CURRENT DIRECTIONS IN HOMOTOPICAL ALGEBRA

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1. May 20, 2019: Hiro Tanaka: broken techniques for disappearing things

Thanks to the organizers for inviting me. Various people have heard versions of this before, I have new things to say but I want to toe the line between new things and background.

Let me tell you an origin story. I started thinking about these structures thinking about Morse theory. Fix X a smooth compact manifold along with $f : X \to \mathbb{R}$ sufficiently generic, and a Riemannian metric. Imagine that X is a two-sphere embedded in $\mathbb{R}^3$ in a sufficiently generic way, and use a height function, and a Riemannian metric inherited from $\mathbb{R}^3$.

Given two critical points x and z in $\mathrm{Crit}(f)$ we can count or "enumerate" the gradient trajectories from x to z. For the sake of my sanity I'll take negative gradient trajectories. Everything is a raindrop falling down the statue. So let's define:

Definition 1.1. Let $\mathcal{M}^\circ(x,z)$ be the set of gradient trajectories, it's the set of $\gamma : \mathbb{R} \to X$ so that the velocity vector at time t is $\dot\gamma(t) = -\nabla(\gamma(t))$, with $\lim_{t\to-\infty}\gamma(t) = x$ and $\lim_{t\to\infty}\gamma(t) = z$. This differential equation is translation invariant, where I want to be at time zero is invariant, so I'll mod out this collection by $\mathbb{R}$.

Let me give an example in pictures. [picture]

One thing I can do to this is compactify this. What happens as I run toward the open point? I get a gradient trajectory that wants to develop a kink. This place it wants to develop a kink is y which I claim will also be a critical point. That point is going to correspond to a pair of gradient trajectories that pass through another critical point. So that's two distinct gradient trajectories. I can also run the movie around the back of the kidney bean [picture] and get something on the other side.

Definition 1.2. Let $\mathcal{M}(x,z)$ be $\mathcal{M}^\circ(x,z) \cup \bigcup(\mathcal{M}^\circ(x,y) \times \mathcal{M}^\circ(y,z))$ — but you might have multiple breaking points. This works if x and z have index difference at most two. So we take

$$\bigcup_n \bigcup_{y_1,\ldots,y_n} \mathcal{M}^\circ(x,y_1) \times \cdots \times \mathcal{M}^\circ(y_n,z).$$

This has a tautological family of broken lines living over it. I've drawn a cartoon [picture].

Definition 1.3. Broken is the stack classifying families of broken lines.

I haven't given a definition of a broken line or a family of them.

Let's say my base is $\mathbb{R}^2_{\geq 0}$. There is a tautological family living over this with unbroken lines over the interior, above a face a line with one break, and over the corner a line with two breaks.

So for a topological space S a map $S \to \mathsf{Broken}$ is the same thing as a family of broken lines over S.

Theorem 1.1. *(Lurie–T.) Any family of broken lines is locally isomorphic to a family over a pullback of a Euclidean octant $\mathbb{R}^n_{\geq 0}$.*

Why am I talking about Morse theory at a workshop about homotopical algebra. So one thing you can do if someone gives you a space is to contemplate sheaves on it.

The theorem is that we can completely classify sheaves on Broken.

Theorem 1.2. *(Lurie–T.) Fix $\mathcal{C}$ a compactly generated ∞-category, say sets or topological spaces or chain complexes over R or spectra.*

Then $\mathrm{Shv}(\mathsf{Broken}\mathcal{C})$ *is equivalent to a very well-known functor category,*

$$\mathrm{Fun}(\Delta_{\mathrm{surj}}, \mathcal{C})$$

Here Δ_{surj} has objects $[n]$ with $n \geq 0$ and morphisms surjections.

This is usually where I get at minute fifty-five, so if I'm going too fast, let me know.

Let me give a sketch given $\mathbb{R}^n_{\geq 0}$ covering Broken.

What is a sheaf $\mathcal{F}$ on Broken, then for any $S \xrightarrow{j} \mathsf{Broken}$, I get $j^*\mathcal{F}$ a sheaf on S. So $\mathbb{R}^n_{\geq 0} \xrightarrow{j_n} \mathsf{Broken}$ gives me $j_n^*\mathcal{F} = \mathcal{F}_n$ a sheaf on $\mathbb{R}^n_{\geq 0}$. So $\mathbb{F}_0 = j_0^*\mathbb{F}$ is a sheaf on $\mathbb{R}^0$, i.e., an object of $\mathcal{C}$, call it V_0.

