INSTITUTE FOR BASIC SCIENCE CENTER FOR GEOMETRY AND PHYSICS SPECIAL LECTURE SERIES, NERO BUDUR

GABRIEL C. DRUMMOND-COLE

1. September 16, Deformation theory

Thank you for the opportunity to speak and visit, it's my first time in Pohang. This is the first of three lectures. These lectures will move from very general to very specific.

Today we'll discuss deformation theory. In fact, I'll spend half of this lecture reminding the classical deformation theory to people who don't know it and then I'll present an enhancement that has to do with trying to answer the following question. Take X to be a topological space and ρ a representation of the fundamental group of X, rank n, $\rho : \pi_1(X) \to GL(n, \mathbb{C})$, and the question that we try to answer is the following. Describe all infinitesimal deformations of ρ. I don't mean just first order. I mean any order but locally, very close to 1. But I want this with cohomological constraints. Fix i and k, and then I want, when you look at the rank n local system L_ρ attached to ρ, a locally constant sheaf, and it has cohomology $H^i(X, L_\rho)$. We want that the rank of this $\mathbb{C}$-vector space is $\geq k$. This is the typical question that we're interested in.

Let me rephrase this. More geometrically, take X a topological space with some finiteness built into it, it's the same homotopy type as a finite CW complex, and $\mathcal{M}$ is the moduli space of all these rank n representations, $Hom_{\mathrm{gp}}(\pi_1(X), GL(n, \mathbb{C}))$, the moduli space of rank n representations of π_1. This is finitely presented, so this is an affine scheme given by some polynomial equations of finite type over $\mathbb{C}$. All this means is you can write down this space in a big affine space as the zero locus of some polynomial equation. This can have multiple components, there are no reduced points.

Just to mention, the difference between this and local systems is that you have to mod out by conjugation, I'll try to sweep this under the rug, you'll see why later.

So inside this scheme we are looking at $\mathcal{V}^i_k$, the substrata of representations with $\dim H^i(X, L_\rho) \geq k$. This is an affine closed subscheme of the moduli space and the quetsion is to describe locally the cohomology jump loci. You want to see the local behavior. You'd like to see the global behavior as well, but one question is the local question, about the formal scheme / analytic germ at ρ, $\mathcal{V}^i_{k,(\rho)} \subset \mathcal{M}_{(\rho)}$.

The classical deformation theory is about the bigger space here and I'm interested in enhancing to get to the smaller space.

Let me give an example theorem

Theorem 1.1. *(Esnault–Schechtman–Viekweg) Take X to be the complement of a hyperplane arrangement (so very special) and look at rank one representations, $n = 1$, which is the same as local systems, there's no conjugation to mod out by. Look at $\rho = \mathbf{1}$, the trivial rank one local system.*

In this case, they were able to describe the cohomology jump loci, the reduced structure $(\mathcal{V}^i_{k,(\mathbf{1})})^{red}$ as $(R^i_{k,(0)})^{red}$, and I'll give you this set theoretically although you can make it a scheme,

$$R^i_k = \{\omega \in H^1(X,\mathbb{C}) | \dim H^i(H^\bullet(X,\mathbb{C}),\omega\cup) \geq k\}$$

The idea is that describing this in terms of the cup product is much easier than describing things in terms of the moduli space. You can describe as cochains on a universal cover with twisted differentials, but that's very hard to compute.

More generally, if $\mathcal{M}$ is a moduli space of objects with some cohomology theory, then you can always define the cohomology jump loci $\mathcal{V}^i_k$, then we fix terminology by *deformation theory* we mean $M_{(\rho)}$, you want to understand the space locally at this object ρ. With *deformation theory with cohomological constraints* I mean you want to look at the local behavior $\mathcal{V}^i_{k,(\rho)}$ instead.

It's well-known how to handle the deformation theory, with Deligne's principle, and we'll use a newer enhanced principle, my joint work with Botong Wang for cohomological constraints.

