimit to a filtered colimit. For the second one, we already defined the fibrations. So we don't know the data being from a model structure but we can still talk about fibrant replacement.

If we decompose a morphism in that way, we can define a "path object" which is a factorization of the diagonal $A \to P(A) \to A \times A$, where the first map is a weak equivalence and the second a fibration. We should be able to find $P(A)$ in a consistent way.

The proposition says that if G and $\mathcal{D}$ satisfy these conditions then this data defines a model structure on $\mathcal{D}$.

Let's see the example of transferred model structure. So for example sMod_R can be viewed as being transferred from sSet. So we can look at $\mathrm{dg\,Alg}_R$ and $\mathrm{Ch}(R)$. If you forget multiplication you get a chain complex.

I want to use the projective model structure, fibrations are degreewise epimorphisms. Then there's a canonical functor, the forgetful functor, and the other way is a free functor. More concretely this is the free tensor algebra functor. So $TA = \bigoplus A^{\otimes i}$. Then the multiplication is tensor product. We don't know the model structure on dg algebras, but we have a model structure on Ch_R and adjoint functors. So we want to define weak equivalences and fibrations as weak equivalences and fibrations under the forgetful functor. So these are algebra morphisms so that f is a weak equivalence as chain complexes. It induces an isomorphism on homology. Then fibrations are degreewise epimorphisms.

The condition of the proposition is that

(1) forget preserves filtered colimits
(2) the category of dg algebras should have fibrant replacement and functorial path objects for fibrant objects.

The first condition is true, even one hour ago I didn't know why. Damien let me know. Our fibrations are degreewise epimorphisms. Any map from a dg algebra to zero is a fibration. So every element is fibrant and the identity is a fibrant replacement functor. What about path objects? This is the situation in which the proposition works. So I want a path object, which is actually quite complicated. This can actually be defined as $A \otimes_R \Omega_{\mathrm{poly}}(\Delta^1)$ if R is characteristic zero. Under these certain assumptions, the path object can be written in this form, and then it's obvious that this is functorial. So this is the case.

13. June 12: Yong-Geun Oh

What I'm going to talk about is, try to give the proof of existence, of the model category of dg categories. So let me denote DCAT as the category of differential graded categories. Somehow I read another paper of Tabuada, he introduces another category DCAT_p, which is the category of dg categories with one initial category, with $\mathcal{O}$, with morphisms of $\mathcal{O} \to \mathcal{C}$ in DCAT.

Theorem 13.1. *(Tabuada) There is a model category structure whose weak equivalences W are quasi-equivalences of dg categories. The generating set of cofibrations is I, I'll write it down in detail later. J is the generating set of trivial cofibrations. So $(W, J^{\boxslash}, {}^{\boxslash}(I^{\boxslash}))$ describe a model structure in DCAT_p.*

Let me describe a few categories. Let's consider $\mathcal{A}$ a category with one object called 3, with an identity endomorphism $3 \to 3$. Then $\mathcal{B}$ has two objects 4 and 5 and each has just the identity morphism, and the morphisms between 4 and 5 are trivial. Then there is a category $\mathcal{K}$, with two objects 1 and 2, and a more complicated set of morphisms. It has morphisms $f : 1 \to 2$ and $g : 2 \to 1$ and $r_{12} : 1 \to 2$. So in this category $\mathrm{Mor}(1,2) \cong \mathrm{Hom}_k^0(1,2) \cong k$, the ground ring, generated by f. Similarly, $\mathrm{Mor}(2,1) \cong k$, spanned by g. These are closed, that is, $df = dg = 0$. $dr_1 = gf - 1$, with $r_1 \in \mathrm{Hom}_k^{-1}(1,1)$ and $r_2 \in \mathrm{Hom}_k^{-1}(2,2)$, with $dr_2 = fg - 1$. Then $dr_{12} = fr_1 - r2_f$. So what this means is, this is a kind of contraction, f intertwines these two morphisms.

So the morphisms in this category are generated by f, g, r_1, r_2, and r_{12}.

This will be one slick way of expressing some condition.

So $\mathcal{K}$ and then there's more. So here's another category $\mathcal{P}(n)$, with objects 6 and 7, and the morphism structure has a morphism space D^n from 6 to 7, this is a complex, so what is D^n? We denote by S^{n-1}, this is a complex defined by

$$S^{n-1}[i] = \begin{cases} k & i = n-1 \\ 0 & \text{else} \end{cases}$$

The analog of D^n will have k in degree $n-1$ and n, with the differential the identity. Now let me introduce one more, $\mathcal{R}(n)$, which is the category of dg functors from $\mathcal{B}$ to $\mathcal{P}(n)$. The objects are the same, but you have, well, $R(n)$ is dg functors from $\mathcal{B}$ to $\mathcal{P}(n)$ that send 4 to 6 and 5 to 7.

Here is another, $\mathcal{C}(n)$ is the dg category with two objects, 8 and 9, that has morphisms S^{n-1} from 8 to 9 and all other spaces minimal (k or 0 as appropriate).

Now I want to denote again by $\mathcal{S}(n)$ the unique dg functor from $\mathcal{C}(n)$ to $\mathcal{P}(n)$ the dg functor that takes 8 to 6 and 9 to 7 and takes S^{n-1} to itself in D^n

[long discussion about whether the differential goes down or up]

So $\mathcal{Q}$ is the dg functor from $\mathcal{O}$ to $\mathcal{A}$. Now I'm going to tell you J.

The generating set of trivial cofibrations is the set of dg functors F which is the functor fro $\mathcal{A}$ to $\mathcal{K}$ such that 3 goes to 1 and $\mathcal{R}(n)$ for $n \in \mathbb{Z}$. For I it's the dg functor $\mathbb{Q}$ and the functors $S(n)$ for $n \in \mathbb{Z}$. Then W is the category of quasi-equivalences. Then the main theorem is that these satisfy the generating criteria for model categories:

(1) W has the two-out-of three property and is closed under retracts.
(2) The domains of I are small relative to I-cells

(3) The domains of J are small relative to J-cells
(4) $J-\text{cell}$ is contained in $W \cap I - \text{cof}$
(5) $I - \text{inj} \subset W \cap J - \text{inj}$
(6) Either $W \cap I - \text{cof} \subset J - \text{cof}$ or $W \cap J - \text{inj} \subset I - \text{inj}$.

You might wonder what the point of these ad hoc categories is. The proof of this can be reduced to the case of the category of complexes by some categorical nonsense, interpreting dg functors, natural transformations, and so on, by categorical nonsense, and then those two theorems, we have two motivating propositions in the category of complexes.

The model structure in complexes is weak isomorphisms and fibrations are surjections. Well, a map p in $\mathrm{Ch}(k)$ is a fibration if and only if p_n is surjective for all n. This can be lifted as a lifting property involving $0 \to \mathcal{P}(n)$ and $X \to Y$ via p. So this is exactly the right lifting property with respect to $0 \to \mathcal{P}(n)$. A map $p : X \to Y$ in chains of k is a trivial fibration if and only if it is J-injective, exactly the same J. This one I didn't check but I think this is exactly what this is.

14. June 12: Kyoung-Seog Lee

Thank you for giving me this opportunity to give a talk. What Damien asked me to do was solve some exercises. Before that, let me briefly tell you what we're going to do. We want to give a model category structure on dg categories. We'll do it by giving two of the three classes. The weak equivalences are quasi-equivalences. We'll define the fibrations, and thes will be $f : T \to T'$ satisfying two conditions.

(1) $f_{x,y} : T(x,y) \to T'(f(x), f(y))$ is surjective in chain complexes, and
(2) for any $u' : x' \to y'$ in $[T']$, there exists $y \in [T]$ such that $f(y) = y'$, there exists an isomorphism $u : x \to y$ in $[T]$ such that $[f](u) = u'$

Theorem 14.1. *(Tabuada) these classes (W, F) determine a model category structure on dg categories.*

I want to prove exercise 14, my mission given by Damien, this is fun. This is four parts.

(1) Let $\mathbf{1}$ have a single object and the morphisms just k in degree 0. Show that 1 is cofibrant. Recall that A is cofibrant if and only if $\varnothing \to A$ is cofibration. This is true if and only if for every map $X \to Y$ that is a trivial fibration, there is a lifting

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d] \\ A \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

Let us prove this. So we have a commutative diagram, so our object in A hits something y in Y, since this is a trivial fibration, the map from $[X]$ to $[Y]$ is essentially surjective. Then there is an object quasi-isomorphic to y, and x' that hits it. By the second property, then there is some x which hits y. I then just define a dg functor which sends my object to x. You can just directly check that this works. You have to check that this is a chain morphism. I can define a map because the differential of the identity is always zero.

(2) Let me introduce another category and show it has another lifting property. This category Δ_k^1 is the k-linear category with two objects 0 and 1 and a morphism. The claim is that this is again cofibrant. This is similar checking. I didn't write everything down. Let me check that.

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta_k^1 \arrow[r] & Y \end{tikzcd}$$

So I have y_0 and y_1 and I want to lift everything. I again us that $X \to Y$ is a trivial fibration. I can assume that there exists x_0 and x_1 which map to each of these, by the same argument, and by fullness I'm surjective on the homotopy class for this morphism, and this gives a way to define this map. I should confess that I didn't check every detail of that.

(3) These are about non-cofibrant objects. He claims that by exercise 7, you can show that the dual numbers $k[\epsilon]$ is not a cofibrant dg category. Let me briefly solve this exercise here. Let me explain this exercise. This says that, let B be a commutative k-dg algebra whose underlying graded k-algebra is a graded commutative polynomial algebra $k[X,Y]$ with X degree 0 and Y degree -1. Let the differential of Y be X^2. This is a dg category with one object. Then the first claim is to show that there exists a natural quasi-equivalence p from $B \to k[\epsilon] = k[X]/X^2$. The second claim is to show that there is no section in dg categories. Then this gives you that $k[\epsilon]$ is not cofibrant. Because I can write

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & B \arrow[d, "p"] \\ k[\epsilon] \arrow[r, "\mathrm{id}"'] & k[\epsilon] \end{tikzcd}$$

We can show that $B \to k[\epsilon]$ is a weak equivalence and the lack of a section indicates that there is no lift so $k[\epsilon]$ is not cofibrant. So I have $k[X,Y]$, and a differential, and in degree 0, I have $k[X]$. In -1 degree I have $k[X]Y$, and in degree -2 I have $Y^2 = 0$ as long as 2 is invertible. Then $d^2(Y) = 0$ so $d(X^2) = 0$ so $XdX + dX \cdot X = 0$. Then I have $f(X)Y$ which goes to $f'(X)dX \cdot Y + f(X)dY = f(X)X^2$. So in here the kernel is $k[X]$ and the image is $k[X]X^2$ so the homology is $k[X]/X^2$. If this is zero then f is zero in this ring. So I computed the cohomology, and $H^0(B) = k[X]/X^2$ and $H^{-1}(B) = 0$, et cetera. So I can naturally define $k[X,Y] \to K[X]/X^2$ naturally. This is a natural quasi-equivalence. This is an isomorphism in homology. The first condition is satisfied. It does not admit a section. If p admits a section, then this is $k[X]+k[X]Y$, it goes to some $(f(X), g(X)Y)$, and it should square to zero, so $f(X) = 0$. Since it is a section, then it can never go back to X.

(4) Let me finish by proving exercise 4, again very simple, I hope. There is a non-cofibrant object and I want to show one more example, T is a dg

category with four objects, x, x', y, y', and

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "u"'] & x' \arrow[d, "u'"] \\ y \arrow[r, "g'"'] & y' \end{tikzcd}$$

From x to y' I have $k\langle u'f\rangle \oplus k\langle g'u$ in degree 0. I put in another $k\langle h\rangle$ with h in degree -1, with boundary $u'f - g'u$.

There is a category $\Delta^1_k \otimes \Delta^1_k$. This has

$$\begin{tikzcd} 0\otimes 0 \arrow[r] \arrow[d] & 1\otimes 0 \arrow[d] \\ 0\otimes 1 \arrow[r] & 1\otimes 1 \end{tikzcd}$$

with diagonal morphism just k. Again the argument says that there is a natural fibration $T \to \Delta^1_k \otimes \Delta^1_k$ with no section, so then $\Delta^1_k \otimes \Delta^1_k$ cannot be cofibrant. There is a natural functor, just take everything to the thing it looks like, and if I construct a functor between them, the only thing I have to check is that $\mathrm{Hom}(x, y') \to \mathrm{Hom}(0\otimes 0, 1\otimes 1)$ which takes $\langle u'f\rangle \oplus \langle g'u\rangle \oplus \langle h\rangle \to k\otimes k$ is a quasi-isomorphism. This takes h to 0 and the other two generators to $1\otimes 1$. So this is a chain map. It's also a quasi-isomorphism. You can check that this is $0 \to k \to 0$ in both cases. So this is a trivial fibration, and there is no section, let me claim, that's the only claim I want to say, this is, this involves a lift

15. 6/20: Morimichi Kawasaki: on T-dg-modules

Today we consider category of functors and natural transformations. It was a little confusing so if I make a mistake please point it out. Let T be a dg category. Then $F : T \to C(\mathbf{k})$ is a T-dg module if F is a dg-functor.