What about $\mathbb{F}_1$? I get a sheaf that's constant on the interior of $\mathbb{R}_{\geq 0}$. Whatever it is, it's an object V_1, the stalk at the origin, and V_0 is the object from before.

Because this is a sheaf it's a presheaf, so I can restrict and get a map $V_1 \to V_0$. There's a unique surjection $V_1 \to V_0$, and that's what's being assigned to this map via the functor.

Let me show you $n = 2$. For $n = 2$ we have a sheaf on $\mathbb{R}^2_{\geq 0}$. Let me make the same observations as before. The stalk is associated to V_0 over the interior. Over the edge I get V_1 and over the corner V_2 I have two restriction maps $V_2 \to V_1$ and one from V_1 to V_0 and the two diagrams commute.

In general whe you look at corners in this way you see that this works with surjective maps of linear sets.

That's kind of algebraic but let me state an even better version of this theorem. Now endow $\mathcal{C}$ with a monoidal structure $\otimes$ preserving colimits in each variable. This doesn't need to be symmetric monoidal. I can look at sheaves on Broken valued in $\mathcal{C}$ and compatible with the monoidal structure. These are *factorizable* sheaves.

Given two families F_1 and F_2 of broken lines living over S, you can glue one family on top of the other to get a new family, $F_2 \bigstar F_1$. So the condition would now say that whatever you assign to V_1 is equivalent to $V_0 \otimes V_0$., and V_n is equivalent to $V_0^{\otimes n+1}$.

Now you can guess what factorizable sheaves on Broken are. Before they were functors out of the simplex category. Now if we use the factorizability property, you get something like

$$V_0 \xleftarrow{m} V_0^{\otimes 2} \leftleftarrows V_0^{\otimes 3} \cdots$$

where these are the two products tensored with the identity. So I get an associative algebra (or A_∞-algebra) and note that this is a *non-unital* associative algebra.

I think this observation wasn't made before and I think it warranted investigation and that's what we did.

Let me give some motivation for broken trees. I can look at gradient trajectories emerging out of x_1 for f_1. Let's choose another critical point x_2 for another point f_2, and we can look at the trajectories that come out of these, and look at their interection, and look at the gradient trajectory for $f_1 + f_2$. If you understand all of these then you understand the Fukaya category of T^*X where I'm doing Morse theory on X.

Theorem 1.3. *Fix $\mathcal{C}$ as before. Let $\mathsf{Broken}_{\text{trees}}$ be the stack classifying families of broken planar trees. Then $\text{Shv}^{\otimes}(\mathsf{Broken}_{\text{trees}}, \mathcal{C})$ is the category of planar non-unital operads. (here non-unital means both no arity zero and no units in arity one)*

Theorem 1.4. *Fix $\mathcal{C}$ without a monoidal structure, and let $\mathsf{Broken}_{\text{para}}$ classify families of "$\mathbb{Z}$-equivariant broken lines" or "families of broken paracyclic lines". Then sheaves on this with values in $\mathcal{C}$ is equivalent to $(\Delta^{\text{op}}_{\text{para,inj}}, \mathbb{C})$*

So for instance when $\mathcal{C}$ is a certain kind of category of $\mathbb{A}_\infty$-categories, this goes to the $S_\bullet$ construction of $\text{Fuk}(X)$.

Now let me try to explain the title. What do I mean by disappearing things?

Remark 1.1 (Remarks on the theorems)**.**

(1) "Disappearing" — associative structures can be encoded by colliding points in $\mathbb{R}$. An coherent association from colliding points in $\mathbb{R}$ is the same as an associative multiplication. Colliding points is hard to model. Critical points never collide. Our things come from the Poincaré dual picture, marking my intervals and multiplying by deleting points.

(2) Where's the Morse theory? The goal of Morse theory is to give invariants of manifolds, so where are the invariants? So we can define the spaces $\mathcal{M} = \amalg_{x,z}\mathcal{M}(x,z) \xrightarrow{p} \mathsf{Broken}$. So $\mathcal{M}$ is a compact manifold with corners which has a constant sheaf.