The next part of the lecture is to remind you about Deligne's principle, and then I'll talk about the new principle.

Deligne's principle, around 1986, in a letter to J. Millson, says the following. "Every deformation problem over a field of characteristic zero is controlled by a differential graded Lie algebra with equivalent Lie algebras giving the same deformation theory."

I have to explain what this means. I'll give the formal definition of a differential graded Lie algebra. What is a differential graded Lie algebra? We'll stick with the complex numbers here. This consists of the following data:

(1) a graded complex vector space $C^* = \bigoplus_{i\in\mathbb{N}} C^i$
(2) a Lie bracket, that is, a bilinear graded skew commutative pairing, so that for $\alpha \in C^i$, $\beta \in C^j$, and $\gamma \in C^k$, we have $[\alpha,\beta] + (-1)^{ij}[\beta,\alpha] = 0$, it satisfies a graded version of the Jacobi identity
$$(-1)^{ki}[\alpha,[\beta,\gamma]] + (-1)^{ij}[\beta[\gamma,\alpha]] + (-1)^{jk}[\gamma,[\alpha,\beta]] = 0$$
(3) a family of linear maps $d^i : C^i \to C^{i+1}$ which forms a differential, $d^{i+1}d^i = 0$, so C is a complex of vector spaces, and satisfies the Leibniz rule
$$d[\alpha,\beta] = [d\alpha,\beta] + (-1)^i[\alpha, d\beta]$$

So that's a differential graded Lie algebra. There are morphisms that you have to define that keep the structure. A homomorphism of differential graded Lie algebras is a linear map preserving grading, brackets, and differentials.

Some more terminology. A differential graded Lie algebra (so I'm trying to build up to the principle here, we need a little more) morphism $g : C \to D$ is a 1-*equivalence* if it induces an isomorphism on cohomology (with respect to the differential) up to degree 1 and a monomorphism on degree 2. Of course you can define an i-equivalence. But it turns out that for the deformation theory this is enough. Let me note that if (C,d) is a differential graded Lie algebra, then $(H^*C, d = 0)$ inherits a Lie bracket and becomes a differential graded Lie algebra itself with induced bracket. This can be checked easily.

We say that two differential-graded Lie algebras *have the same* 1-*homotopy type* if they are connected by a zig-zag of 1-equivalences.

So let me explain the meaning of "controlled" in this principle now. We have our moduli space and our object there, we have to change the point of view, view $\mathcal{M}_{(\rho)}$ as a functor. Let me try to give some geometric intuition [picture]. So $\mathcal{M}_{(\rho)}$ is a functor from ART, local Artinian $\mathbb{C}$-algebras of finite type to SET, sets, taking A to $Hom_{\mathbb{C}\text{-schemes}}(\operatorname{Spec} A, \mathcal{M}_{(\rho)}) = Hom_{\mathbb{C}\text{-alg}}(\widehat{\mathcal{O}_{M_{(\rho)}}}, A)$.

This functorial approach has an advantage because moduli spaces might not exist, but you can always ask about what deformations are.

To say that this deformation problem $\mathcal{M}_{(\rho)}$ is controlled by the differential graded Lie algebra (C, d) if as functors $\mathcal{M}_{(\rho)}$ is equivalent to the canonical deformation functor $\operatorname{Def}(C, d)$, which takes A a local Artinian algebra to the so-called Maurer–Cartan elements up to gauge. Let me give a slightly more intutive description. Let me give a set theoretical equality (that has to be made sense of).

$$Def(C,d)(A) = \{omega \in C^1 \otimes_{\mathbb{C}} \mathfrak{m}_A | d\omega + \frac{1}{2}[\omega, \omega] = 0\}/C^0 \otimes \mathfrak{m}_A.$$

This is not literally true, you have to make this categorical, but this is the idea.