Remark 15.1. The category of chain complexes, $C(\mathbf{k})$, has a dg-category structure $C_{dg}(\mathbf{k})$ or $\underline{C(\mathbf{k})}$, so in this derived seminar, this was explained by Yoosik Kim. So recall this first. We can

In other words, for any object x of T, there is a chain complex F_x, and this satisfies $F_x \otimes T(x, y) \to F_y$ and this satisfies the usual associativity and unit conditions (these being that for any x and $c \in F_x$ that $c \otimes 1_x \to c$, where 1_x is the unit of $T(x, x)$).

The proof of the "in other words" is, take x, y, z in the objects of T and s in $T(x, y)$ and t in $T(y, z)$. Then by the definition of a dg functor, we have the following diagram:

$$\begin{tikzcd} T(x,y)\otimes T(y,z) \arrow[r] \arrow[d] & T(x,z) \arrow[d] \\ C(\mathbf{k})(F_x, F_y)\otimes C(\mathbf{k})(F_y, F_z) \arrow[r] & C(\mathbf{k})(F_x, F_z) \end{tikzcd}$$

and this implies that $F_{(t\circ s)} = F_t \circ F_s$, which means that $F_x \otimes (t\circ s) = (F_x \otimes t)\otimes s$. Here the bilinear map $\otimes$ is $c \otimes s \mapsto F_s(c)$.

Now the unit condition on dg functors, we have

$$\begin{tikzcd} \mathbf{k} \arrow[r, "e_x"] \arrow[dr, "e_{F_x}"'] & T(x,x) \arrow[d, "F_{x,x}"] \\ & C(\mathbf{k})(F_x, F_x) \end{tikzcd}$$

and for any k in $\mathbf{k}$, $F_{x,x}(e_x(k)) = e_{F_x}(k)$. Since $e_{F_x}(1_x) = 1_{F_x}$ in $C(\mathbf{k})(F_x, F_x)$, then $c \otimes 1_x = c$.

We want to consider the category $T-\ \mathrm{mod}$ of dg T-modules. First we, to consider a category, we have to define morphisms. The set of morphisms between T-dg modules F and F' from T to $C(\mathbf{k})$ is the set of natural transformations between dg functors. This means we have commutativity of the following diagram:

$$\begin{tikzcd} F_x \otimes T(x,y) \arrow[r] \arrow[d] & F_y \arrow[d] \\ F'_x \otimes T(x,y) \arrow[r] & F'_y \end{tikzcd}$$

Then $T-\ \mathrm{mod}$ consists of the category of T-dg modules. Now we define a model structure on T-dg modules. The computation here is simple. For f in $T-\ \mathrm{mod}\ (F, F')$, we say f is an equivalence if for all x in $\mathrm{ob}(T)$, the component $f_x : F_x \to F'_x$ is an equivalence in $C(\mathbf{k})$ (that is, a quasi-isomorphism).

Second, we define fibrations. A morphism $f \in \mathrm{Mor}(F, F')$ is a fibration if for all x, $f_x : F_x \to F'_x$ is a fibration in $C(\mathbf{k})$. Here fibration means surjection. This induces the model structure on $T-\ \mathrm{mod}$. Toën's pdf did not give a description of cofibrations, but I think it's hard to describe.

Today we will not check that this is a model structure.

Definition 15.1. The derived category of T-modules, $D(T)$ is $\mathrm{Ho}(T-\ \mathrm{mod}\ ) \coloneqq W^{-1}(T-\ \mathrm{mod}\ )$, the localization with respect to the weak equivalences.

Next we'll give a definition. I gave only the definition of T-dg modules. Let me give examples.

(1) The trivial $C(\mathbf{k})$-dg module from $C(\mathbf{k})$ to itself (the identity) is a trivial module.
(2) The next examples come from Toën, pages 15 and 16. Let T be a dg category. Define $f \coloneqq h^x : T \to C(\mathbf{k})$ by $y \mapsto T(x, y)$, which is an object of $C(\mathbf{k})$. Moreover, for a morphism a, you get $b \mapsto a \circ b$.
(3) For T any dg category, let $h_x : T^{\mathrm{op}} \to C(\mathbf{k})$, this takes y to $T(y, x)$ and a morphism a to $b \mapsto (b \circ a)$.

These three are the three main elementary examples. As an exercise, let T be a dg category. We'll prove exercise 2, that $x \mapsto h_x$ defines a functor $[T] \mapsto D(T^{\mathrm{op}})$.

Proof. First, we construct the functor $h : T \to T^{\mathrm{op}} -\ \mathrm{mod}$. If x is an object of T, then h_x is a T^{op}-module. So this was our example three. For a morphism, fix x and y in the objects of T^{op}. Then choose a in $T^{\mathrm{op}}(y, x) = T(x, y)$. We can define h_a, a map in $T^{\mathrm{op}} -\ \mathrm{mod}\ (h_x, h_y)$ by the following. For $z \in \mathrm{ob}(T)$ we define $h_a(z)$, a map in $C(\mathbf{k})(h_y(z), h_x(z))$ by $h_a(z)(b) = b \circ a$ for any $b \in T(y, z)$. In diagrams I can write

$$a \mapsto (z \mapsto ((y \xrightarrow{b} z) \mapsto (x \xrightarrow{a} y \xrightarrow{b} z))$$

Consider the restriction of h to $Z^0(T(x,y)) : Z^0(T(x,y)) \to T^{\mathrm{op}} - \mod (h_x, h_y)$. Recall that $H^0(T(x,y)) = Z^0 T(x,y)/B^0(T(x,y))$. We still have to show the well-definedness. Assume a, a' in $Z^0(T(x,y))$, and assume that $[a] = [a']$ in $H^0(T(x,y))$, which means that $a - a' \in B^0(T(x,y))$. We have to prove that $h_a = h_{a'}$ in $D(T^{\mathrm{op}})$. Since $a - a'$ is in $B^0(T(x,y))$, there is an α in $T^{-1}(x,y)$ with $a - a' = d\alpha$. Then for any b in $T(y,z)$, we have

$$b \circ a - b \circ a' = b \circ (a - a') = b \circ d\alpha = d(b \circ \alpha) - db \circ \alpha.$$

Therefore for all b in $T(y,z)$ with $db = 0$, we have $b \circ a - b \circ a' \in \mathfrak{I}(d) \subset T(x,z)$. Therefore $h_a = d'_a$ in $D(T^{\mathrm{op}})$. □

I'll finish here. Thank you.

16. Youngjin Bae

Let me briefly give you what I want to do in one hour. We are considering the model categories and dg categories, and we want to consider M a model category enriched by chain complexes over $\mathbf{k}$, and this is a model category and also has a dg category structure, and I want to say, if I consider the dg category, I can consider $\mathrm{Int}(M)$, and I can consider the model category, let me say M. I can consider the homotopy category of the dg-category $\mathrm{Ho}\,\mathrm{Int}(M)$. We can also consider the derived category of this model category $W^{-1}M$, and I want to compare these. At the last point I want to define a kind of Yoneda embedding in the case of dg categories. That's my brief outline and introduction.

Definition 16.1. Let me recall the $C(\mathbf{k})$-model category structure. This consists of the data of:

(1) A $C(\mathbf{k})$-module structure on M, $\otimes : C(\mathbf{k}) \otimes M \to M$ satisfying the associativity and unit conditions, $E \otimes (E' \otimes X) \cong (E \otimes E') \otimes X$ and $\mathbf{k} \otimes X \cong X$ for all E and E' in $C(\mathbf{k})$ and X in M.
(2) For any pair X and Y in my model structure, I have $\underline{\mathrm{Hom}}(X,Y) \in C(\mathbf{k})$ for all X and Y in M satisfying that $\mathrm{Hom}(E, \underline{\mathrm{Hom}}(X,Y)) \cong \mathrm{Hom}(E \otimes X, Y)$
(3) M already has a model structure so we want a compatibility of the model and the module structure. $i : E \to E'$ is a cofibratino in $C(\mathbf{k})$ and $j : A \to B$ is a cofibration in M, then we want

$$E \otimes B \sqcup_{E \otimes A} E' \otimes A \to E' \otimes B$$

to be a cofibration in M

Definition 16.2. Let M be a $C(k)$-model category and T a dg-category. Then M^T is called the T-dg module with coefficients in M, for objects dg functors from T to M, which is the same thing on objects as $F_x \in M$ for $x \in T$. This satisfies the following compatibility: $F_x \otimes T(x,y) \to F_y$ with

$$\begin{tikzcd}
F_x \otimes T(x,y) \otimes T(y,z) \arrow[r] \arrow[d] & F_x \otimes T(x,z) \arrow[d] \\
F_y \otimes T(y,z) \arrow[r] & F_z
\end{tikzcd}$$

and the unit condition $F_x \otimes \mathbf{1}_x \xrightarrow{\sim} F_x$.

The morphisms from $F \to F'$ are collections $f_x : F_x \to F'_x$ for $x \in T$ with the compatibility condition

$$\begin{tikzcd} F_x \otimes T(x,y) \arrow[r] \arrow[d, "f_x \otimes \mathbf{1}"'] & F_y \arrow[d, "f_y"] \\ F'_x \otimes T(x,y) \arrow[r] & F'_y \end{tikzcd}$$

So these are just natural transformations.

Remark 16.1. M^T admits a model category structure. I considered some *dg* category but it has a model category structure. f is a weak equivalence in M^T if f_x is a weak equivalence in M for all x. Also f is a fibration in M^T if f_x is a fibration in M. By the lifting property, the cofibrations of M^T are ${}^{\boxslash}(W_{M^T} \cap \mathrm{Fib}_{M^T})$. This defines a model category structure (under some conditions, i.e., that M is cofibrantly generated)

Remark 16.2. M^T admits a $C(\mathbf{k})$-model structure

Some exercise. Let T and T' be dg categories and M a $C(\mathbf{k})$-model category. Then I want to give an idea of the proof that $M^{T \otimes T'} \cong (M^T)^{T'}$ as $C(\mathbf{k})$-model categories.

In order to make this make sense, we need $M^{(T \otimes T')}$ to be a dg model category.

Proof. We're going to define a map Φ from $M^{(T\otimes T')} \to (M^T)^{T'}$. So on objects, $F \mapsto \Phi(F)$.

So $F : T \otimes T' \to M$ is a dg functor, with $F_{(x,x')}$ such that $F_{(x,x')} \otimes (T \otimes T')((x,x'),(y,y')) \to F_{(y,y')}$ with associative and unit conditions.

We want to define $\Phi(F) : T' \to M^T$. This $\Phi(F)_{x'}$ for x' in T' with

$$\Phi(F)_{x'} \otimes T'(x',y') \to \Phi(F)'_y$$

with $\Phi(F)_{x'}$ a functor $T \to M$, which is a dg functor with the following data $(\Phi(F)_{x'})_x$ for x in T, such that there are maps

$$(\Phi(F)_{x'})_x \otimes T(x,y) \to (\Phi(F)_{x'})_y$$

for x and y in T. The answer is then quite obvious, we choose $\Phi(F)$ satisfying $(\Phi(F)_{x'})_x = F_{x,x'}$ in M. You can directly check that using the morphism $(T \otimes T')((x,x'),(y,y')) \cong T(x,y) \otimes T'(x',y')$ and this gives your compatibility, you can cook up the things you need.

The morphisms are also roughly the same.

$$\Phi : M^{(T\otimes T')}(F,F') \to (M^T)^{T'}(\Phi(F),\Phi(F'))$$

and maybe this is boring and I'll skip it. □

Okay, so now for M a $C(\mathbf{k})$-model category, we have $\underline{M}$ a dg-category with objects the objects of M and morphisms $\underline{M}(x,y) = \underline{\mathrm{Hom}}(x,y)$ from the $C(\mathbf{k})$-enrichment. But we actually want $\mathrm{Int}(M)$, which is the full sub-dg-category whose objects are M^{cf}, the fibrant and cofibrant objects of M, that is, the ones where $\varnothing \to X$ is a cofibration and $Y \to *$.