Theorem 1.5. *$p_!\mathbb{K}$ can be made a factorizable sheaf on Broken.*

Conjecture 1.1. An orientation on $\mathcal{M}$ yields from the algebra $p_!\mathbb{K}$ a chain complex equivalent to $C_*(X, \mathbb{K})$.

To talk about $p_!$ you need a triangulated or stable category. The orientation is to talk about this in a coherent way.

(3) Let me talk about this in terms of the operad example and the Fukaya category, I can push forward in the same way for broken trees, and I claim I get the Fukaya category enriched in spectra.

Remark 1.2 (Quick remarks on the Fukaya category)**.** There is a simple well-studied class of Fukaya categories, those of D_2 with f_n, a collection of $n+1$ points on the boundary of the disk. These are symplectic, not holomorphic. For each one, I can look at its Fukaya category. With one marked point I get 0. For two poitns I get $D^b(\text{Vect})$ or chain complexes, and with $n+1$ points on the boundary, it's representations of the A_n quiver in Vect. This, I claim, is a well-known sequence.

This wants to form the K-theory of a base ring R. The association $[n] \mapsto \text{Rep}(A_n)$ defines a simplicial system of categories called the $S_\bullet$ construction of R-modules where R is the base. If R was the integers, this is K-theory of the integers.

Somehow the question is, I have these Fukaya categories of these spaces, can I get the functors between them to give myself the simplicial maps?

If I restrict myself to only the injective maps, this can be realized geometrically by forgetting points, modulo isotopy of points.

Theorem 1.6. *For any Liouville manifold X the association $(D^2, f_n) \mapsto \mathrm{Fuk}(X \times (D^2, f_n))$ defines a sheaf on $\mathsf{Broken}_{\mathrm{para}}$ equivalent to $S_\bullet \mathrm{Fuk}(X)$. Any Lagrangian with boundary between these marked points gives a K-theory class and that gives an explanation for why such Lagrangians give rise to such classes.*

2. Damien Lejay: linear cogebras up to homotopy

[I do not take notes at slide talks]

3. Andrew Macpherson: symmetries of enriched category theory

Thank you for the introduction and the invitation. As Rune said I'm going to talk about the autoequivalence group of $\mathcal{V}$-enriched ∞-categories.

I'll start with a bit of motivation; my motivation came from model independence. When you write a paper about $(\infty, 1)$-categories or $\mathcal{V}$-categories or (∞, n)-categories, you say these objects are defined materially in terms of some external theory, and everything you do is going to be a statement about some simplicially flavored things. This may not involve manipulation these simplicial things at all but instead do natural operations inspired by things in ordinary category theory. For example:

- functor categories
- mapping objects
- opposites
- the Grothendieck construction

and relations among these things.

When you've done your paper you might find that you've applied a sequence of these kinds of constructions and facts about these constructions; you've never had to manipulate the simplicial objects at all. The actual choice of model is irrelevant. If someone happened to choose a different model you'd need to know that your models can be compared.

A second situation is that some of your operations have been formulated in one model and others in another model, and you need to be able to work in a single model. For example, it's easy in (∞, n)-categories modeled by n-fold Segal spaces to formulate the opposite. On the other hand, mapping objects and the composition are more obvious in an enriched category setting, thinking of these as $(n-1)$-category-enriched categories. You need to understand the equivalence between models to make statements relating these.

The ideal situation for model independence is that you know the complete set of rules that any of these constructions satisfy, a *complete axiomatization* suitably aligned with the things you are going to do.

In practice we just take the models we have and try to compare them. We try to classify models.

I'd like to begin with an example taken from the work of Barwick and Schommer-Pries. This is something like "on the unicity of homotopy theory of n-categories" or something like this.

There are multiple models, one that comes in is Barwick's n-Segal spaces and another Θ_n-spaces. The work of Barwick and Schommer-Pries tells us that these are equivalent up to the automorphisms of $n\,\mathrm{Cat} = (\mathbb{Z}/2\mathbb{Z})^n$ (taken by taking opposites).

The way this proceeds is first by identifying generating objects of $n\,\mathrm{Cat}$, cut out by purely categorical properties, the autoequivalences must preserve it, and then you can compute the autoequivalences of K and it's this. I won't go into the specific generator they chose but that's how they do it.

I will say that we can make Φ unique by fixing an inclusion of K. There's another model that I like more, iterated enriched categories, which is $(n-1)-\mathrm{Cat}$-enriched categories. Rune has constructed an equivalence there to n-Segal spaces. If someone came along and constructed another equivalence straight to Θ_n-spaces, then it's imperative to know that the triangle commutes.