That's what it means for a differential graded Lie algebra to control a deformation problem.

In practice the differential graded Lie algebra you build first is infinite dimensional. What you can do is for example in some nice cases, replace this differential graded Lie algebra with its cohomology, this typically is finite dimensional, there's no differential, so this last part is absolutely fundamental which is why this next theorem is called

Theorem 1.2. *(The fundamental theorem of deformation theory) (Deligne, Goldman–Millson, Schlessinger–Stasheff) The deformation functor $Def(C)$ depends only on the 1-homotopy type of C.*

More precisely, if you have two differential graded Lie algebras connected by a 1-equivalence, there's an induced map which is an isomorphism of functors between the deformation sets.

The nicest case you can stumble on is the case when C is 1-*formal*, where C is 1-homotopy equivalent with its cohomology differential graded Lie algebra with no differential. Then one can replace $Def(C)$ with $Def(H^*C)$. One gets just algebraic equations, not only algebraic but quadratic, for the Maurer–Cartan element. Usually the cohomology is finite dimensional.

That's a principle, it has to be illustrated. That's what Goldman–Millson had done. Let me illustrate this principle.

We'll keep two examples running in this talk. The first is about representations of π_1.

Theorem 1.3. *(Goldman–Millson, Simpson) Let X be a compact Kähler manifold and $\mathbb{R}(X, n)$ the moduli space of representations of π_1 of rank n, and we'll take ρ in that space of representations. If ρ is semisimple (which doesn't mean that nearby representations are semisimple) then $\mathbb{R}(X, n)_{(\rho)}$ are described by some linear algebra, there's a space $Q(\rho)$, the* quadratic cone *and the space of representations is $Q(\rho)_{(0)}$.*

Let me describe the quadratic cone set theoretically, it's

$$\{\eta \in \text{“}H^1(X, \operatorname{End}(R_\rho))\text{''} | \eta \wedge \eta = 0 \in H^2(X)\}$$

which is almost right but you need to look instead at group cocycles

$$\{\eta \in \text{“}Z^1(\pi_1(X), gl(n,\mathbb{C})_{\mathrm{ad}\,\rho})\text{''} | \eta \wedge \eta = 0 \in H^2(X)\}.$$

It's still manageable, it's a vector space. In particular, this has quadratic singularities.

How does the proof go? All one needs to do is provide the differential graded Lie algebra governing this. This is the de Rham complex with coefficients in End, well, almost.

$$\mathbb{R}(X,n)_{(\rho)} \text{“}=\text{''}\ Def(A^{\bullet}_{DR}(X, \mathrm{End}(L_\rho)), d)$$

up to issues of augmentation.

The assumption that ρ is semisimple implies by Simpson's theorem that this differential graded Lie algebra is formal. But for deformation functors you only need C^1 and C^2 at most. This is in fact ∞-formal so you can replace it with its cohomology and zero differential, in which case, again, you're left just with the quadratic equations.

You'll see because of this amazing principle the setup will be different but the end result will be similar.

Look at holomorphic vector bundles.

Theorem 1.4. *(Goldman–Millson, Nadel) Let X be a compact Kähler manifold and $\mathcal{M}(X,n)$ is the moduli space of stable rank n vector bundles on X with vanishing total Chern class. I'll take one such vector bundle. Then the statement is that the deformations of this again are described by some linear algebra,*

$$\mathcal{M}(X,n)_{(E)} \cong Q(E)_{(0)}$$

for a quadratic cone Q with a similar description:

$$Q(E) = \{\eta \in H^1(X, \mathrm{End}(E)) | \eta \wedge \eta = 0 \in H^2\}.$$

The proof again involves displaying the Lie algebra governing this problem, which is the Dolbeault Lie algebra $(A^{0,*}_{Dol}(X, \mathrm{End}(E)), \bar\partial)$. This is formal because of the assumptions we made. Because it's formal, you can replace it with its cohomology $d = 0$.

