INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS C^0-SYMPLECTIC TOPOLOGY AND DYNAMICAL SYSTEMS CONFERENCE JANUARY 20–24, 2014

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1. Marie-Claude Arnaud

I want to work with certain kinds of maps. Let me give two classes: symplectic twist maps of the two dimensional annulus and Tonelli Hamiltonians on T^*M for any closed M.

Let me start with f a twist map of the annulus, analytic and symplectic, $\mathbb{T}^\times\mathbb{R} \to \mathbb{T} \times \mathbb{R}$, and if this is filled with invariant essential (continuous) loops. A question of J. Mather is: are the loops analytic? We don't even know that the loops are C^1.

What is not related to this question? Consider the rigid pendulum, with Hamiltonian $H(q,p) = \frac{1}{2}p^2 + as2\pi q$. Then you have an invariant curve that is not C^1 but it's not a counterexample because your loops are non-essential.

If f is C^ℓ and has a C^k foliation into C^r loops that are invariant with $\ell \geq r \geq k$. Then

(1) for all Γ in the foliation, we have that $f|_\Gamma$ is C^r conqugate to a C^r orientation preserving diffeomorphism of $\mathbb{T}$.
(2) We know things about these; they have rotation numbers. If $rot(f|_\Gamma)$ is rational, then $f|_\Gamma$ is completely periodic and Γ is C^ℓ.
(3) If the rotation number is irrational and $r \geq 2$ then $f|_\Gamma$ is C^0-conjugate to a rotation.
(4) If $k \geq 1$ then there is aC^{k-1}-symplectic diffeomorphism h such that $h \circ f \circ h^{-1}(q,p) = (q + z(p), p)$ with $z : \mathbb{R} \to \mathbb{R}$ which is C^{k-1}.

In general if f is a smooth symplectic twist map that is C^0-integrable, then we can say

(1) If Γ is in the foliation and $rot(f|_\Gamma)$ is rational, then Γ is smooth and $f|_\Gamma$ is C^∞ conjugate to a rotation.
(2) there exist a dense G_δ subset of the foliation such that Γ in this subset is C^1. This is a 2009 result.
(3) There exists a dense invariant subset U of $\mathbb{T} \times \mathbb{R}$ with positive Lesbegue measure such that for every invariant Γ, Γ is smooth and $f|\Gamma$ is smoothly conjugated to a Diophantine rotation.

[Some discussion about what a twist map is.]

Theorem 1.1. *(Bialy, 1993) We know we can define symplectic coordinates for a billiard map. If the billiard map is C^0 integrable, then the billiard table is a standard round disk and the billiard map is C^∞ integrable.*

There are cases where the one kind of integrability implies the other.

Now I will speak about Tonelli Hamiltonians on T^*M. Maybe I will recall what is a Tonelli Hamiltonian. Let $H \in C^2(T^*M, \mathbb{R})$ with flow φ_t^H. Then H is C^0 integrable if there is an invariant foliation into C^0 Lagrangian graphs, the image a continuous closed 1-form on M.

Remark 1.1. *Such an invariant is Lipschitz.*

So saying it's Lagrangian is just saying it's differentiable Lebesgue everywhere and then saying it is Lipschitz is saying the tangent space is Lagrangian.

Definition 1.1. *We call H C^k, C^ℓ integrable ($\ell > k$) if it has an invariant C^k foliation into C^ℓ-Lagrangian graphs. We say it's C^k integrability if there exist D Hamiltonians $H_1, \ldots, H_D$ such that at every point $DH_1, \ldots, DH_D$ are independent and $H_1, \ldots, H_D$ commute in the Poisson sense (so that the flows commute) [ed: I believe D is the dimension of M?]*

The following are well-known:

(1) For $k \geq 2$, C^k, C^k integrability implies C^k integrability.
(2) If H is C^k integrable then $H_1 = C_1, \ldots, H_D = C_D$ are embedded Lagrangian tori and $\varphi_t^H|\Gamma$ is C^{k-1}-conjugated to a rotation flow.
(3) Because H is Tonelli, in this case, if H is C^k-integrable, then M is $\mathbb{T}^D$.

There is a remark. Because, well, this is called multidimensional Birkhoff theorem, from 2010, which says that any invariant C^1 manifold that is Hamiltonian-ly isotopic to a Lagrangian graph is a graph. You can change the definition of C^k integrability but they will actually be graphs. You can prove in fact that C^k, C^k integrability is exactly C^k integrability if $k \geq 2$. In the smooth or C^2 case, you know the whole dynamics. So my question is what happens for C^0 integrability.