If I consider $[\mathrm{Int}\, M]$, I can also consider $\mathrm{Ho}(M) \coloneqq M^{cf}/\sim$ and I want to say something about the $\sim$ homotopy relation for M^{cf}.

So I have maps f and g from x to y, and I want a cylinder $x \sqcup x \xrightarrow{i \sqcup j} C(x) \xrightarrow{p} x$ where $i \sqcup j$ is a cofibration and p is a fibration and $p \circ i = p \circ j$ is the identity on x, and then we want

$$\begin{tikzcd} x \arrow[d, "i"'] \arrow[dr, "f"] & \\ C(x) \arrow[r, dotted, "h"] & y \\ x \arrow[u, "j"] \arrow[ur, "g"'] & \end{tikzcd}$$

This is "left" homotopy

There is also "right" homotopy, using a path object, a factorization of $Y \xrightarrow{s} P(Y) \xrightarrow{p \times q} Y \times Y$ which factorizes the diagonal into a trivial cofibration followed by a fibration, and then you want

$$\begin{tikzcd} & y \\ x \arrow[ur, "f"] \arrow[r, "h"] \arrow[dr, "g"'] & P(y) \arrow[u, "p"'] \arrow[d, "q"] \\ & y \end{tikzcd}$$

So at the object level both $[\operatorname{Int} M]$ and $M^{cf}/\sim$ are the same. The morphisms of $[\operatorname{Int} M]$ are $H^0(\underline{\operatorname{Int}} M(x,y))$ where $\underline{\operatorname{Int}}$ is the same as $\underline{\operatorname{Hom}}$ on our full subcategory and so we want to show

$$H^0(\underline{\operatorname{Hom}}) \cong \operatorname{Hom}_M(x,y)/\sim$$

So this was very hard for me, but the fact you use is that

$$\operatorname{Hom}_M(x,y) \cong \operatorname{Hom}_M(k \otimes x, y) \cong \operatorname{Hom}_{C(\mathbf{k})}(\mathbf{k}, \underline{\operatorname{Hom}}(x,y))$$

so the right hand side are degree zero chain maps in $\underline{\operatorname{Hom}}(x,y)$. So these are $Z^0(\underline{\operatorname{Hom}}(x,y))$.

So the left hand side is $Z^0(\underline{\operatorname{Hom}}(x,y))/B^0(\underline{\operatorname{Hom}}(x,y))$, and so the only thing we need to do is to compare the model category homotopy relation with the quotient by $B^0(\underline{\operatorname{Hom}}(x,y))$.

So on the left we say $f - g = dh$, and then I have the following, suppose I have a right homotopy, $\underline{\operatorname{Hom}}(x,y)\underline{\operatorname{Hom}}(x,P(y)) \rightrightarrows \underline{\operatorname{Hom}}(x,y)$, and I want to say that this uses $f - g = dh'$ [Some discussion about how $\underline{\operatorname{Hom}}(x,P(y)) \cong P(\underline{\operatorname{Hom}}(x,y))$ and this can be used to reduce to chain complexes.]

17. June 27: Tae-su Kim

I'm going to talk about the Yoneda embedding, which was part of Youngjin's talk last time, but he skipped it. So $\mathbf{k}$ will be our ring, a commutative unital ring, and T a dg category defined over $\mathbf{k}$. Then a T-dg module is nothing but a dg functor $F : T \to C(\mathbf{k})$. What we're going to do is construct a dg functor from T to another category which is quasi-fully faithful, and this will be $\operatorname{Int}(T^{\mathrm{op}} - \text{ mod })$.

Let me construct this functor. I'll denote a functor $\underline{h} : T \to T^{\mathrm{op}} - \text{ mod}$ which will give rise to the one I want later. So we have $x \mapsto \underline{h}_x : T^{\mathrm{op}} \to C(\mathbf{k})$. This functor $\underline{h}_x$ takes y to $T(y,x)$. Moreover, a morphism $\alpha : x \to x'$ gives a natural transformation given using the composition map, from $\underline{h}_x$ to $\underline{h}_{x'}$. So for each z this defines a map $T(z,x) \to T(z,x')$ and it's just composition with α.

I want to show that this is a fibrant and cofibrant object in $T^{\mathrm{op}} - \mod$ to say that it lands in the interior. To talk about this I need to define the model structure on $T^{\mathrm{op}} - \mod$. If I have F and F' objects in $T^{\mathrm{op}} - \mod$, then morphisms between them are natural transformations between functors. This $F \to F'$ is a fibration if $F_x \to F_{x'}$ is a fibration if it's a fibration in $C(\mathbf{k})$ and similarly for weak equivalences.

This is our model category structure on $T^{\mathrm{op}} - \mod$. We can also find an initial and terminal object, which is a 0 object, which to any object in x assigns the zero chain complex. My claim is that $\underline{h}_x \to 0$ is a fibration. This is easy to show because what we have to show is that $(\underline{h}_x)_y \to 0$ is a degreewise surjection. But with 0 as a target it will be surjective, so this is a fibrant object.

The more tricky part is to show that $\underline{h}_x$ is cofibrant. What we have to show is, we need this kind of diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & F' \arrow[d, "p"] \\ \underline{h}_x \arrow[r] \arrow[ur, dotted] & F \end{tikzcd}$$

for F, F' objects of $T^{\mathrm{op}} - \mod$ and p a trivial fibration. So to each object y I need to assign maps

$$T(x,y) \to F'_y \to F_y$$

What do I get for $y = x$? I can assign id_x to something in F'_x because $F'_x \to F_x$ is a trivial fibration so a degreewise surjection. Now for α in $T(y,x)$, I want to give something in F'_y. But α defines a map $F'_x \to F'_y$. Since F is a T^{op}-module, there is a map ϕ'_α which takes a to $\phi'_\alpha(a)$. Then my lift will be $\alpha \mapsto \phi'_\alpha(a)$.

This data will define a natural transformation as I desired. Let me check. I have $p_y \cdot g_y = f_y$. So $p_y \cdot g_y(\alpha) = p_y \cdot \phi'_\alpha(a)$. By the definition, since p is a natural transformation, I have

$$\begin{tikzcd} F'_x \arrow[r, "\phi'_\alpha"] \arrow[d] & F'_y \arrow[d] \\ F_x \arrow[r, "\phi_\alpha"'] & F_y \end{tikzcd}$$

Then

$$p_y \cdot \phi'_\alpha(a) = \phi_\alpha p_x(a) = \phi_\alpha \cdot f_x(\mathrm{id}_x) = f_y \phi^h_\alpha(\mathrm{id}_x) = f_y(\alpha).$$

Here ϕ^h is a transformation between f_x and f_y.

So we have checked that $\underline{h} : T \to T^{\mathrm{op}} - \mod$ is fibrant and cofibrant in the category, so instead we can write it as landing in $\mathrm{Int}(T^{\mathrm{op}} - \mod)$. Our next claim is that this is a quasi-fully faithful functor between these two dg categories.

What does this mean? It means that the morphism level map

$$T(x,y) \to \mathrm{Hom}(\underline{h}_x, \underline{h}_y)$$

is a quasi-isomorphism. Let me check this. I'll directly construct a map ψ_z for every z in the objects of T from

$$Z(T(x,y))/B(T(x,y)) \to Z(\mathrm{Hom}(T(z,x),T(z,y))/B(\mathrm{Hom}(T(z,x),T(z,y))).$$

So suppose I have $[u]$ in the domain, then $\Psi_z([u]) := [\phi_z(u)]$ where $\phi : T(x,y) \to \{Hom(T(z,x), T(z,y))\}$ is defined by composition with u. Let me check that this Ψ_z is an injective map. So $\phi_z(u)(b) = (d\alpha, b) = d(\alpha(b)) \pm \alpha(db)$. If for all b we have this equality, for $z = x$ we get $\phi_x(u)(\mathrm{id}_x) d\alpha(\mathrm{id}_x) \pm \alpha(d\mathrm{id}_x) = d\alpha(\mathrm{id}_x)$. This is u on the left, so u is zero in the domain. So the map is injective. [Some problems with the setup]

Surjectivity is more tricky, I'll explain after that talk.

Assume we have defined this, then we'll call this the Yoneda embedding.

The next topic I'm going to talk about is about $[T, \mathrm{Int}(M)]$. I want to state a proposition here and Gabriel will prove it later. This is isomorphic to $\mathrm{Iso}\,\mathrm{Ho}(M^T)$. Here M is a $C(\mathbf{k})$-model category and T is our dg category, and we'll assume two conditions, M is cofibrantly generated. This means that there is some set of cofibrations and trivial cofibrations, suitably small, that generates the cofibrations. The second thing is, if $E \to E'$ is a quasi-isomorphism in $C(k)$ and X is cofibrant in M. Then we demand that $E \otimes X \to E' \otimes X$ is an equivalence. Under these two conditions, we can show this, we can prove this. I want to talk about a lemma, for f a quasi-equivalence from T to T', quasi-fully faithful and quasi-essentially surjective, then, under the conditions of the hypothesis, the homotopy categories of $\mathrm{Ho}(M^T)$ and $\mathrm{Ho}(M^{T'})$ are equivalent, witnessed by f^* and $f_!$.

Let me quickly mention how to prove this lemma and then I'll stop. The first one is about any object in the homotopy category. Any object in M^T can be written as a homotopy colimit of something. Let I be a category and let $c : M^T \to (M^T)^I$ be the constant functor. Then this induces a map on the homotopy level $\mathrm{Ho}(M^T) \to \mathrm{Ho}((M^T)^I)$ which has a left adjoint, the homotopy colimit functor. Any object in M^T can be written $\mathrm{hocolim}(F)$ with $F(i) := \underline{h}^{x_i} \otimes X_i$ where X_i are cofibrant.

The second fact is that hocolim and $\mathbb{L}f_i$ (and f^*) commute. So then it's enough to prove it for just cofibrant elements $\underline{h}^{x_i} \otimes X_i$. Then he proves the statement in this case. We have to show there is a natural isomorphism $\mathbb{L}f_! f^* \to \mathrm{id}$ and $f^* \mathbb{L} f_! \to \mathrm{id}$.

These follow similar logic so let me show one case. So what are we trying to do? We're trying to show an isomorphism $\mathbb{L}f_! f^*(\underline{h}^{x_i} \otimes X_i) \cong \underline{h}^{x_i} \otimes X_i$.

So I want to say that this is isomorphic in the homotopy category for some x' to $\mathbb{L}f_! f^*(\underline{h}^{f(x')} \otimes X)$ by quasi-essential surjectivity. Then this is isomorphic to $\mathbb{L}f_!(\underline{h}^{x'} \otimes X)$, from quasi-full faithfulness. Then the object we have is coifbrant and then we can erase L, and then $f_!(\underline{h}^{x'} \otimes X)$, by adjointness, this is the same as $\underline{h}^{f(x)} \otimes X \cong \underline{h}^x \otimes X$.

The opposite direction can be shown in a similar way. Then we have the equivalence of categories between $\mathrm{Ho}(M^T)$ and $\mathrm{Ho}(M^{T'})$. I should have gone into more detail but I didn't have much time, I think this is where I should stop.

18. July 18: Byunghee An: Localization of dg categories

I'll talk about localization of dg categories. So what's the localization? Let T be a dg category. We consider $[T]$ the homotopy category of T and let S be a subset of the morphisms in $[T]$. So then our goal is to define a *localization* of T along S, another dg category $L_S T$. This should satisfy some condition, regarding morphisms contained in S as isomorphisms in some sense. This localization in some sense is a *dg* localization. What does that mean? Consider a functor, sorry, okay, from section two, we consider some functor $F_{T,S}$ from $\mathrm{Ho}(\mathrm{dg}\,\mathrm{Cat}) \to \mathrm{Ho}(\mathrm{Cat})$. If we have a dg category T' then the target category is something $F_{T,S}(T') \subset [T,T']$. Basically it is a collection of dg functors up to homotopy, and f is in $F_{T,S}(T')$ if and only if the induced functor $[f]:[T]\to[T']$ sends S to isomorphisms in T'.

So then L_ST is a dg category such that for any T', $[L_ST,T']$ is a subcategory of $[T,T']$ and $[L_ST,T']$ is $F_{T,S}(T')$.

Define a functor $\ell : T \to L_ST$. We call ℓ a localization of T along S if it satisfies some universal condition. For any $f:T\to T'$ such that $[f](S)$ lives in the isomorphisms of T', then f factors through ℓ, unique up to natural isomorphism.

Proposition 18.1. *For any dg category T and for any S in the morphisms of $[T]$ there exists a localization $\ell : T \to L_ST$.*

So ℓ is actually a morphism in $\mathrm{Ho}(\mathrm{dg\,Cat})$.