Let me make a remark, you may know this, if X is a smooth projective variety, then Simpson has constructed a few moduli spaces. The Betti moduli space $\mathcal{M}_\beta(X,n)$ is the irreducible local systems of rank n, which doesn't need a smooth projective variety. But the other two, you can look at the de Rham moduli space $\mathcal{M}_{DR}(X,n)$, stable vector bundles with flat connection $\nabla : E \to \Omega^1_X \otimes E$, and then the Dolbeaut version of this $\mathcal{M}_{Dol}(X,n)$ which is the space of stable Higgs bundles with $c = 0$, where a Higgs bundle [missed some]

Analytically the first two are the same. There's formality hidden here. One can describe locally, these things have quadratic singularities, the Betti moduli space is the one I put down in the example. You also know the same thing for the de Rham moduli space. In the third case the Lie algebra is the Higgs complex, using $\bar\partial + End(\theta)$ instead of d.

There are many other formality statements in the literature. This was a principle, by now it's a theorem. The hard part is writing down the axiomitization. This was done by [unintelligible]and [unintelligible]in derived algebraic geometry. This might not quite coincide with the classical notion.

This also needs very little information from the differential graded Lie algebra. The rest of the Lie algebra has to know about the derived algebraic geometry.

But we'll see how the cohomological jump loci knows about this without passing to derived algebraic geometry, just staying with classical stratified algebraic geometry.

So let me go now to the new principle, dealing with cohomology jump loci. Before I state the new principle, let me note that if you look at the space of rank one representation $\mathbb{R}(X,1)$, this is $Hom(\pi_1(X),\mathbb{C}^*)$, and you can replace π_1 with $H_1(X,\mathbb{Z})$, the Abelianization, and so this is just $(\mathcal{C}^*)^{b_1(X)} \times G$ for a finite Abelian group G, since there could be torsion in H_1. All the story is trivial in this case. But $\mathcal{V}^i_k$ in $\mathbb{R}(X,1)$ could be nontrivial.

So we're addressing a *different* problem which could be nontrivial in this simple case.

The new principle is the following:

"Every deformation problem (over a field of characteristic zero) with cohomology constraints is governed by a pair (C,M) of a differential graded Lie algebra C with a differential graded Lie module M over C with equivalent pairs giving the same deformation theory with cohomological constraints."

So (M,d_M) is a differential graded Lie module over the differential graded Lie algebra (C,d_C), which is the data

(1) a graded vector space $M = \bigoplus_{i\in\mathbb{N}} M$ with a bilinear pairing $C\times M \to M$ written $(a,\xi)\mapsto a\xi$, satisfying some conditions:

$$C^i \times M^j \to M^{i+j}$$

$$\text{for } \alpha \in C^i, \beta \in C^j, \xi \in M,$$

$$[\alpha,\beta]\xi = \alpha(\beta\xi) - (-1)^{ij}\beta(\alpha\xi)$$

(2) a family of linear maps $d^i_M : M^i \to M^{i+1}$ with $d^{i+1}_M d^i_M = 0$ and the Leibniz rule

$$d_M(a\xi) = d_C(\alpha)\xi + (-1)^i\alpha(d_M\xi).$$

That's the formal definition.

Let me define a homomorphism between differential graded Lie modules over (C,d_C). Such a homomorphism $f:(M,d_M)\to(N,d_N)$ is a linear graded compatible with the differentials and the multiplication from C.

The equivalence that we're looking for in the statement is as follows. Well, first, as before, (C,M) will be called a DGLA pair and the cohomology with zero differentials $((H^*C,0),(H^*M,0))$ inherits the structure of a DGLA pair.

A map of pairs $g:(C,M)\to(D,N)$ is a the data of a differential graded Lie algebra homomorphism $g_1 : C\to D$, and a linear map $g_2 : M\to N$ which is a C-module map via the C-module structure given by g_1. I could make this more precise but I'll just keep it like this.