I have to mention some rigidity theorems. There are some cases where you know that C^0 integrability implies smooth integrability. If your Tonelli Hamiltonian is a Riemannian metric, well, if a Riemannian C^∞ metric on the torus is C^0-integrable then it is flat and C^∞ integrable. In this case, C^0-integrability implies smooth integrability. The same year (1994), Heber proved that a Riemannian metric with no conjugate point is C^0-integrable.

With several collaborators, we proved the same results for Torelli Hamiltonians with no conjugate points on $\mathbb{T}^\ell$.

Now let me give some results in the general case.

Theorem 1.2. *If H is C^0 integrable (assume it's Tonelli) then*

(1) *There is a dense G_δ invariant subset of T^*M that is filled by C^1 invariant graphs.*
(2) *If H is C^3, The Lyapunov exponents of every invariant invariant [missed] the metric and topological entropy is [missed]*

Theorem 1.3. *Let H be smooth and defined on the cotangent bundle of the torus of dimension D. Say it's C^0 integrable. There is a dense invariant subset with positive Lesbegue measure such that any invariant graph contained in it is smooth and the dynamics is conjugate to a Diophantine rotation flow. From that there is a G_δ subset of invariant graphs on which the dynamics are strictly ergodic. In fact we cannot prove that it is a rational flow, but it is something, it has some common ponit. It has a unique invariant Borel probability measure for this restriction supported [missed].*

To prove these theorems, there are two steps, to prove a normal form and then to apply a standard argument. To obtain a normal form, what we can do is apply a result for Riemannian metrics. I will explain what are the tools for the first theorem. Green bundles were introduced to prove in 1958 [some result]. They are defined so that $V(x) = KerD\Pi(x) \subset T_x(T^*M)$ with $\Pi : T^*M \to M$. We have $G_+(x)$ as the limit under flow of $D\varphi_t V(\varphi_{-t}x)$. With $G_-(x)$ you interchange $\pm t$. There are some properties. [List of properties]. The set where $G_- = G_+$ is G_δ and they are continuous there. There is a dynomical criterion for belonging to G_+ and G_-.

So what do you do. There is a dense subset of invariant C^∞ graphs filled by periodic orbits. These are in both G_- and G_+, you have a dense G_δ set, and by standard arguments you get places where they are equal. Then you use another result that I didn't write down. After for the other result, you use the other facts and another argument. I'll stop here.

2. M. Mazzucchelli, On the Multiplicity of Isometry-Invariant Geodesics

The celebrated closed geodesics conjecture says that every closed Riemannian manifold of dimension greater than one has infinitely many closed geodesics. This problem goes back to Poincaré and Hadamard and inspired Morse to develop Morse theory.

The conjecture is a theorem in many cases, for many classes of smooth manifolds. Let me mention the most relevant for my talk. So it is a theorem for non-simply connected manifold as soon as the fundamental group is abelian and infinite. If the fundamental group contains $\mathbb{Z}^2$ it's an exercise, but $\mathbb{Z}$ plus a finite group it's already hard. This was due to Bamgert-Hingston. I mean for every Riemannian metric on such a manifold. What about the simply connected case? The conjecture is still true provided that the cohomology with rational coefficients has at least two generators (as a ring). This was a celebrated result of Gromoll and Meyer in the 60s. After this, what's left is all the manifolds that don't satisfy this. Spheres, projective spaces. The last, most relevant for our conjecture is due to Bungert-Franks-Hingston, for S^2.

Metric geodesics, I mean Riemannian metrics and geodesics, not Finsler. The third result doesn't hold in the same way in S^2. Finding three closed geodesics doesn't work as well there.

If n is large, say, greater than five, we can't even find a second geodesic. My talk will not be about Hamiltonian dynamics but not about C^0 topology at all. This is the closed geodesics problem. What I'm going to do is study it using symmetry.

Let me give you the setting of the problem.

This goes back to Grove, who formulated it in the 70s. You have a Riemannian manifold (M, g) and some symmetry, an isometry I. If you study closed geodesics and have a symmetry, you'd like to study the image in the quotient. But that's not a manifold. So the interesting thing is to look for so-called I-invariant geodesics, which are curves $\gamma : \mathbb{R} \to M$, immersed, such that they are invariant by I in the strong sense. $I(\gamma(t)) = \gamma(t + \tau)$ for all $t \in \mathbb{R}$ and τ is a positive constant. The fact that the constant is nonzero is crucial. Every geodesic is invariant by the identity, but in this sense the only identity-invariant geodesics are the closed ones.