Before showing the proof, I want to give an easy example. Recall the category $\mathbf{1}$ with one object and $\mathbf{k}$ as self-morphisms. Then $\Delta^1_{\mathbf{k}}$ has two objects 0 and 1 and a distinguished morphism u from 0 to 1.

What if T is $\Delta^1_{\mathbf{k}}$? A very simple example. T is $[T]$ and $S=\{u\}$. I'll construct $\ell:\Delta^1_{\mathbf{k}}\to\mathbf{1}$. This sends both objects to $*$ and this functor is obvious. I claim that ℓ is a localization of T along S. In other words it satsifies the universal condition

$$\begin{tikzcd} & T \arrow[dl, "\ell"'] \arrow[dd, "f"] \\ L_ST \arrow[dr, dotted] & \\ & T' \end{tikzcd}$$

The universal condition is equivalent to the fact that the $\ell^*:[L_ST,T']\to[T,T']$ is injective and the image of ℓ^* is functors f which take S to isomorphisms.

The induced functor $\ell^* : [\mathbf{1},T'] \to [\Delta^1_{\mathbf{k}},T']$ is by precomposition. These were computed in a previous exercise. We have $[\mathbf{1},T']\cong\mathrm{Iso}(T')$, the isomorphism classes of objects of T'. On the other hand, $[\Delta^1_{\mathbf{k}},T']$ is the same as isomorphism classes of morphisms in $[T']$. If you have a morphism in $[T']$, the isomorphism class means that you can pre- and post-compose with isomorphisms. This functor sends the isomorphism class of an object to the class of the identity on that object, $[x]\mapsto[\mathrm{Id}_x]$. This is obviously injective.

For the second condition, let's see what the isomorphism condition is for this situation. Let f be a morphism in $[\Delta^1_{\mathbf{k}},T']$ so $[f]$ is the same as a morphism $\Delta^1_{\mathbf{k}}\to[T']$ and the condition is that $[f](u)$ is an isomorphism. The image consists of functors in which u is taken to an identity morphism, and thus an isomorphism. Then the image of ℓ^* is a functor which takes u to an isomorphism.

This is a simple example. Let me prove the existence of the localization in the general case. I want to define L_ST as a homotopy pushout

$$\begin{tikzcd} \amalg_S\Delta^1_{\mathbf{k}} \arrow[r] \arrow[d] & \amalg_S\mathbf{1} \\ T & \end{tikzcd}$$

So let S actually be the set of representing morphisms. We have a canonical morphism $\Delta^1_{\mathbf{k}} \to \mathbf{1}$, the canonical one, and we also have the map $\Delta^1_{\mathbf{k}} \to T$ taking u to the morphism s.

I'll give a definition of a homotopy colimit. Let $\mathcal{D}$ be a diagram like this, a category, then we can consider $\mathcal{C}$ a model category and we can consider the diagram category $\mathcal{C}^{\mathcal{D}}$, and we can take the colimit $\mathcal{C}^{\mathcal{D}} \to \mathcal{C}$. We can construct a functor the other way by a constant functor. Then actually these two are adjoint to each other and are a Quillen adjunctions so induce functors on the homotopy category. Then the left derived functor of the colimit is the homotopy colimit. If $\mathcal{D}$ is just the pushout diagram, then this is just the homotopy pushout.

Now we have an explicit definition of the localization. So how to see that this is a localization? We have the map from T to $L_S T$ from it being a (homotopy) pushout. How about the universal property? Let's consider T'? Then f sends all morphisms f to isomorphisms.

Before we have a break, I want to give an exercise. Show that $L_u \Delta^1_{\mathbf{k}}$ is equivalent to $\mathbf{1}$. So this is saying that $\mathbf{1} \sqcup^{\mathbb{L}}_{\Delta^1_{\mathbf{k}}} \Delta^1_{\mathbf{k}}$ but this is a cofibrant diagram (trust me on this), and this is just the regular pushout, but pushing out along the identity gives the other object.

Okay, the next thing I want to talk is an exercise. Let T and T' be two dg categories and S and S' collections of morphisms in the homotopy categories of T and T' as usual, containing all identities. Then the derived tensor $L_S T \otimes^{\mathbb{L}} L_{S'} T'$ is equivalent to $L_{S \otimes^{\mathbb{L}} S'} T \otimes^{\mathbb{L}} T'$ in the homotopy category of dg categories. We know $T \otimes^{\mathbb{L}} T'$ but we need to define $S \otimes^{\mathbb{L}} S'$.

What is the derived tensor product $T \otimes^{\mathbb{L}} T'$? I erased the definition. This is, in our paper, this is $Q(T) \otimes Q(T')$, where Q is a cofibrant replacement functor. So what's the right definition of $S \otimes^{\mathbb{L}} S'$? This is a set of morphisms in the homotopy categories of morphisms. We can safely use the representing morphisms. We can in some sense remove the brackets. Then these, I want to say something like $Q(S) \otimes Q(S')$, this is not defined but let me write this, consider $s : x \to y$ in S. If we take Q, then we get $Qx \to x$ and $Qy \to y$. We want $S \otimes^{\mathbb{L}} S' \subset \mathrm{Mor}([T \otimes^{\mathbb{L}} T'])$.

How to prove the exercise? I think I'm wrong, but let's look at the right hand side. Let M be a $C(\mathbf{k})$-model category. Look at $[L_{S \otimes^{\mathbb{L}} S'}(T \otimes^{\mathbb{L}} T'), \int(M)]$, this sits in $[T \otimes^{\mathbb{L}} T', \int(M)]$, this should be injective and the image has some property. We have injectivity here, and by using something we know, by using the universal property of the internal homomorphisms, we can move this to $[T, \mathbb{R}\underline{\mathrm{Hom}}(T', \int(M))]$ which is the same as $[T, \int(M^{T'})]$ which is the same as $\mathrm{Iso}(\mathrm{Ho}((M^{T'})^T))$, and then we can switch it around, this is the same as $\mathrm{Iso}(\mathrm{Ho}(M^{T \otimes T'}))$, and what I've done is gotten rid of $\mathbb{L}$.

[some discussion of alternate methods]

So instead let's take $\ell \otimes^{\mathbb{L}} \ell' : T \otimes^{\mathbb{L}} T' \to L_S T \otimes^{\mathbb{L}} L_{S'} T'$. So suppose we have a functor F to T'' and suppose it takes morphisms in $S \otimes^{\mathbb{L}} S'$ to isomorphisms, then we want to lift to $L_S T \otimes^{\mathbb{L}} L_{S'} T'$.

[some more discussion]

The next one is the following proposition.

Proposition 18.2. *Let M be a cofibrantly generated $C(\mathbf{k})$-model category and $\underline{M}$ is M viewed as a dg category. Then there is an isomorphism in* $\mathrm{Ho}(\mathrm{dg\,Cat})$ $\mathrm{Int}(M) \cong L_W M$.

If we invert every weak equivalence in a dg sense, then we get $\mathrm{Int}(M)$.

The proof is actually not hard, relatively easier than this exercise. We have a functor $\mathrm{Int}(M) \to M^f \to M$, the $\mathrm{Int}(M)$ is cofibrant and fibrant objects, and M^f is the fibrant objects. We denote the inclusions

$$\begin{tikzcd} & M^f \arrow[dr, "k"] & \\ \mathrm{Int}(M) \arrow[ur, "j"] \arrow[rr, "i"'] & & M \end{tikzcd}$$

and we can define fibrant replacement r and cofibrant replacement q. The existence of these functors q and r comes from cofibrant generation of M (but you could just assume them). Then $(q \circ j)(x) = q(x) \to x$ and $(j \circ q)(x)$ is again $q(x)$, what I mean is there are natural transformations $(q \circ j) \to \mathrm{id}$ and $(j \circ q) \to \mathrm{id}$ and then $\mathrm{id} \to r \circ k$ and $\mathrm{id} \to k \circ r$. These are natural weak equivalences.

Then there are isomorphisms

$$L_W \mathrm{Int}(M) \to L_W M^f \to L_W M$$

in $\mathrm{Ho}(\mathrm{dg\,Cat})$. So what we want to prove is that $L_W(M) \cong \mathrm{Int}(M)$, and my claim is that $L_W(\mathrm{Int}(M)) \cong \mathrm{Int}(M)$. Localization means that we want to invert some morphisms in M. We want to declare certain morphisms to be isomorphisms. But W was already, well, $[\mathrm{Int}(M)] \cong \mathrm{Ho}(M) \cong M[W^{-1}]$. By the definition of localization we have a functor $\mathrm{Int}(M) \xrightarrow{\ell} L_W \mathrm{Int}(M)$. Then if we have $f : \mathrm{Int}\, M \to T'$ satisfying some condition then we get a lift from $L_W \mathrm{Int}(M)$. But what's the condition? It's that when we have $[\mathrm{Int}(M)] \to [T']$ it sends W to isomorphisms. But this condition is vacuous. So there's a correspondence.

The next one is the last one:

Proposition 18.3. *Let T be a dg-category and S some subset of $\mathrm{Mor}([T])$, then $\ell : T \to L_S T$ induces a functor $\ell^* : D(L_S(T)) \to D(T)$ which is fully faithful and so that $F : T \to C(\mathbf{k})$ is in $\mathrm{im}\ell^*$ if and only if $F(S)$ lands in quasi-isomorphisms.*

Proof. $D(T) \coloneqq \mathrm{Ho}(T-\ \mathrm{mod}\ )$ and $D(L_S T) \coloneqq \mathrm{Ho}(L_S T-\ \mathrm{mod}\ )$, and this is the same as $[\mathrm{Int}(C(\mathbf{k})^{L_S T})]$ which in turn is the same as $[\mathbb{R}\underline{\mathrm{Hom}}(L_S T, \mathrm{Int}(C(\mathbf{k})))]$. We are comparing this to $[\mathbb{R}\underline{\mathrm{Hom}}(T, \mathrm{Int}(C(\mathbf{k})))]$. Then the functor is well-defined by functoriality of $\mathbb{R}\underline{\mathrm{Hom}}$. To prove fully faithfulness, we first consider the functor without brackets, ℓ^* from $\mathbb{R}\underline{\mathrm{Hom}}(L_S T, \mathrm{Int}(C(\mathbf{k}))) \cong \int (C(\mathbf{k})^{L_S T}) \subset L_S T-\ \mathrm{mod}$ to $\mathbb{R}\underline{\mathrm{Hom}}(T, \int (C(\mathbf{k}))) \cong \int (C(\mathbf{k})^T) \subset T-\ \mathrm{mod}$. We have ℓ^* from $L_S T-\ \mathrm{mod}\ \to T-\ \mathrm{mod}$, and fully faithfulness there (or the quasi-version) should imply it at the lower level. But that I didn't show.

The second condition is easy. The image of ℓ^* are the ones that factor through $L_S T$, and so something in S, they must go to quasi-isomorphisms in order to be isomorphisms in the homotopy category. I couldn't show the proof for full faithfulness. But I think this is not too hard. □

I want to introduce one last exercise without an answer. The final exercise is like this.

Let ℓ be the $T \to L_S T$ a localization of dg categories. Then we can consider, this induces, $f : T \to T'$ induces $M^T \leftarrow M^{T'}$, this is natural because the objects are functors and we can compose with f. But actually there is a left adjoint $f_!$, here

M is a $C(\mathbf{k})$-model category. This pair is a Quillen adjunction and moreover if f is an equivalence (isomorphism in the homotopy category) then these are a Quillen equivalence. This part is Lemma 1, but it uses this fact about $f_!$. Especially if f is a localization, then the left adjoint $\ell_!$ exists and its derived functor $\mathbb{L}\ell_!$, well, first $\ell_! : T-\ \mathrm{mod}\ \to L_S T-\ \mathrm{mod}$, then the derived functor is on the homotopy categories. So this is $D(T^{\mathrm{op}}) \to D(L_S T^{\mathrm{op}})$. We've got a functor between derived categories. Let W_S be the morphisms in $D(T^{\mathrm{op}})$ such that $\mathbb{L}\ell_!(u)$ is an isomorphism. The first question is about classifying the elements of W_S. I don't want to talk about it. The second is more interesting. We can invert W_S (which satisfies a 2 out of 3 property in $D(T^{\mathrm{op}})$) and the second statement is that $D(T^{op})[W_S^{-1}] \xrightarrow{\mathbb{L}\ell_!} D(L_S T^{\mathrm{op}})$ and that's an equivalence of categories. So you can localize either before or after passing to modules.

I wanted to solve this but I don't have enough time. I'll stop.