The point is to talk about doing this for enriched categories. This is work in progress, including some statements that I don't know how to prove.

The enriched category situation is similar. Here $\mathcal{V}$ is a monoidal category, and we have $\underline{\mathcal{V}-\mathrm{Cat}}^{GH}$ due to Gepner–Haugseng, and then there's $\underline{\mathcal{V}-\mathrm{Cat}}^{Hin}$ due to Hinich and there's $\underline{\mathcal{V}-\mathrm{Cat}}^{Simp}$ due to Simpson. There are equivalences constructed on this triangle, by me, Rune and David, and then Hinich, and we want to calculate $\mathrm{Aut}(\mathcal{V}\,\mathrm{Cat})$ by identifying generators in order to see that this commutes around the triangle.

So $\mathcal{V}$ and $\otimes$ is a monoidal category, then Gepner–Haugseng and Hinich have constructed $\mathcal{V}-\mathrm{Cat}$, a full subcategory of what I'll call $\mathcal{V}$-algebroids, these are multi-object algebras in $\mathcal{V}$. They don't satisfy the univalence property, they are like the Segal spaces to the complete Segal spaces of categories. If $\mathcal{V}$ is presentably monoidal there is a completion functor in the other direction.

There's a forgetful functor from $\mathcal{V}$-algebroids, take the Grothendieck construction over all spaces

$$\int_{X\in\mathrm{Spc}} \mathrm{Fun}(X\times X,\mathcal{V}).$$

The fiber over X is algebras in $(\mathrm{Fun}(X\times X,\mathcal{V}),\otimes)$.

This construction is natural in $\mathcal{V}$, so in particular automorphisms of $\mathcal{V}$ act on $\mathcal{V}$-enriched categories.

Theorem 3.1. *Each autoequivalence of* $\mathcal{V}-\mathrm{Cat}$ *(also $\mathcal{V}$-algebroids) lifts uniquely to an equivariant structure.*

Then there is a sequence of groups $1\to\mathrm{Aut}(V,\otimes)\to\mathrm{Aut}(\mathcal{V}-\mathrm{Cat})\to\mathbb{Z}/2\mathbb{Z}$ *is exact. Moreover the fiber of the signature map over* $[\mathrm{op}]$ *is one to one with the class of reversals of $\mathcal{V}$, equivalences of $\mathcal{V}$ with $\otimes$ to $\mathcal{V}$ with* $\otimes\,\mathrm{op}$ *Splittings of signature correspond with involutive things.*

What are the objects of my generators? They're free $\mathcal{V}$-strings. I have a totally ordered sequence, so I take $v_1,\dots,v_n$ in $\mathcal{V}$, and the free $\mathcal{V}$-string on this is the algebroid on a space, $n+1$ elements, and $\mathrm{Hom}(i,j)=\bigoplus_{k=i+1}^{j} v_k$.

I erased that I can define alegbroids over X as algebras in some monoidal structure on functors $[X^{\mathrm{op}}\times X,V]$. This works for X a one-category.

In order to state the thing I'm going to write down I need X to be a 1-category, in which case [unintelligible]

So $\mathcal{V}$-algebroids with $n+1$ points marked has a forgetful functor to $\mathcal{V}^n$ and so $\mathcal{V}$-strings map down to Δ.

Proposition 3.1. (1) *Any autoequivalence of* $\mathcal{V}-\mathrm{Cat}$ *(or the other) restricts to an autoequivalence of* $\mathcal{V}$ *strings (this is where there is a gap).*
(2) *The projection is a Cartesian fibration, and the associated functor from* $\Delta^{\mathrm{op}} \to \mathrm{Cat}$ *is just* $(\mathcal{V}, \otimes)$.

So what I really want to do is identify $\mathcal{V}$-strings categorically, replacing $\mathcal{V}$ with presheaves on $\mathcal{V}$. I want to say that these are connected, compact, and projective. But I don't see, epimorphisms in Cat, you have to say what kind you want to use. So the problem is characterizing Δ^n in Cat.

Now for the second part, the fiber from $\mathcal{V}^{\times n}$ to $\mathcal{V}$-strings is fully faithful, and then you just need to look at the free property to figure this out.