The problem I want to study is the multiplicity of these isometry-invariant geodesics. Maybe there are always infinitely many. This turns out to be too optimistic. One can immediately find a counterexample. One of the first results of Morse theory was the result that that every Riemannian manifold has at least one closed geodesic. So here there are examples with no isometry invariant geodesics. Take the torus $\mathbb{T}^2$ with the flat metric, the squares, and identify the edges pairwise. Take the isometry that rotates the square ninety degrees. It's easy to see that there are no I-invariant geodesics. If you look closely at this example, what goes wrong is that the isometry I is not homotopic to the identity.

We will see in a while that when an isometry is homotopic to the identity we can always find one I-invariant geodesic. So the next conjecture is that when an isometry is homotopic to the identity there are always infinitely many I-invariant geodesics. This is also false.

Take the round sphere, and a rotation of the sphere. The only I-invariant geodesic is the equator. The situation is definitely different for closed geodesics. One can formulate the problem for Finsler geodesics, but I'll just discuss the Riemannian case.

In view of these examples, we have the question, when is it true that there are infinitely many? This question was raised by Grove, who also proved the first theorem in this direction. This is not very hard to prove but somehow clever.

Theorem 2.1. *If in this setting there exists a non-closed I-invariant geodesic γ then see γ as a submanifold and take its closure. Then the closure contains infinitely many other I-invariant geodesics. It's not hard to prove that it contains at least one other, and then a Baire category argument shows that there must be uncountably many.*

Before discussing this theorem further, let me tell you how we attack this conjecture.

2.1. **Variational setting.** The closed geodesics are the critical points of the energy, and we want to make a setup like that here.

The space of isometry-invariant curves $\Lambda(M, I) = \{\gamma : R^{W^{1,2}_{loc}} \to M | I(\gamma(t)) = \gamma(t+1)\}$ (reparameterized so that $\tau = 1$).

The energy of γ is

$$E(\gamma) = \int_0^1 g(\gamma, \gamma) dt$$

The critical points of E, integrating from 0 to 1 is the same as integrating from t to $t+1$ so the critical points are exactly the I-invariant geodesics.

Now $\mathbb{R}$ acts on Λ by reparameterization. $(\tau \cdot \gamma)(t) = \gamma(t + \tau)$.

This function is invariant by the action of $\mathbb{R}$. Whenever you find a critical point, you have a full line of critical points, that's not a big deal. The orbit will be denoted $\mathbb{R} \cdot \gamma$. Either γ is a nonperiodic curve in which case you get $\mathbb{R}$ embedded in your manifold. Otherwise you get S^1. So Grove's theorem tell us that the only case we have to keep in mind is the S^1 case because in the other case we already have infinitely many. The isometry maps γ_0 to γ_1. and then comes back to γ_0 with period at least 1.

Here's a warning, here's the main issue. Every time you have such a γ, it comes with infinitely many critical orbits of the energy. Why is this the case? The reason is, let me give you notation. Let me say that $\gamma^\tau(t) = \gamma(\tau t)$. Then γ^{np+1}, in period

1, winds around itself n times and then joins γ_0 and γ_1. This is a different point of our space. If we detect critical points, we want to identify all critical points of this form. We want distinct geometric objects.

As in the closed geodesics problem, the question becomes, what are the properties of the sequence of critical points coming from one of these geodesics. This was addressed by Grove and his student Tanaka. Even though this seems very different, the properties are the same as in the closed geodesics problem. There one studied the behavior of the index of the energy. Iterating is taking γ^{mp+1} and seeing its index as a function of m. This was first studied by Bott in the case of closed geodesics. The result is that it grows linearly. This would be enough to prove theorems about the multiplicity of closed geodesics for generic Riemannian metrics. I'm interested in the non-generic setting. Then the critical points may be degenerate. The Hessian can have a kernel. You study along with the index the nullity of the critical point, the nullity of the Hessian. He found that the nullity is bounded uniformly. There's a third index that describes degenerate critical points (these are very complicated and hard to classify) so another index is the local homology of γ^{mp+1}, the rank of that local homology. I don't want to give you the definition but you should have in mind that it's the homology you need to attache to the domain when you pass the critical point. For non-degenerate ones, you attach a handle, and you get the homology of the handle. For degenerate critical points you get more complicated things.