19. July 25: Tae-Su Kim: Triangulated dg categories

Let T be a dg category over $\mathbf{k}$. I'll remind you that $T^{\mathrm{op}}-\ \mathrm{mod}$ is the category of *dg* functors $T^{\mathrm{op}} \to C(\mathbf{k})$. We have $\underline{h} : T \to T^{\mathrm{op}}-\ \mathrm{mod}$, the Yoneda embedding, which takes x to $y \mapsto T(y,x)$ and morphism via compositions.

Now some facts about this Yoneda embedding are:

- $\underline{h}(x)$ is cofibrant and fibrant for all x,
- That is, $\underline{h}$ is a functor from T to $\mathrm{Int}(T^{\mathrm{op}}-\ \mathrm{mod}\ )$, the fibrant and cofibrant objects here; we'll denote $\mathrm{Int}(T^{\mathrm{op}}-\ \mathrm{mod}\ )$ by $\widehat{T}$.
- $\underline{h}$ is quasi-fully faithful, so
- $[\underline{h}] : [T] \to [\widehat{T}]$ is fully faithful.

Definition 19.1. F in $T^{\mathrm{op}}-\ \mathrm{mod}$ is *quasi-representable* if F is in the essential image of $[\underline{h}]$, that is, there exists x in T so that $F \cong \underline{h}_x$ in $[\widehat{T}]$.

For F in $T^{\mathrm{op}}-\ \mathrm{mod}$, define a functor $\chi_F : T^{\mathrm{op}}-\ \mathrm{mod}\ \to \mathbf{k}-\ \mathrm{mod}$ takes G to $\mathrm{Hom}_{[T^{\mathrm{op}}-\ \mathrm{mod}\ ]}(F,G)$ and morphism by compositions.

Definition 19.2. F in $T^{\mathrm{op}}-\ \mathrm{mod}$ is *compact* if $\chi_F(\bigoplus_i G_i) \cong \bigoplus_i \chi_F(G_i)$

Here the direct sum of functors $T^{\mathrm{op}} \to C(\mathbf{k})$ is evaluated as the direct sum objectwise in T^{op}.

We claim that quasi-representability implies compactness. How to show this? Assume that we have $\chi_{\underline{h}_x}(G) \cong H^0(G_x)$ and $G = \bigoplus_i G_i$ and F is quasi-representable with x as its representing object, then $\chi_F(G)$ is $\chi_{\underline{h}_x}(G)$ which is $H^0(G_x)$ which is $H^0((\bigoplus G_i)_x)$ which is $\bigoplus H^0((G_i)_x)$ which is eventually isomorphisc to $\bigoplus \chi_F(G_i)$.

So we should show this isomorphism $\chi_{\underline{h}}(G) \cong H^0(G_x)$? I'll construct an isomorphism between $\mathrm{Hom}_{[\widehat{T}]}(\underline{h}_x, G)$ and $H^0(G_x)$. So first we'll construct Φ from $\mathrm{Hom}_{\widehat{T}}(\underline{h}_x, G) \to G_x$. So such a morhpism is α a natural transformation, which is $\{\alpha_y : T(y,x) \to G_y\}$. We send this α to $\alpha_x(\mathrm{id}_x)$. This is a chain map because $d\alpha = \{d\alpha_y\} \mapsto (d\alpha)_x(\mathrm{id}_x) = d(\alpha_x(\mathrm{id}_x)) \pm \alpha_x(d\mathrm{id}_x)$; the second term vanishes because id_x is always closed. So this is $d(\alpha_x(\mathrm{id}_x))$. So this Φ is a chain map, also, degree zero.

This then induces a map on cohomology, and taking the degree zero part, we get a map from $H^0(\mathrm{Hom}_{\widehat{T}}(\underline{h}_x, G)) \to H^0(G_x)$, and by definition the left hand side here is $\mathrm{Hom}_{[\widehat{T}]}(\underline{h}_x, G)$.

How do we define an inverse Ψ? Take $a \in G_x$ to $\overline{\Psi}(a)_y : (\underline{h}_x)_y \to G_y$, where this sends ϕ in $T(y,x) = (\underline{h}_x)_y$ to $G(\phi)(a)$. We can show that this is a chain map.

$$\begin{aligned} d(\overline{\Psi}(a)_y)(\phi) &= d(\overline{\Psi}(a)_y(\phi)) \pm \overline{\Psi}(a)_y(d\phi) \\ &= d(G(\phi)(a)) \pm G(d\phi)(a) \\ &= d(G(\phi)(a)) \pm dG(\phi)(a) \\ &= \pm G(\phi)(da) = \pm \overline{\Psi}(da)_y(\phi) \end{aligned}$$

so we have that this is a chain map up to a sign that I'll leave as an exercise.

Then we can define the inverse Ψ as $[a] \mapsto [\{\overline{\Psi}(a)_y\}]$. Now for x in T, we can consider the composition $H^0(G) \to \mathrm{Hom}_{[\widehat{T}]}(\underline{h}_x, G) \to H^0(G_x)$, which sends $[a]$ first to the class of $\{\phi \mapsto G(\phi)(a)\}$ which goes to $G(\mathrm{id}_x)(a)$ which is $[\mathrm{id}(a)] = [a]$. We checked that $\Phi \circ \Psi$ is the identity. Similarly we can show the other direction is the identity but I'll omit that.

Definition 19.3. A dg category T is *triangulated* if every compact object in $\widehat{T}$ is quasi-representable.

We showed that quasi-representables are compact but we consider here the opposite direction. If every compact object is quasi-representable, then we call this triangulated. We call the full category of triangulated dg categories $\mathrm{dg-cat}^{\mathrm{tr}}$. It's easy to see that we have the inclusion $\iota : \mathrm{Ho}(\mathrm{dg-cat}^{\mathrm{tr}}) \hookrightarrow \mathrm{Ho}(\mathrm{dg-cat})$.

[discussion of what the homotopy category of triangulated dg categories means].

Inside $\widehat{T}$ we have $\widehat{T}_{\mathrm{pe}}$, the full subcategory of compact objects in $\widehat{T}$. We have $\underline{h} : T \to \widehat{T}$ actually has its essential image in $\widehat{T}_{\mathrm{pe}}$, because quasi-representability implies compactness. So we'll use the same notation $\underline{h}$. If T is triangulated, then every compact object is quasi-representable. Then there is an isomorphism between T and $\widehat{T}_{\mathrm{pe}}$ in the homotopy category of dg-cat. Conversely, if $\underline{h} : T \to \widehat{T}_{\mathrm{pe}}$ is essentially surjective (this is the essential image), then compact objects are quasi-representable so that T is triangulated.

Now I want to show that $\widehat{T}_{\mathrm{pe}}$ is triangulated. One way to show this is using a theorem from Toën's other paper, namely that the Yoneda embedding induces a quasi-equivalence

$$\mathbb{R}\,\mathrm{Hom}(\widehat{T}_{\mathrm{pe}}^{\mathrm{op}}, \mathrm{Int}(C(k))) \to \mathbb{R}\,\mathrm{Hom}(T^{\mathrm{op}}, \mathrm{Int}(C(k)))$$

whose left hand side is $\widehat{\widehat{T}_{\mathrm{pe}}}$. Inside of this thing we can find $\widehat{\widehat{T}_{\mathrm{pe}}}_{\mathrm{pe}}$. So we get a functor $\widehat{(\;)}_{\mathrm{pe}}$ from $\mathrm{Ho}(\mathrm{dg-cat})$ to $\mathrm{Ho}(\mathrm{dg-cat}^{\mathrm{tr}})$, the *triangulated hull of* T. This pe is "perfect." For R a $\mathbf{k}$-algebra and the category BR, then

$$[\widehat{BR}_{\mathrm{pe}}] \cong H_{\mathrm{perf}}(R) \subset D(R).$$

Triangulated in this sense implies that there is a natural triangulated structure in the ordinary sense on $[T]$.

Let me continue. So now we have two functors between $\mathrm{Ho}(\mathrm{dg-Cat})$ and $\mathrm{Ho}(\mathrm{dg\,cat}^{\mathrm{tr}})$, ι and $\widehat{(\;)}_{\mathrm{pe}}$. We can show that under certain conditions ι is right adjoint. To show this we need a lemma. For more detail you can check Toën's paper, Lemma 7.3, which says

Lemma 19.1. *Let T' be triangulated. Under certain conditions ($T'^{\mathrm{op}}-$ mod is cofibrantly generated and for $F \in T'^{\mathrm{op}}-$ mod , F_x is projective for all x. Then $\underline{h}_*$: $(T'^{\mathrm{op}}-\text{ mod })^{\widehat{T}_{\mathrm{pe}}} \to (T'^{\mathrm{op}}-\text{ mod })^{T}$ is a Quillen equivalence which then induces an* *equivalence on homotopy categories, so we have a bijection of isomorphism classes of objects in these homotopy categories.*

We assumed that T' is triangulated, so that it is isomorphic to $\widehat{T'}_{\mathrm{pe}}$ in the homotopy category. Consider this diagram.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,cat}^{tr})}(\widehat{T}_{\mathrm{pe}}, T') \arrow[r] & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,cat})}(\widehat{T}_{\mathrm{pe}}, \mathrm{Int}(T'^{\mathrm{op}} - \quad \mathrm{mod}\ )) \arrow[d, "\cong"] \\
& \mathrm{Iso}(Ho((T'^{\mathrm{op}} - \quad \mathrm{mod}\ )^{\widehat{T}_{\mathrm{pe}}})) \arrow[d] \\
& \mathrm{Iso}(\mathrm{Ho}((T'^{\mathrm{op}} - \quad \mathrm{mod}\ )^{T})) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,cat})}(T, \iota T') \arrow[r] & \mathrm{Hom}(\mathrm{Ho}(\mathrm{dg\,cat}))(T, \mathrm{Int}(T'^{\mathrm{op}} - \quad \mathrm{mod}\ ))
\end{tikzcd}$$

The image of the top horizonal map are the morphisms which factor $\widehat{T}_{\mathrm{pe}} \to T' \to \mathrm{Int}(T'^{\mathrm{op}} - \quad \mathrm{mod}\ )$ and the same for the bottom map. It is actually clear that the property of factorizing is independent of the bijection between the isomorphism classes of the homotopy categories, so these two criteria of the inclusion, are the same and the two hom sets are the same.

The last thing I have to talk about is Morita equivalence.

Definition 19.4. $f : T \to T'$ a morphism in $\mathrm{Ho}(\mathrm{dg\,Cat})$ is a *Morita equivalence* if $\hat{f}_{\mathrm{pe}} : \widehat{T}_{\mathrm{pe}} \to \widehat{T'}_{\mathrm{pe}}$ is an isomorphism.

Denote by W_{Mor} the set of all Morita equivalences. We consider localization at Morita equivalences. We want to consider

$$\begin{tikzcd}
\mathrm{Ho}(\mathrm{dg\,cat}) \arrow[r, "\widehat{(\ )}_{\mathrm{pe}}"] \arrow[d, "\ell"] & \mathrm{Ho}(\mathrm{dg\,cat}^{\mathrm{tr}}) \\
W_{\mathrm{Mor}}^{-1}\,\mathrm{Ho}(\mathrm{dg\,cat}) \arrow[ur, dotted] &
\end{tikzcd}$$

and you can actually show that this dotted arrow is an equivalence. It follows from some formal steps which is related to the localization process.

Proposition 19.1. *Let $f : T \to T'$ be a morphism in* $\mathrm{dg\,cat}$. *Then the following are equivalent:*

(1) *f is a Morita equivalence.*
(2) *For T_0 is triangulated we have that $[T', T_0] \to [T, T_0]$ is a bijection.*
(3) *The induced functor $f^* : D(T') \to D(T)$ is an equivalence of categories*
(4) *The induced functor $\mathbb{L}f_! : D(T) \to D(T')$ is an equivalence of categories after restriction to compact objects.*

One of the steps here is hard but you can do it.

[Toën took the simplest definition, being equivalent to the category of perfect objects, but it's better than being triangulated to be isomorphic to this. This is like triangulated plus idempotent complete.]

20. 8/8: Weonmo Lee: Morita equivalence

Remember the Yoneda embedding $\underline{h} : T \to \hat{T} \coloneqq \mathrm{Int}(T^{\mathrm{op}} - \mathrm{Mod})$ which takes x to $\underline{h}_x : T^{op} \to \mathbf{k} - \mathrm{Mod}$. We showed that $\underline{d}$ is quasi-fully faithful so that $[\underline{h}] : [T] \to [\hat{T}] = D(T^{op})$ is fully faithful.

We say that $F \in D(T^{op})$ is *quasi-representable* if F is in the essential image of $\underline{h}$, that is, if $F \cong \underline{h}_x$ or some x.