Such a map is a *q-equivalence* if g_1 is 1-equivalent and g_2 is q-equivalent. Then you can define the *q-homotopy type of a pair* and *q-formality* of a pair.

Let's go over what it means to govern a deformation problem. For any differential graded Lie algebra you had this functor $Def(C) : \mathsf{ART}\to\mathsf{SET}$ given by Maurer–Cartan elements up to gauge. We define a subfunctor $Def^i_k(C_1M)$ which takes (C,M) to

$$\{\omega\in C^1\otimes\mathfrak{m}_A | d_C\omega + \frac{1}{2}[\omega,\omega] = 0; \mathcal{J}^i_k(M\otimes_{\mathbb{C}} A, d_M\otimes \mathrm{id} + \omega\wedge) = 0\}/C^0\otimes A$$

Here $\mathcal{J}^i_k$ is the *cohomological jump ideal* which I'll define (on any complex) shortly.

What is controlling the deformation problem? You have $\mathcal{M}_{(\rho)}$ and this is the same as the deformation functor given by $\mathrm{Def}(C)$. Then with in this you have $\mathcal{V}^i_{k,(\rho))}$, so this usbset in $\mathrm{Def}(C)$ acts on something, this is M, and this is $\mathrm{Def}^i_k(C,M)$. That's the solution that we propose.

The cohomology jump ideal is the crucial tool. Let me introduce those in a more general context. Let me recall a more classical statement you're all familiar with in algebra. If M is a finitely generated R-module and R is a commutative ring, take a free presentation $F^{-1} \to F^0 \to M \to 0$

Theorem 1.5. *(Alexander, Fitting) The ideal I given by minors of seize k in R, such that $I_{\mathrm{rk}(F^0)-k}(d)$, depends only on M and k.*

These are the so-called *Fitting ideals* of the module. They give subschemes, strata. The generalization I'll need is

Theorem 1.6. *Let M be a bounded above complex of R-modules, and right now our theorem is just for R Noetherian with M having finitely generated cohomology. Take a free resolution of this $F^\bullet \to M^\bullet$ where is a bounded above complex of free R-modules of finite rank. These always exist, and this is not necessarily well-known. Then*

(1) *the $\mathcal{J}^i_k(M^\bullet) = I_{rk(F^i)-k+1}(d_F^{i-1} \oplus d_F^i)$ depends only on M, i, k. This gives the Fitting ideal as $\mathcal{J}^0_{k-1}(M)$. Let me tell you more some properties, why this deserves the name cohomology jump ideals.*
(2) *The second part is, it depends only on the class of M in the derived category. If M and N are quasi-isomorphic then they have the same cohomology jump ideals.*
(3) *If M is a complex as above of flat R-modules, and if S is an R-algebra (also Noetherian), then you can change the base as expected, $\mathcal{J}^i_k(M)\cdot S = J^i_k(M \otimes_R S)$.*
(4) *As before, if $S = R/\mathfrak{m}$, then $J^i_k(M) \subset \mathfrak{m}$ if and only if $\dim_{R/\mathfrak{m}} H^i(M^\bullet \otimes_R R/\mathfrak{m}) \geq k$.*

This is easy. It all comes down to computing the dimension of the complex. How do you compute the dimension of the cohomology, the dimension? This dimension is $\dim\ker d^i - \dim\operatorname{im} d^{i-1}$. Then this is $r_i - \dim\operatorname{im} d^i - \dim\operatorname{im} d^{i-1}$, and then to compute the ranks, you take minors of the appropriate size.

Back to the condition, you have $\mathcal{J}^i_k(M\otimes A, d_M + \omega)$, which is 0 in A, what does this mean? This means that $\mathrm{Spec}(A)$, we know it maps to $M_{(\rho)}$, and what do you want? The cohomology jump loci contains this little curve, the map factors through $\mathcal{V}^i_{k,(\rho)}$. That's why this is the answer to the deformation problem with cohomological constraints.