Right away they found the corollary on isometry invariant geodesics.

Corollary 2.1. *Suppose M is simply connected, $H(M)$ has more than one generator as a ring, and I is homotopic to the identity (not via isometries). Then there are infinitely many I-invariant geodesics.*

The third condition tells you that the space of isometry invariant curves is homotopy equivalent to the loop space. This is an exercise. What is not trivial is that the second condition, due to [unintelligible] and Sullivan, is equivalent to the fact that the Betti numbers of the loop space are unbounded. Therefore the Betti numbers of the domain of our energy are unbounded. Each point in the sequence gives a bounded contribution to the homology, so that's roughly the proof.

I want to come to my contribution. I studied the non-simply connected case. On the Riemannian manifolds whose fundamental groups contain $\mathbb{Z}^2$, it's an exercise to find the infinitely many I-invariant geodesics. So the problem becomes hard when you don't have that.

Theorem 2.2. *There are infinitely many I-invariant geodesics if one of the following is true:*

(1) *$I \cong id$ and $M = M_1 \times M_2$, with $dim M_2 \geq 2$ and $rk(H_1(M_1)) \neq 0$*
(2) *$I \cong id$ and $\pi_1(M)$ is Abelian and contains [is?] $\mathbb{Z}$ and another nontrivial group. The hard case is when H is finite. The conjecture is that I expect that H trivial will be fine as well.*

Instead of discussing the proof, I'll discuss a generalization to the contact setting.

Are there questions on the statements?

The geodesic flow can be realized as a Reeb flow on the unit cotangent bundle and isometries can be lifted too. So it's natural to restate this in the contact setting in terms of invariant Reeb orbits.

Take a contact manifold Y^{2d+1}, $d \geq 1$ with a contact form α. I'm sure everyone here knows the definition: $\alpha \wedge (d\alpha)^d \neq 0$, you get a volume form this way. You have Reeb dynamics from the Reeb vector field R which is defined by means of $\alpha(R) = 1$ and $(R, d\alpha) = 0$.

There is a conjecture due to Weinstein, let me call ρ_t the Reeb flow (along this vector field), that ρ_t has closed Reeb orbits. There are some cases where this is known. Very often there are several closed Reeb orbits.

The generalization would be the generalization in terms of symmetry. So there should be a contactomorphism, and more, we should have $\phi^*\alpha = \alpha$. It's easy to see that $\rho_t \circ \phi$ will tequal $\phi \circ \rho_t$ in this case. So we should study Reeb orbits that close up in the quotient.

Definition 2.1. *The curve $\gamma : \mathbb{R} \to Y$ is a ϕ-invariant Reeb orbit if it is a Reeb orbit ($\gamma(t) = \rho_t(\gamma(0))$) and $\phi(\gamma(t)) = \gamma(t + \tau)$ for all $t \in \mathbb{R}$ for some constant τ.*

There are contact manifolds with no Reeb orbits. Let's say a manifold has a contactomorphism isotopic to the identity. Then this lifts to the cotangent bundle.

Conjecture 2.1. *ρ_t should have at least one ϕ-invariant Reeb orbit provided ϕ is contact isotopic (via contactomorphisms) to the identity. I guess this is a good time to stop. Thank you.*

3. Topological contact dynamics and its applications

It's great to be back at POSTECH. Let me start with some references. Everything is contained in a series of three papers, altogether about 100 pages, Topological Contact Dynamics I, II, and III. This is all joint work with Peter Spaeth. I'll have to pick and choose. I have to pick some notation. Let H be a time dependent function $H : [0, 1] \times M \to \mathbb{R}$, not necessarily smooth. I'll be specific if I remember about when it is smooth. Then Φ will be an isotopy ϕ_t for $0 \leq t \leq 1$, $\phi_t : M \to M$. We will have ϕ_t either a diffeomorphism or a homeomorphism. If it's smooth then such Φ are in one to one correspondence with time dependent vector fields (this is an ODE with initial condition).

Okay, let me recall some smooth dynamics. We either have Hamiltonian dynamics or Hamiltonian and contact or contact dynamics. Let me make a table.