There was a fact, if F is quasi-representable, then F is compact. If the opposite implication holds, then we call T triangulated.

Proposition 20.1. $\mathrm{Ho}(\mathrm{dg\,Cat}^{\mathrm{tr}}) \to \mathrm{Ho}(\mathrm{dg\,Cat})$ *has a left adjoint* $(\hat{\ })_{\mathrm{pe}}$*, which takes* T *to* $\hat{T}_{pe}$*, the full subcategory of compact objects of* T*.*

Last time we defined Morita equivalence:

Definition 20.1. We cal $f : T \to T'$ a *Morita equivalence* if $\hat{f}_{\mathrm{pe}} : \hat{T}_{\mathrm{pe}} \to \hat{T'}_{\mathrm{pe}}$ gives an isomorphism in $\mathrm{Ho}(\mathrm{dg\,Cat})$.

The localization $W^{-1}_{\mathrm{Mor}} \mathrm{Ho}(\mathrm{dg\,Cat})$ is equivalent to $\mathrm{Ho}(\mathrm{dg\,Cat}^{\mathrm{tr}})$.

Proposition 20.2. *Let* $f : T \to T'$ *be a morphism in* $\mathrm{dg\,cat}$*. Then the following are equivalent:*

(1) f *is a Morita equivalence.*
(2) *For* T_0 *is triangulated we have that* $[T', T_0] \to [T, T_0]$ *is a bijection.*
(3) *The induced functor* $f^* : D(T'^{\mathrm{op}}) \to D(T^{\mathrm{op}})$ *is an equivalence of categories*
(4) *The induced functor* $\mathbb{L}f_! : D(T^{\mathrm{op}}) \to D(T'^{\mathrm{op}})$ *is an equivalence of categories after restriction to compact objects.*

To try to prove this, to get between the first and the second, we have $[T', T_0] = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat})}(T', T_0)$, and then by adjointness this is $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}^{\mathrm{tr}})}(\hat{T'}_{\mathrm{pe}}, T_0)$.

$$\begin{tikzcd}
{[T', T_0]} \arrow[r] \arrow[d, equal] & {[T, T_0]} \arrow[d, equal] \\
\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat})}(T', T_0) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat})}(T, T_0) \\
\mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}^{\mathrm{tr}})}(\hat{T'}_{pe}, T_0) \arrow[r] & \mathrm{Hom}_{\mathrm{Ho}(\mathrm{dg\,Cat}^{\mathrm{tr}})}(\hat{T}_{pe}, T_0)
\end{tikzcd}$$

[long discussion]

21. October 17: Damien Lejay

Today we have the last talk about dg categories in this session. I want to say some interesting things, summarize what we've seen, and so on, like a free open discussion. In the second part we are going to discuss what we want to do for the next year, we have lost quite a few people. As what happened this year, you will decide what we're going to do.

So let me just recall that we started with a notion of dg categories which was a good categorical notion because there is nothing in this definition, we've changed the morphisms from being sets to chain complexes. With this notion we can have a property of a dg category, which is to be triangulated. This was one of the main goals of introducing everything, which is to get to triangulation. Let me recall quickly what is a triangulated category. Recall that you can always embed A a dg-category into its category of A^{op}-modules, which is like the category of modules on a dg-algebra. This is triangulated and we can do whatever we want. Sitting inside A^{op}-modules are the compact modules, x is compact if $\mathrm{Hom}(x, \quad)$ commutes with all direct sums. This goes from $\mathrm{Ho}(A^{\mathrm{op}} - \mod) \to \mathbf{k} - \mod$, and you check that this commutes with direct sums. All the representable functors become compact objects. If this map from A to compact A^{op}-modules is an equivalence, we call this triangulated. You can describe this in another way by saying that the category of compact objects is the smallest subcategory of modules containing A, sums, retracts, and cones.

With this notion it was possible to do some things that it was impossible to do in the beginning.

(1) We have functorial cones,
(2) the natural link between enriched Hom and the shift,

- you can take the tensor product of dg categories, and so of triangulated dg categories. You recomplete, so if T and T' are triangulated dg categories, then you take the regular tensor product and then re-add everything you are lacking, $\widehat{T \otimes T'}$ gives triangulated dg categories a monoidal product,
- which has a right adjoint $R\,\mathrm{Hom}$, the internal hom of triangulated dg categories.
- Since we have been describing model category structures, we can compute all (homotopy) limits and colimits. It's always complicated to actually compute, but we at least have the theory,
- and as a direct application that we have seen long ago, if you take X and Y two smooth proper schemes in algebraic geometry you get a derived category of vector bundles, and on the tensor product, you get $L_{pe}(X) \otimes L_{pe}(Y) \cong L_{pe}(X \times Y)$. People in algebraic geometry said that this was expected but that this wasn't provable with triangulated categories without dg enhancement. Nowadays people use dg categories. I don't know how much of the papers rely on this.

I realize I started with numbers and moved to itemization.

I want to make a comment about the kind of thing that Calin has been doing. Between dg categories and triangulated dg categories there is a forgetful and a completion functor, you look at A^{op}-modules and take perfect modules in it. This is described as compact objects. This is not concrete. I haven't given you a description, and this is where the twisted complexes are going to come.

A bit of notation. Toën says that *triangulation* is the same as having the zero object, sums, triangles, and idempotent-complete. In the 80s Bondal–Kapranov tell you about *pre-triangulated*, which means stable under $[+1]$, $[-1]$, and cones. This doesn't have zero objects, sums, or idempotent completeness. I'd even use the definition of Lurie, who uses *stable* for zero objects, sums, and triangles. No idempotent-complete.

So people are not assuming the same properties on the categories. You can compose your completion functors and just get the things that you want.

Say you start with A a dg-category and the first thing you want to do is add the sums and the zero object. You add, formally, the sums, you replace the objects of A with sums, so collections X_i, finite, and the new Hom is $\mathrm{Hom}(X_i, Y_j)$ is

$\prod_{i,j} \mathrm{Hom}(X_i, Y_j)$. This gives you an adjunction between dg-categories and dg-categories with finite sums.

Normally the next thing you would do is add triangles, but since this is difficult I'll do it last and I'll say how you add completion of idempotents. Take any element X and an idempotent π so $\pi^2 = \pi$. You say the category is idempotent complete if you can compute the kernel or cokernel of π. There is a way to build a category where it will exist. You create a new category $\mathcal{C}^{\mathrm{idem}}$ out of $\mathcal{C}$ where the objects are pairs (X, π), an object and an idempotent. The maps commute with the idempotents. There is an embedding of $\mathcal{C}$ in $\mathcal{C}^{\mathrm{idem}}$ because the identity is idempotent. A truly easy exercise is that the kernel of the cokernel of π is (X, π), and you created the kernels that you want to have. This works before or after taking the sums. For the hard step I'll only use, I want to build cones, shifts, and everything, I should have chain complexes. Chain complexes of elements of your category don't make sense, at the map level it makes sense but not for objects. So they build $\mathrm{PreTri}(A)$, which are families $\{X_i\}$ with endomorphisms that satisfy some relations. You have to use the dg structure you started with, and the idea is, if this is already a category of chain complexes, what should I add, how should I write it down. If you take chain complexes here, you're creating chain complexes of chain complexes. When you have a bicomplex, you want to create the totalization of it, adding up the two differentials. You have $d + d' = d''$. Then what you wish for is that $(d'')^2 = 0$. For a bicomplex you ask that d^2 and $(d')^2 = 0$, and for commutation between the two, and so the sum that comes out is zero.

I have a complex of complexes, and $d^2 = 0$. Maybe I don't want $(d')^2 = 0$, and if you try to compute what is $(d'')^2$ you get $d^2 + \partial d' + (d')^2$. You ask that $\partial d' + (d')^2 = 0$ and actually you don't need to start with chain complexes, I don't need a complete chain complex, and I recreate everything, these are the twisted chain complexes. I don't want to give the details but maybe I can ask about the cones.

If you have a map $f : X \to X'$ between those two constructions and you want to know what is the cone of f, it should be a pair (X'', d''), and you suppose you have sums in your category, and so the cone is $X''_i = X_i \mathrm{op} lus X'_{i-1}$, and $d''_{ij} = \begin{pmatrix} d_{ij} & f_{ij} \\ 0 & d'_{ij} \end{pmatrix}$ and it's the same proof as usual. They show a lot of other things, that this has lots of nice categorical properties, and this is a constructive approach. Normally if you start with a dg category and add sums, cones, and then idempotents, this should coincide with the hull defined by Toën.

That's what I wanted to say about dg categories. Let's ask any question and then just discuss our wishlist for this year.

22. November 7: Christophe Wacheux: Homotopy Algebras

(My understanding of) A_∞ algebras and A_∞ categories. I'll define what is an A_∞ algebra and I don't know if I'll define the categories, we'll see along the way. For reference, I'm following the work of Keller, if you take A_∞ algebra the first work you thought of is Keller, his student K. Lefévre-Hasegawa, but not Kontsevich, which apparently adopts a very different approach. I'm convinced it has its merits and all, but it was inaccessible and doesn't give a good introduction.

Okay, now I'm going to set $\mathbf{k}$ a field, V a graded $\mathbf{k}$-vector space, so maybe sometimes I'll just say GVS, so I mean I have $V = \bigoplus_{p \in \mathbb{Z}} V^p$ and I define $V[q]$ by

$(V[q])^p := V^{p+q}$ and you will see that I'll make an important use of this shift. If $v \in V^p$, then it is said to be homogeneous of degree p and we say $|V| = p$.

The category G, I don't know if there is conventional notation GrV, has objects graded **k**-vector spaces and morphisms, let M and L be two graded vector spaces, then $\mathrm{Hom}_{\mathrm{GrV}}(M, L)$ is a graded vector space, in category theory it means something I guess when the homs are again an object, with component

$$\mathrm{Hom}_{\mathrm{GrV}}(M, L)^r := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{Vec}(M^p, L[r]^p).$$

So f is said to be of degree r.

So of course you have to pay attention to how you define your morphisms even though they might look the same. If you shift then the degree will change.

Now I will define what is called the monoidal structure. First I'll define $M \otimes L$ to be a graded vector space with

$$(M \otimes L)^n = \bigoplus_{p+q=n} M^p \otimes_{\mathbf{k}} L^q$$

where here we have the tensor product of vector spaces, the usual tensor product.

Now if $f : M \to M'$ and $g : L \to L'$, I also need to define what is $f \otimes g$, this will go from $M \otimes L \to M' \otimes L'$, and I define this so that $|f \otimes g| = |f| + |g|$, which is a consequence of my definition, I can define it by saying that for v and w homogeneous

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w).$$

This is some trick because to get the symbols from the one order to the other order you should permute the g and the v. This amounts to a choice of a map $M \otimes L \to L \otimes M$, $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$.

The neutral element for $\otimes$ is $e := \begin{cases} e^0 = k \\ e^n = \{0\}, \ n \neq 0. \end{cases}$ So now GrV is a symmetric monoidal category. An interesting point here is that the morphisms of graded vector spaces are again graded vector spaces. Now if I have (M, d_M) a cochain complex, meaning that $d_M \in \mathrm{Hom}_{\mathrm{GrV}}(M, M)^1$ satisfying $d_M^2 = 0$, and for (M, d_M) and (L, d_L) two complexes, we equip $\mathrm{Hom}_{\mathrm{GrV}}(M, L)$ with differential δ where $\delta^r : \mathrm{Hom}_{\mathrm{GrV}}(M, L)^r \to \mathrm{Hom}_{\mathrm{GrV}}(M, L)^{r+1}$, with $\delta^r(f) = d_L \circ f - (-1)^{|f|} f \circ d_M$ of degree f.

I guess, then, f and f', morphisms of graded vector spaces (or maybe I'd better reduce to maps of complexes), are homotopic if $f - f' = \delta(h)$ for some $h \in \mathrm{Hom}_{\mathrm{GrV}}(M, L)^{|f|-1}$.

A note is that h and h' homotopic induce the same maps on cohomology. Normally I'm also supposed to set, if I shift, I set $d_{V[1]} = -d_V$.

Now we are ready for A_∞ algebras.

Definition 22.1. An A_∞ algebra is a graded vector space A with maps $b_n : (A[1])^{\otimes n} \to A[1]$ such that the degree of b_n is 1, for $n \geq 1$.

Here I should stop and make a big comment. Sometimes you want maps $A^{\otimes n}$ to A of degree $2 - n$. Understanding the difference of signs is sometimes an annoying thing.