Theorem 1.7. *The functor $Def^i_k(C,M)$ depends only on the i-homotopy type of the pair.*

This generalizes the previous theorem for differential graded Lie algebras. Note that if you phrase things in terms of the endomorphisms of M, it's not so intuitive how to get this step.

As I said, the only new thing here is this construction.

The nicest case is again the formal pair case. For (C,M) a formal pair, then you can replace the cohomology jump deformation functors by those for the cohomology with trivial differentials $Def_k^i(C,M) = Def_k^i(HC,HM)$, and you can prorepresent this functor by equations. Let's say for simplicity that C^0 acts trivially on C^1. Then in this case what you get

$$Def(HC) \cong Q \text{ given by } [\omega_{univ}, \omega_{univ}] = 0$$

with $\omega_{univ} = \sum e_i \otimes x_i$ where e_i is a base of H^1C and x_i the dual base. These are equations in $\mathcal{O}_{H^1C}$, polynomials in the x_i. Setting this equal to zero gives you a set of quadratic equations in the x_i.

This is prorepresented by the formal scheme at zero of this cone. In the module case, this is $\mathrm{Spec}(\mathcal{O}_Q/\mathcal{J}_k^i(HM \otimes \mathcal{O}_Q, \omega_{univ}))$, the formal scheme at 0. This ω is linear in the x_i, so these are minors of appropriate size of matrices of linear forms in x_i. So that's the first note.

What I'm trying to say is that in the formal pair case, things become very explicit. All you have to do is pick a basis and write down a matrix.

Let me write down what one gets in particular cases in the last few minutes.

Theorem 1.8. *Remember X was compact Kähler and we look at $\mathbb{R}(X,n) = Hom(\pi_1(X), GL(n,\mathbb{C}))$, and then $\mathcal{V}_k^i = \{\rho : \dim H^i(X, L_\rho) \geq k\}$ for ρ semisimple.*

(1) *Then $\mathcal{V}^i_{k,(\rho)}$ is*

$$\{\eta \in Q(\rho) | \dim H^i(H^*(X, \mathrm{End}(L_\rho)), \eta\wedge) \geq k\}$$

This is written down set theoretically, but on the right hand side this is actually the jump ideal and can be given a scheme theoretic expression.

(2) *The second part is, if you look at $\mathcal{V}_k^i \backslash \mathcal{V}_{k-1}^i$, the ones which have dimension exactly equal to k, you get quadraticity.*

Let me sketch the proof. What is the differential graded Lie algebra pair controlling the problem? There's the discrepancy between local systems and π_1 representations, but that can be dealt with. Now $(A_{DR}(X, \mathrm{End}(L_\rho)), A_{DR}(X, L_\rho))$ is our pair, it's formal so you can replace it with its homology

$$(H(X, End(L_\rho)), H(X, L_\rho))$$

which gives you the first part and then to calculate $\dim H^i(X, L_\rho) = k$, this is giving you the ideal generated by the entries of the matrices that appear, you cut down bilinear equations in the ambient space, which is quadratic to begin with.

I'm out of time, let me just state the second case of this illustration.

Theorem 1.9. *We have X a compact Kähler manifold and $\mathcal{M}(X,n)$ as before the moduli space of holomorphic vector bundles with vanishing total Chern class. Then $\mathcal{V}_k^{p,q}$ is $\{E | \dim H^q(X, E \otimes \Omega_X^p) \geq k\}$. So then $\mathcal{V}^{p,q}_{k,(E)}$ is $\{\eta \in Q(E) | \dim H^i(H(X, E \otimes \Omega^p), \eta\wedge) \geq k\}$*

Let me just remark, we didn't prove any new formality results. There are other formality results in the literature, we have not looked at what happens there to cohomology jump loci.