Hamiltonian	Hamiltonian and contact	contact
(M^{2n}, ω) symplectic, $\omega^n \neq 0$		$(M, \xi = ker\alpha)$ contact (cooriented): $\alpha \wedge (d\alpha)^{n-1}$
$\omega(X_H^t,) = dH_t$		$d\alpha(X_H^t,) = (R_\alpha H)\alpha - dH_t$ and $\alpha(X_H^t) = H$

So we have a function giving us a vector field. We want to look at coordinate transformations which in the symplectic world is pulling back the symplectic form and pulling back the contact form, which may not be preserved (although its kernel is)

We have a one to one correspondence between H and Φ_H if we apply some normalization. In the symplectic case we choose mean value zero with respect to ω^n and in the other side we have a choice of α but not mean value zero because constant functions correspond to nontrivial isotopies.

Everything depends on the choice of normalization. Many things turn out though to be independent of α. The precise dependence on α or independence of it, in topological contact dynamics is the same as in the smooth case.

There is a transformation law when I pull back an isotopy, it tells me what the corresponding function is. I'll come back to that more precisely.

Let's get to definitions, comparing this to topological Hamiltonian dynamics. I want to write that once and only once, I'll say THD henceforth. There are a lot of things this has in common with topological contact dynamics (TCD). The left part is joint with Yong-geun Oh. The right column is joint with Peter. You have a sequence of objects Φ_H, H which limit in some sense (via Cauchy sequences) to Φ, H. In the TCD world, I'll talk about what we have in a little bit.

Let's start with these functions. Assume these functions converge to some H which might not be smooth, in the sense $||H_i - H|| \to 0$, where $||H|| = \int_0^1 osc(H_t)dt$, the average oscillation. On the TCD side you have a correction term, $||H|| = \int_0^1 (osc(H_t) + |mv(H_t)|)dt$ where the new term corresponds to the "Reeb direction" which is the constant functions. The integrand is equivalent to the norm, $max(H_t)$. If you look at prequantization bundles, the relation of these two norms is apparent. This makes it make sense not to look at the max. This is not smooth. It's continuous for almost every value of [missed].

For the isotopies there's not much to say. We just look at uniform convergence or C^0 convergence. It won't really matter. The limiting object is an isotopy but phi_t is just a homeomorphism, continuous in t.

So what do you get on the TCD side for the coordinate transformations, I get [missed], what do the conformal factors g_i and h_i go to? If you look at this for the symplectic form, you get constant functions which are zero on a compact manifold. In the contact world they matter a lot.

Let's take a look at the transformation law, [can't read the board]. There are constant functions, so if $H_i = 1$ for all i, then we get e^{-g_i}, the Hamiltonian of $\phi_i^{-1}\Phi_{H_i}\phi_i$. I want the conjugate to also be a Hamiltonian system, so I need g_i to converge uniformly to some g. To get a theory with coordinate transformations. After coordinate change it's still Cauchy. It's quite similar. It's actually very similar for the group structure. We need the h_i to converge to some h uniformly.

[missed some.]

4. Morimichi Kawasaki: Superheavy Lagrangian immersion in 2-torus

Thanks to the organizers for giving me this opportunity. Today's topic is the following. First, notation, then review, and then results and then proof. Let's start.

4.1. Notation. Today we consider (M, ω) a closed, symplectic manifold and if $H \in C^\infty(M)$, we define the Hamiltonian vector field X_H hich is defind by $\omega(X_H, V) = dH(V)$ for all V. If $H \in C^\infty(M \times [0,1])$ then there is a Hamiltonian flow ϕ_H^t where $\phi_H^0 = id$ and $\frac{d\phi_H^t}{dt} = X_{H_t}$. Then we can define $Ham(M, \omega)$ to be diffeomorphisms such that there is an H such that $h = \phi_H^1$. We can define $Symp(M, \omega)$ to be diffeomorphisms such that $h^*\omega = \omega$.

Now $QH_*(M, \omega)$ is quantum homology of (M, ω) with coefficients in $\mathbb{C}$. Then $c(a, F)$ are spectral invariants $a \in QH_*(M, \omega)$ and $F \in C^\infty(M)$. I'll omit the definition. It's $\mathbb{R}$-valued.

4.2. Review. Assume that $a \in QH_*(M, \omega)$ and $a * a = a$, where this is the quantum product. We have ζ_a, the functional $C^\infty(M) \to \mathbb{R}$ given by $\zeta_a(H) = \lim_{\ell \to \infty} \frac{c(a, \ell H)}{\ell}$. Now define for X closed in M and a as above that X is a-heavy if $\zeta_a(H) \geq inf_X H$. We say X is a-superheavy if $\zeta_a(H) \leq sup_X H$.