So I just wanted to say that the link between m_n and b_n, if what I read in Lefèvre-Hasegawa, there is a formula linking the b_n and the formula linking the m_n, there are no pluses or minuses linking the b_n, but for m_n there are signs, and he said that, yes, there is no precise, no canonical choice of signs between the m_n, and, which, I think this is, uh, [some discussion]

Let's write the formula b_n satisfies.

$$\sum_{i+j+\ell=n} b_{i+1+\ell} \circ (\mathbf{1}^{\otimes i} \otimes b_j \otimes \mathbf{1}^{\otimes \ell}) = 0$$

for all $n \geq 1$.

Several comments. The advantage of defining b_n like this, now I have maps that are all of the same degree, and also, because I take this as a convention, with this I don't have sign troubles, but I'll have sign issues.

I never apply it to an element, I've never come across it. Okay, so now a representation, if I take, or realization, [pictures].

In this case he has a sign of $(-1)^{ij+\ell}$, but he says there's no canonical choice, so.

Okay so what do we have? For $n = 1$ you have $b_1 \circ b_1 = 0$ so $(A[1], b_1)$ is a complex.

For $n = 2$, I can have $b_2(\mathbf{1} \otimes b_1) + b_2(b_1 \otimes \mathbf{1}) + b_1(b_2) = 0$. If you remember the formula I erased, we know that b_2 goes from $A[1] \otimes A[1] \to A[1]$. Then $A[1] \otimes A[1]$ is a complex with differential $d_{A[1]} \otimes \mathbf{1} + \mathbf{1} \otimes d_{A[1]}$. Now if I write $\delta(b_2)$ I get that it is $d_{A[1]} b_2 - (-1)^{|b_2|} b_2 \circ (d_{A[1]} \otimes \mathbf{1} + \mathbf{1} \otimes d_{A[1]})$, and we know that this is equal to zero by the A_∞ equation (A_2). This means that b_2 is a morphism of complexes.

Now this is where it gets funny. This is also supposed to be like the graded Leibniz rule, because b_2 is actually the multiplication but here b_2 is defined on $A[1]$ so you have to get back, that's the discussion we had with you, so actually $m_2(x, y) = (-1)^{|x|} s^{-1} b_2(sx, sy)$. Normally if we check the formula, we should find out that, I'm going to switch it, I'm going to change in the formula, so I have

$$d_A \circ m_2(x, y) = m_2(d_A(x) \otimes y) + (-1)^x m_2(x, d_A(y))$$

which is graded Leibniz. Next, (A_3) implies that

$$\begin{aligned} & b_2 \circ (b_2 \otimes \mathbf{1} + \mathbf{1} \otimes b_2) \\ & \qquad + b_1 \circ b_3 + b_3 \circ (b_1 \otimes \mathbf{1} \otimes \mathbf{1}) + b_3 \circ (\mathbf{1} \otimes b_1 \otimes \mathbf{1}) + b_3 \circ (\mathbf{1} \otimes \mathbf{1} \otimes b_1) \\ & \qquad = 0 \end{aligned}$$

and when you switch to m_2 you get associativity of m_2 up to a homotopy which is more or less m_3.

For $n > 3$ you have a quadratic equality up to higher homotopy. Also, a consequence of what I said, if $b_n = 0$ for all $n \geq 3$, then we have a dg algebra and vice versa a dg algebra gives you an A_∞ algebra with $b_n = 0$ for $n \geq 3$.

What about $n = 0$? If I try to adapt the formula, allowing $n = 0$? Then applying bluntly what happens, in that case, $b_0 : \mathbf{k} \to A[1]$, you get that this would modify all the equations, and (A_0) now says that $b_1 \circ b_0 = 0$ but (A_1) tells youo that $b_1^2 = -b_2(\mathbf{1} \otimes b_0 + b_0 \otimes \mathbf{1})$ so this is what is called, this is not zero, this is what is called weak A_∞-algebra or curved A_∞ algebra. In Keller and Lefévre-Hasegawa, they say little is known. We'll speak about the rest of this next week.

23. November 14: Christophe Wacheux: Homotopy algebras II

So first I'll come back on A_∞ algebra. I had stopped giving the definition of an A_∞ algebra in particular. I'll rewrite the equation I called A_n, which is

$$\sum_{i+j+\ell=n} b_{i+1+\ell}\circ(\mathbf{1}^{\otimes i}\otimes b_j\otimes \mathbf{1}^{\otimes \ell}) = 0$$

I had a problem interpreting how this formula changes when you switch b_n (which is an operation $(A[1])^{\otimes n}\to A[1]$) to m_n (an operation $A^{\otimes n}\to A$), how we could interpret A_2 as a graded Leibniz rule on A and A_3 as an associativity up to homotopy. So it turns out that I had, I was told that there was no canonical choice in the sign between b_n and the m_n. But it's only half true. There is no canonical choice of sign, but it amounts to precisely choosing what they call the braiding, $M\otimes L\to L\otimes M$, and I choose this isomorphism $x\otimes y\mapsto (-1)^{|x||y|}y\otimes x$, and this makes my braided monoidal category into a symmetric monoidal category.

So I'd just like to make it up to you and show how the graded Leibniz rule appears for m_2, there's a little game that depends on whether I evaluate on elements of A or not. If I just set, if I try to recover the graded Leibniz rule of A, m_1, m_2 from A_2.

So first let's set $s: A\to A[1]$, if I have a homogeneous element of degree p, I go $v\in A^p$ goes to v in $A[1]^{pi-1}$, so s is of degree -1.

Then $m_2 \coloneqq s^{-1}\circ b_2\circ(s\otimes s)$ and $m_1\coloneqq s^{-1}\circ b\circ s$. These really, these start from $A\otimes A$ to A or $A\to A$. On each of these I could put a plus or minus, and that wouldn't change whether these are derivations or chain complexes or whatever. Now I have given myself a braiding and I didn't write any elements, and when I write those I'll have to be careful.

So now let's write the Leibniz rule, let a and b be in A, and then I'm going to show that the graded Leibniz rule is equivalent to A_2. So I write

$$m_1\circ m_2(a\otimes b) = m_2(m_1(a)\otimes b) + (-1)^{|a|}m_2(a\otimes m_1(b)).$$

This is my graded Leibniz rule. Now I want to write this right hand side as

$$m_2\circ(m_1\otimes\mathbf{1})(a\otimes b) + m_2\circ(\mathbf{1}\otimes m_1)(a\otimes b)$$

because of the formula $(f\otimes g)(u\otimes v) = (-1)^{|g||u|}f(u)\otimes g(v)$. Then I have

$$m_1\circ m_2 = m_2\circ(m_1\otimes\mathbf{1}+\mathbf{1}\otimes m_1).$$

Now I just exchange the formulas:

$$\begin{aligned}&s^{-1}\circ b_1\circ s\circ s^{-1}\circ b_2\circ(s\otimes s)\\ &\qquad\qquad = s^{-1}\circ b_2\circ(s\otimes s)\circ((s^{-1}\circ b_1\circ s)\otimes\mathbf{1}+\mathbf{1}\otimes(s^{-1}\circ b_1\circ s))\end{aligned}$$

and so expanding and cancelling (say, the s^{-1} at the beginning) I have

$$\begin{aligned}b_1\circ b_2\circ(s\otimes s) &= b_2\circ((-1)^{|s||s^{-1}b_1 s|}(b_1\circ s)\otimes s + s\otimes b_1\circ s)\\ &= b_2\circ(-(b_1\otimes\mathbf{1})\circ(s\otimes s) - (\mathbf{1}\otimes b_1)\circ(s\otimes s))\end{aligned}$$

which is

$$b_1\circ b_2\circ(s\otimes s) = -b_2\circ(b_1\otimes\mathbf{1}+\mathbf{1}\otimes b_1)\circ(s\otimes s)$$

and then cancelling the $s\otimes s$ you get precisely A_2.

This was a bit tedious but I found it a bit instructive. But let's just do this once.

Going on, I have an example taken from Keller but I'll skip it for later, especially since I cannot push it until the end, and let's just define morphisms and quasi-isomorphisms of A_∞ algebras.

Definition 23.1. A morphism of A_∞ algebras $f : (A, b^A) \to (B, b^B)$ is given by a family of homogeneous maps f^n with $|f^n| = 0$ which go from $A[1]^{\otimes n} \to B[1]$ and verify for all $n \geq 1$

$$\sum_{i+j+\ell=n} f_{i+1+\ell}(\mathbf{1}^{\otimes i} \otimes b_j^A \otimes \mathbf{1}^{\otimes \ell}) = \sum_{i_1+\cdots+i_s=n} b_s^B(f_{i_1} \otimes \cdots \otimes f_{i_s}).$$

Note that $\tilde{f}_1 \coloneqq s^{-1} \circ f_1 \circ s : (A, m_1) \to (B, m_1)$ is a morphism of complexes. It is compatible with m_2 up to a homotopy given by $\tilde{f}_2 \coloneqq s^{-1} \circ f_2 \circ (s \otimes s)$.

In particular, $\tilde{f}_1$ induces a morphism of algebras at the homological level, $H[f_1] : H^*(A) \to H^*(B)$. I should have said that $(H^*(A), H[m_2^A])$ and $(H^*(B), H[m_2^B])$ are (non-unital) algebras. Remember that if you look at A_3, this is something like $m_2 \circ (\mathbf{1} \otimes m_2) = m_2 \circ (m_2 \otimes 1) + \delta(m_3)$, so that on the homology you get true associativity. So now there is a notion of a homological unit, which is called, for instance in Keller.

Definition 23.2. A *homological unit* is a map $\eta_A : \mathbf{k} \to A$ such that $d_A \eta_A = 0$ and it induces a unit for $(H^*(A), H[m_2^A])$.

You can also define a strict unit, where $f : \mathbf{k}[1] \to A[1]$ has f^1 essentially the unit and $f^n = 0$ for $n \geq 2$. [There is something wrong here]. Let me try again. A strict unit is a map $\mathbf{k} \to A[1]$ of degree -1 such that $b_2(\mathbf{1} \otimes e) = \mathbf{1} = b_2(e \otimes \mathbf{1})$ and $b_n(\mathbf{1}^{\otimes i} \otimes e \otimes \mathbf{1}^{\otimes j}) = 0$ if $i + j$ is not 1.

Now you should check immediately that this is a unit for H and the remark I made last week is that in Fukaya–Oh–Ohta–Ono, there is another notion of homological unit, where this one only uses m_2^A and not the higher homotopy, and there there is another definition and this, apparently, when you do, Lefèvre-Hasegawa says when you look a cohomology over a field all of the kinds of units are the same. These other people (Lyubashenko, e.g.) work over a ring. It could turn out to be important because I think the Fukaya category is written over a ring, what they call the Novikov ring, and if at some point you look at the Fukaya category it might turn out that the unit will bother us.

So just, we say f is an A_∞ *quasi-isomorphism* if f_1 is a quasi-isomorphism. I define

$$(f \circ g)_n = \sum_{i_1+\cdots+i_s=n} f_s \circ (g_{i_1} \otimes \cdots \otimes g_{i_s})$$

and the identity morphism of $A[1]$ has $f_1 = \mathbf{1}$ and all higher components zero.

This defines A_∞ algebras as a category. Of course it's not yet an A_∞ category and I promised Damien that I'd give a definition before the end so I'll rush.

Actually there is a lot of stuff, when you can write, explaining the relation between A_∞ algebras and dg algebras, there are propositions relating the two, the fact that if you have an A_∞ algebra and a map to a complex V then you can push the structure of the A_∞ algebra, there is a structure on V so that the map is a map of A_∞ algebras. I won't write it, I'll come back to it if I have time, but I want to be sure that I've written the definition of an A_∞ category. Is that okay?

Somewhere I read that A_∞ categories are a "horizontal categorification" of A_∞-algebra, meaning that an A_∞ category, well you ask yourself if you know what is an A_∞ algebra, then an A_∞ algebra is an A_∞ category with one object. There is no well-defined "horizontal categorification" but anyway. You can define an A_∞ category as an A_∞ algebra over some category, but it required some stuff that I didn't think we should go through.

Definition 23.3. An A_∞ category $\mathbb{A}$ is the data of

- A set $\operatorname{obj}(\mathbb{A})$ of objects,
- For A and A' objects, a graded vector space $\operatorname{Mor}(A, A') =: \mathbb{A}(A, A')$
- For any sequence $A_0, \ldots, A_n$ of objects, an operator b_n
$$\mathbb{A}(A_0, A_1)[1] \otimes \cdots \otimes \mathbb{A}(A_{n-1}, A_n)[1] \to \mathbb{A}(A_0, A_n)[1]$$
of degree 1 (for $n \geq 1$) that verify the relations A_n (this is maybe not quite so easy to understand I guess):
$$\sum_{i+j+\ell=n} b_{i+1+\ell} \circ (\mathbf{1}^{\otimes i} \otimes b_j \otimes \mathbf{1}^{\otimes \ell}) = 0$$
and I think this is misleading because all the $\mathbf{1}$ have different domains, say. I'd rather write

$$\sum_{i+j+\ell=n} b_{i+1+\ell} \circ (\mathbf{1}_{\mathbb{A}(A_0,A_1)[1]} \otimes \cdots \otimes \mathbf{1}_{\mathbb{A}(A_{i-1},A_i)[1]} \otimes b_j \otimes \mathbf{1}_{\mathbb{A}(A_{i+j},A_{i+j+1})[1]} \otimes \cdots \otimes \mathbf{1}_{\mathbb{A}(A_{n-1},A_n)}) = 0$$

I have like ten minutes left. I can stop here or give a couple of properties I mentioned about dg algebras or the example in Keller related to Hochschild cohomology.

24. November 28: Weonmo Lee /Byung Hee An

I talked about the model category, where W consists of the quasi-isomorphisms, F consists of the degreewise surjective morphisms and the cofibrations are those that lift against the acyclic fibrations. This gives a model structure on differential graded algebras over $\mathbf{k}$. I mentioned five axioms last time. The last thing I didn't prove was the fourth condition, which describes, for any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

where $A \to B$ is a cofibration, $X \to Y$ is a fibration, and one of the two is a weak equivalence, then there exists h from B to X making the diagram commute. So for the proof we will use the lemma, we had the lemma last time, by following Jardine's paper, he assumes that i is a trivial cofibration. Then he describes, the fifth axiom, I proved this last time

Lemma 24.1. *Any morphism $f : X \to Y$ can be factorized as a cofibration followed by a fibration, $q \circ j$ either one of which can be assumed acyclic.*

So the proof of the fourth axiom, suppose we have the trivial cofibration from A to B, then it can be factored as j and then q. By construction, the thing in the middle was $A * (*_{b\in B} G(b))$ which I will denote $\bar{B}$. This is a kind of filtered colimit of the tensor algebra. So you can write for any trivial cofibration $A \to B$ the following commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \bar{B} \arrow[d] \\ B \arrow[r] & B \end{tikzcd}$$

and then $\bar{B} \to B$ is an acyclic fibration and so we have a map from B to $\bar{B}$ making the diagram commute. So then i is a retract of j. This means that i and j fit into a diagram as follows:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"] \arrow[rr, bend left, "id"] & A \arrow[r] \arrow[d, "j"] & A \arrow[d, "i"] \\ B \arrow[r] \arrow[rr, bend right, "id"'] & \bar{B} \arrow[r, "q"] & B \end{tikzcd}$$

So one thing that is not clear to me is that in the proof we just mentioned, this is a trivial fibration and so you have a lifting.

[long discussion]

24.1. **The bar functor.** Okay, my title is the bar functor. I want to define functors between the categories of augmented algebras and the coaugmented coalgebras. I'll also use A_∞ algebras. I have a non-full subcategory inclusion of algebras and A_∞ algebras; I also have a full subcategory inclusion of cofibrant fibrant coalgebras and coalgebras. I have bar and cobar between algebras and coalgebras. I want to define a bar functor B_∞ from A_∞ algebras to the cofibrant fibrant coalgebras which is an equivalence of categories. I want to show that all of these functors induce equivalences on the homotopy categories.

Actually, using this, one can think, well,

Proposition 24.1. *Let A be an A_∞ algebra over $\mathbf{k}$. Then there exists an algebra $U(A)$ such that $A \to U(A)$ is a quasi-isomorphism, this is just the composition of the three functors, $\Omega B_\infty A$.*

Let's start with algebras. $\mathbf{k}$ is a field and algebras are always augmented and unital. Unital means it has an element 1. Augmented means there is an map $\epsilon : A \to \mathbf{k}$ which sends 1 to 1. This has a model structure with weak equivalences the quasi-isomorphisms and fibrations the surjections. The cofibrations have some lifting property.

There is a theorem, that these three define a model structure on differential graded (augmented unital) algebras.

Now I want to define coalgebras and the bar and cobar construction. Here I want augmented, dg coalgebras, it's basically a chain complex, it has a differential and a grading, and it has another operation, called a coproduct, $(C, \Delta, d, \epsilon, \eta)$. This is $\Delta : C \to C \otimes C$ satisfying $\Delta \circ d = (1 \otimes d + d \otimes 1) \circ \Delta$. Our ϵ is a coaugmentation $\mathbf{k} \to C$ and η is a counit $C \to \mathbf{k}$. This satisfies $\eta \circ \epsilon = 1_{\mathbf{k}}$.

Now let V be a complex. Then $TV = \bigoplus V^{\otimes n}$. One can assign a coproduct here like this: $\Delta : TV \to TV \boxtimes TV$ (introducing $\boxtimes$ to separate the tensors) then

$$(v_1 \otimes \cdots \otimes v_n) \mapsto \sum (v_1 \otimes \cdots \otimes v_i) \boxtimes (v_{i+1} \otimes \cdots \otimes v_n).$$

This sum ranges from 0 to n, and one can prove that this defines a coproduct.

I want to define a coaugmentation and unit map. You have an inclusion $k \to V^{\otimes 0}$ and the projection is the counit.

So to show that this has a coalgebra structure I write T^cV, regarding this as a coalgebra.

So if you have a complex V then we can make TV an algebra, and this construction is kind of a free functor, from complexes to algebras, and this satisfies some universal property. We have a canonical map $V \to TV$, and given a map from $V \to C$, a dg map to an algebra map C.

As a coalgebra TV is not cofree. If you have a map $T^cV \to V$, even if you have a map, a $\mathbf{k}$-linear map $C \to V$, we might not be able to fill with an arrow $C \to T^cV$.

So maybe our, there is another construction to make something slightly bigger, a bigger coalgebra which is cofree, but in this talk I want to shrink our category of coalgebras. Instead of thinking of all coalgebras, I want to consider some "cocomplete" coalgebras. We say C a coalgebra is cocomplete if $C = \bigcup \ker(C \to C^{\otimes n} \to (C/\mathbf{k})^{\otimes n})$. If you apply the coproduct $(n-1)$ times, then mod out by the scalar part at each factor, the kernel here means that if we take some iterated coproduct, then there is eventually some scalar in each factor. Then Cog is our category of cocomplete coalgebras.

One can prove that $T^cV \to V$ is now a cofree cocomplete coaugmented coalgebra. Any element is a sum up to a finite length of tensors. If you take the coproduct more than n times you will have at least one scalar factor. So it's cocomplete (conilpotent). Now it's cofree in this category.

Now let A be a dg algebra and consider a chain complex of $\mathbf{k}$-linear maps $\mathrm{Hom}_{\mathbf{k}}(C, A)$, and these are both graded, and we want a differential on here and a multiplication. So $d(f) = d \circ f - (-1)^{|f|} f \circ d$. Then the product $f * g$ (for μ the product on A) is $\mu \circ (f \otimes g) \circ \Delta$. We need to check that this differential is a derivation with respect to this product, but having done so, then $\mathrm{Hom}_{\mathbf{k}}(C, A)$ is a dg algebra.

Here's a definition.

Definition 24.1. A *twisting cochain* τ in $\mathrm{Hom}^1(C, A)$ is a element that satisfies $d\tau + \tau * \tau = 0$

So then $Tw(C, A)$ will be the set of twisting cochains, and this gives for fixed A a functor from Cog to sets. This is a subset of $\mathrm{Hom}(C, A)$. If it defines a functor it should be contravariant.

So given $f : C \to D$, we can ask if $\tau \circ f$ is a twisting cochain for τ twisting in D. Then we see $d(\tau \circ f) + (\tau \circ f) * (\tau \circ f) = d \circ \tau \circ f + \tau \circ f \circ d + \mu \circ (\tau \circ f) \otimes (\tau \circ f) \circ \Delta$ and since f commutes with coproducts and differentials this is $(d\tau + \tau * \tau) \circ f$ which is zero.

So this is really a functor. This functor is (co)representable by BA, so $Tw(C, A) \cong \mathrm{Hom}(C, BA)$, and so replacing C with BA we get a special (universal) twisting cochain τ_0.

So $BA = T^c(SA)$ along with a differential D which has two components, one which looks like $1^{\otimes a} \otimes d_{sA} \otimes 1^{\otimes b}$ and the other given by $1^{\otimes a} \otimes b_2 \otimes 1^{\otimes (b-1)}$. You have another functor, fixing C instead of A, defining a set $Tw(C, A)$, and one can prove that this is a functor. The representation is denoted by ΩC, so that Hom (in algebra) between ΩC and A is in bijection with $Tw(C, A)$. Indeed ΩC is the tnsor algebra of $T(s^{-1}C)$ along with a differential.

So Ω and B are adjoint to each other.

Theorem 24.1. *(Lefévre–Hasegawa) these form a Quillen equivalence. This means that they preserve the model category structure and moreover induce an equivalence of categories on the homotopy level.*

Secondly, all algebras are fibrant and all coalgebras are cofibrant. So A is cofibrant if and only if it is a retract of a cobar of something; C is fibrant if and only if it is quasi-free.

The homotopy relation between two maps, f and g are morphisms between A and A' fibrant and cofibrant objects. Then $f \sim g$ if and only if there exists a homotopy $h : A \to A'$ of degree -1 such that (some augmentation condition is satisfied) and $h \circ \mu_A = \mu_B \circ (f \otimes h + h \otimes g)$ and $f - g = dh + hd$.

I didn't say anything about the model category structure in coalgebras. In coalgebras, the weak equivalences are those whose image $\Omega(f)$ is a weak equivalence in algebras. The cofibrations are injections. The fibrations have the lifting property.

Let's see how this result is related to A_∞-algebras and minimal models. An A_∞ algebra (maybe with augmented strict unit), as before, C is a coalgebra, cocomplete (or conilpotent) and we want to consider all the $\mathbf{k}$-linear maps from C to A. By using the A_∞ algebra structure, we can define an A_∞ structure on $\mathrm{Hom}(C, A)$ as $b_n(f_1 \otimes \cdots \otimes f_n) = b_n^A \otimes (f_1 \otimes \cdots \otimes f_n) \circ \Delta^{(n)}$.

As before I want to define twisting cochains, as the set of morphisms τ satisfying the Maurer–Cartan equation,

$$\sum b_n(\tau, \ldots, \tau) = 0$$

Now fix A. Whenever we choose a coalgebra C, we can assign a set $\mathrm{Tw}(C, A)$, and one can prove that this is functorial and moreaever representable, by $B_\infty A$. Then $\mathrm{Tw}(C, A) \cong \mathrm{Hom}_{\mathrm{Cog}}(C, B_\infty A)$. Practically, $B_\infty A$ is $T^c(sA)$ with some differential. Then actually, in fact, if you use V a graded vector space then there is a bijection of sets between A_∞ structures on V and differentials on the coalgebra structures on $T(sV)$. Another fact, the hom sets, $\mathrm{Hom}_{A_\infty}(A, A') \cong Hom_{\mathrm{Cog}}(B_\infty A, B_\infty A')$, so B_∞ as a functor from A_∞ algebras to coalgebras is fully faithful.

Moreover, there's a theorem

Theorem 24.2. *(Lefèvre–Hasegawa) This functor B_∞ is essentially surjective to cofibrant-fibrant objects.*

So any cofree coalgebra has some A_∞ algebra with equivalent B_∞.

Secondly, for each C there exists a minimal model and if A has $A_{\min}$, then $B_\infty(A_{\min})$ is the minimal model of $B_\infty(A)$.

So we have a diagram that I drew at the beginning.

$$\begin{tikzcd}
\mathrm{Alg} \arrow[r] \arrow[d, shift left, "B"] & \mathrm{Alg}_\infty \arrow[d, "B_\infty"] \\
\mathrm{Cog} \arrow[u, shift left, "\Omega"] & \mathrm{Cog}_{cf} \arrow[l, "\text{fully faithful}"']
\end{tikzcd}$$

I was supposed to say more but let me just give one more example. I mentioned about the classification of fibrant and cofibrant objects. Actually in Alg, when A is cofibrant then it must be free as a graded algebra but the converse is not true. So say $|1| = 1$, then $\mathbf{k} \oplus \mathbf{k}1 + \cdots$ and $d(1) = 1 \otimes 1$. Then by using this you can show that this has homology that vanishes except at the bottom, and so you have a trivial "cofibration" from the trivial algebra. Then it's clear that $\mathbf{k}$ is fibrant cofibrant, so that we should be able to invert the map up to homotopy. But we can't.