Theorem 4.1. *If X is a-heavy then X is non-displaceable by Hamiltonian diffeomorphisms. That is, there is no $\phi \in Ham(M, \omega)$ such that $X \cap \phi(X)$ is empty.*

If X is $[M]$-superheavy then it is non-displaceable by symplectomorphisms. If X is superheavy then X is heavy.

There are two easy examples. Example one is the two-sphere. This has the standard form ω_0. Let h be the height $h(x, y, z) = z$. Then let $C_\epsilon = h^{-1}(\epsilon)$. Then C_ϵ is $[S^2]$-superheavy, so non-displaceable by symplectomorphisms (if $\epsilon = 0$). If ϵ is non-zero then it's displaceable by $Symp = Ham$ so it's not $[S^2]$-heavy.

Another example is the two torus with form $dp \wedge dq$. So here $M = \{p = 0\}$ and $L = \{q = 0\}$. Then M, L are $[T^2]$-heavy so non-displaceable by Ham. But they're displaceable by symplectomorphisms, so they're not $[T^2]$-heavy.

4.3. **Results.** Our main result is the following:

Corollary 4.1. *$M \cup L$ is $[T^2]$-superheavy. This is non-displaceable by symplectomorphisms.*

This is a trivial result, since $M \cup L$ is non-displaceable by homeomorphism. We can obtain non-trivial results. To obtain non-trivial results, we use the following theorem. For $i = 1, 2$, (M_i, ω_i) closed and symplectic, if X_i is a_i-superheavy in (M_i, ω_i), then $X_1 \times X_2$ is $a_1 \otimes a_2$-superheavy in $M_1 \times M_2$.

Corollary 4.2. *$(M \cup L) \times C_0$ in $T^2 \times S^2$ is $[T^2 \times S^2]$-superheavy. So $M \cup L \times C_0$ is non-displaceable by symplectomorphisms.*

This is a non-trivial result.

Definition 4.1. *An open subset U of M is a-null (let $a = a * a$) if $\zeta_a(H) = 0$ for all H compactly supported on U. We say U is strongly a-null if $\zeta_a(F + G) = \zeta_a(G)$ if F is compactly supported on U and G is any function such that $\{F, G\} = 0$. Strongly a-null implies a-null by using $G = 0$. So X is strongly a-null if there is a U containing X which is strongly a-null.*

Remark 4.1. *If X is displaceable by Hamiltonian diffeomorphism, then X is strongly a-null.*

Theorem 4.2. *Let $F_i, \ldots, F_k$ be C^∞ functions on M which Poisson commute. Let Φ be a function $F_1, \ldots, F_k : M \to \mathbb{R}^k$. Fix f_0. Assume for all $y \neq y_0$ that $\Phi^{-1}(y)$ is strongly a-null. Then $\Phi^{-1}(y_0)$ is a-superheavy. This is called the stem.*

Our theorem is that any open subset of $T^2 \backslash (M \cup L)$ is strongly $[T^2]$-null. We can prove, then, our corollary.

The proof of our theorem (which as stated is a very special case) is as follows. Let Φ be $F : T^2 \to \mathbb{R}$ with $\Phi^{-1}(0) = M \cup L$. Then for all $\epsilon \neq 0$, $\Phi^{-1}(\epsilon)$ is $[T^2]$-null.

Then $\Phi^{-1}(0)$ is $[T^2]$-superheavy.

Theorem 4.3. *Let (M, ω) be a rational closed symplectic manifold. and $H \in C^\infty(M \times [0, 1])$. For $\alpha \neq 0$ in $[S^1, M]$, assume that $\Phi_H^1|_U$ is the identity, that for all x in U, $t \mapsto \phi_H^t(X)$ is α, and $\alpha \notin [S^1, U]$. Then U is a-null for any a. This comes from Irie's result on Hofer-Zehrder capacity. Thank you for your attention.*

5. Sobhan Seyfaddini: The displaced disks problem via symplectic topology

I'd like to begin by thanking the organizers. It's been a great pleasure and I'm very happy to be speaking. I'll be speaking mainly about the two-sphere S^2 which will be equipped with a volume form ω. Area preserving diffeomorphisms are symplectomorphisms, and since S^2 is special, these are all Hamiltonian. I'll let $\mathcal{H}$ be area and orientation preserving *homeomorphisms* of S^2. A simple consequence of what I've said is that every area preserving homeomorphism can be approximated by Hamiltonians, so it's the closure of $Ham(S^2)$ in the C^0 setting.

The displaced disks question was posed by Beguin, Crovisier, and Le Roux. Take a to be a positive number and $\mathcal{H}_a$ be the set of θ in $\mathcal{H}$ that displaces a disk of area a. This means there exists a disk D_a of area a so that $\theta(D_a) \cap D_a = \emptyset$.

Question: Is Id in the closure of $\mathcal{H}_a$? Can you have arbitrarily small morphisms that displace a disk of area a?

The answer is no. The identity is not in here. If you want to displace a disk of some area, you need some C^0 norm. The goal of the talk is to sketch a proof.

Let me show you a nice corollary first. Consider the conjugacy class of an area preserving homeomorphism $C(\theta)$. The corollary is that $C(\theta)$ is never C^0-dense in $\mathcal{H}$. If θ is not the identity, then $\theta \in \mathcal{H}_a$ and then so is its conjugacy class.

This was the original reason for posing this question.

Now a few remarks. Neither of the above is true if you drop the area-preserving requirement. If you consider just homeomorphisms, these aren't true. It's not hard to see that a non-area preserving homeomorphism can displace a disk of large area. Take a spiral that fills up the disk. Then take a map that pushes the spiral into the space between it. You can in fact have a homeomorphism of S^2 whose conjugacy class is dense in all homeomorphisms is H^2.

You might ask about surfaces other than S^2. Then an appropriate version holds for homeomorphisms in the C^0-closure of Ham, vanishing flux, but not for general homeomorphisms that are area preserving. You can look at a torus. The corollary holds for all homeomorphisms. The flux homeomorphism is [missed]. The real question is about what happens in the kernel of the flux. Can something in the kernel have conjugacy dense in the kernel. The answer is no, and I think while my methods answer this, it was already known by other means.

Questions? Here then is the proof of theorem one. It uses spectral distance. Viterbo Schwarz and Oh showed there was a γ-norm $Ham(M) \to \mathbb{R}$. This is true for M a closed manifold. Here γ has the following properties.

(1) $\phi \in Ham(M)$ and $\phi(B_r) \cap B_r = \emptyset$, then $\gamma(\phi) \geq \pi r^2$.
(2) $\gamma(id) = 0$ (it's also a norm but I don't need the rest of that)

Step two of the proof. Here, this is what I did. If M is a surface, a closed surface, then γ is C^0-continuous.

Putting these two together solves the problem. I'll spell it out but it's easy to see at this point. Suppose you have a sequence of area preserving maps which displace disks of area a and which converge to the identity. Without loss of generality, we assume they are smooth. Then the first fact implies that $\gamma(\theta_i) \geq a$ and the second that $\gamma(\theta_i)$ converges to zero.

I'll spend the rest of the talk saying what γ is and proving that it is continuous in the case of a surface. The formula that was proposed in some talks in the summer, computing it can be very difficult, and a formula was proposed.

So spectral numbers and γ.

Let $\Omega = \{(z, u)\}$ where z is a loop in M, contractible, and u is a contracting disk for z. Then for $\mathcal{H} \in C^\infty([0, 1] \times M)$ we get $A_{\mathcal{H}} : \Omega \to \mathbb{R}$ given by $(z, u) \mapsto \int H_t(z_t)dt - \int_u \omega$. Now $Crit(A_{\mathcal{H}})$ consists of 1-periodic orbits of $\phi^t_{\mathcal{H}}$ and $Spec(\mathcal{H})$ consists of critical values of $A_{\mathcal{H}}$.

Theorem 5.1. *There exists a function $c : C^\infty([0, 1] \times M) \to \mathbb{R}$ with the following properties.*

1. $c(\mathcal{H}) \in spec(\mathcal{H})$
2. $|c(\mathcal{H}) - c(\mathcal{G})| \leq ||\mathcal{H} - \mathcal{G}||_\infty$
3. *triangle inequality. For $\mathcal{G}$ and $\mathcal{H}$, define $\mathcal{H}\#\mathcal{G}$ as H $Hh(t, \gamma) + G(t\phi^t_{\mathcal{H}})^{-1}(\gamma)$. Flow along this is the composition of flow. Then $c(\mathcal{H}\#\mathcal{G}) \leq c(\mathcal{H})\#c(\mathcal{G})$.*

Definition 5.1. *Say you have a Hamiltonian diffeomorphism ψ. Then $\gamma(\psi)$ is $inf c(\mathcal{H}) + c(\bar{\mathcal{H}})$ where this ranges over all Hamiltonians with $\phi^1_{\mathcal{H}} = \psi$.*

Now γ is a norm. Seeing that $\gamma(\psi) \geq 0$ is easy, since $c(\mathcal{H})+c(\bar{\mathcal{H}}) \geq c(\mathcal{H}\#\bar{\mathcal{H}}) = 0$. Seeing that it is invariant under inverses, satisfie the triangle inequality, and is nondegenerate aren't much harder.

Say you have a sequence that converges to the identity, do the spectral numbers converge to zero. You need to normalize, and with the standard normalization it's not continuous.

Now M is any closed symplectic manifold. Supposed B is an open ball in M, fixed. Say $\mathcal{H}_i$ and $\mathcal{H}$ vanish on B and the flow of $\mathcal{H}_i$ converges to the flow of $\mathcal{H}$. Then $c(\mathcal{H}_i)$ converges to $c(\mathcal{H})$. A remark is that if you remove the assumption that $\mathcal{H}_i|_B = 0$, then the theorem is not true. This restriction is a big restriction. It's still another step to show that it's continuous on S^2. I'll come back to this statement if I have time. I'll call this theorem two.

Now I'll show that γ is C^0 continuous on $Ham(S^2)$.

I'll show that if ϕ is C^0-close to the identity then $\gamma(\phi)$ is small.

Pick two disks covering S^2. I apply a C^0-fragmentation theorem by Entov Polterovich and Py that says there exist two diffeomorphisms, Hamiltonian diffeomorphisms, such that

1. The support of ψ_i is in the disk D_i
2. $\psi_1 \circ \psi_2 = \phi$, and
3. ψ_I is C^0-close to the identity.

To apply the previous theorem, I need an entire path that is C^0-close to the identity.

Here's a small lemma. There exist two Hamiltonians F_i supported in D_i so that the time one map of F_i is ψ_i and secondly the C^0 distance from Id to $\phi^t_{F_i}$ is less than the distance to ψ_i. You apply the Alexander isotopy and do a small trick.

So I'll show that $\gamma(\phi)$ is small. So ϕ is $\phi^1_{F_1} \circ \phi^1_{F_2}$ and $\gamma(\phi)$ by the triangle inequality is smaller than $\gamma(\phi^1_{F_1}) + \gamma(\phi^1_{F_2})$ and this is smaller than or equal to $c(F_1) + c(F_2) + c(\bar{F_1}) + c(\bar{F_2})$ and these are small by theorem two.

Now I will prove the theorem two. Assume that $\omega|_{\pi_2} = 0$. I'll show that, supposing $\mathcal{H}|_B = 0$, and $\phi^t_{\mathcal{H}}$ is C^0 close to the identity. Then I'll show that $|c(\mathcal{H})|$

is small. You'll see that all I need is that $\phi^1_{\mathcal{H}}$ is close to the identity (this is because $\omega|_{\pi_2} = 0$).

Pick a Morse function f on M which is C^2-small and so that all critical points are in B. Then we know there exists an $\epsilon > 0$ such that the distance between x and $\phi^1_t(x) \geq \epsilon$ for all x outside the ball B.

If $d_{C^0}(Id, \phi^1_H) < \epsilon$, I'll show that $c(H)$ is smaller than ϵ.

Let's consider ϕ^1_H and ϕ^1_f. I claim that $\phi^1_H \circ \phi^1_f$ has the same fixed points as ϕ^1_f. Clearly the containment of the fixed points of ϕ^1_f is clear. So in the support of $\mathcal{H}$, you move at least ϵ and then it can't be moved back. These two have the same fixed points. Then you can check that the spectra of $\mathcal{H}\#f$ is the same and the spectrum of f. Then $c(\mathcal{H}\#f)$ is in the values, in $spec(f)$. Using $\omega|_{\pi_2} = 0$ you get that this is just critical values of f. So $c(\mathcal{H}\#f)| \leq ||f||_\infty$. On the other hand, $|c(\mathcal{H}\#f) - c(\mathcal{H})| \leq ||f||_\infty$ so $|c(\mathcal{H})| \leq 2||f||_\infty$.

Thank you.