For pullbacks, well, there are three kinds of maps, inert maps, multiplication maps, and unit maps. For example if I look at a map from Δ_1 into Δ_2 [pictures]

Something related to this appears in Rune and David's paper, so in particular we get that the $\mathcal{V}$-strings generate $\mathcal{V}$-algebroids and $\mathcal{V}$-categories.

Now the remaining points are, first, calculate autoequivalences of $\mathcal{V}$-strings. It's not completely obvious how to relate this, you have to recover the structure of the functor on Δ. This will only get you to a presheaf kind of picture so you have to reduce down to the idempotent completion. I should have been taking idempotent comlpetions the whole time. There's another reduction from algebroids in $\mathcal{V}$ to $\mathcal{V}-\mathrm{Cat}$. This follows mainly from that you know what they are and what they do.

Let me say more about how to reduce in these ways. Let's say a little bit about automorphisms of $\mathcal{V}$-strings. This is where we show that autoequivalences will lift on the forgetful functor to $1-\mathrm{Cat}$. So there's a map, I claim, from $\mathrm{Aut}(\mathcal{V}-\mathrm{strings} \to \Delta)$ to $\mathrm{Aut}(\mathcal{V}-\mathrm{strings})$, so there is anyway a map to $\mathrm{Aut}(\mathcal{V}-\mathrm{strings}) \times \mathrm{Aut}(\Delta)$. Then this has kernel Aut(forget), this is stupidly trivial because it lands somewhere with no equivalences. So now I'm in a much more straightforward situation of asking which elements of $\mathrm{Aut}(\mathcal{V}-\mathrm{strings}) \times \mathrm{Aut}(\Delta)$ to $\mathrm{Aut}(\mathcal{V}-\mathrm{strings} \to \Delta)$.

So construct a map from $\mathcal{V}-\mathrm{strings}$ to Δ^{un}, the "convex sets" of linearly ordered finite sets mod order direction.

So I can define $X \to Z \leftarrow Y$ in $\mathcal{V}-\mathrm{strings}$ is *disjoint* if the cones on it is empty. So being disjoint means that the maps are to a disjoint pair of simplices. If the vertices are maps from a simple object Q in, then for example, for Δ^n we have $n+1$ for $\mathrm{Map}(Q, X)$. Then you split up the slices and get a partically ordered set and compare this to these unordered simplices, and then lifting from unoriented to oriented, that's taking either the same direction or a reverse.

4. Danny Stevenson: model structures for correspondences and bifibrations

Thanks very much for the invitation. Let me, I'm talking about model structures for correspondences. The concept underlying this is the concept of a profunctor. This is a useful tool in category theory. If you have categories A and B, you can form the categories of presheaves on those, and this is a cocontinuous functor from $P(A)$ to $P(B)$ i.e., a functor from A to $P(B)$, the presheaf category, which by adjunction is a functor $B^{\mathrm{op}} \times A \to \mathrm{Set}$. There are various other characterizations of these things. Any functor into sets you can regard as a discrete opfibration, sometimes you'd call this a *distributor* from A to B. There's also the notion of correspondence. We recover a correspondence from this functor here, a correspondence is a certain type of functor with codomain the interval. so C_F the domain is the disjoint union of the objects of A and B. If two objects both belong to A, then the morphisms are the morphisms in A from x to y. If they both belong to B then they're the B-morphisms. If x is in A and y is in B then there's no morphisms, and if x is in B and y is in A then it's $F(x,y)$. Composition is a little complicated, you need to use a coend formula.

Then there's another notion, that of bifibration, that I'll talk about a little later. I want to talk about how these work in the context of ∞-categories. So I'll replace the category of sets with the infinity category of spaces. It's sort of obvious what to do for distributors. For correspondences or bifibrations it might be less obvious. This all might be known to experts (known to Joyal)—there's a model-independent treatment as well in a paper of Ayala–Francis so this is not, I'm not making a great claim to originality. So anyway I want to describe model structures and Quillen equivalences relating these model categories.

All right, so let me begin by talking about correspondences in a little more detail. SUppose that I've got two simplicial sets A and B, and I want to say what I mean by a correspondence from A to B. It's a little bit annoying the way this works out, this should be considered as the same as a simplicial map $p : X \to \Delta^1$, so that $\phi^{-1}(0) = B$ and $p^{-1}(1) = A$. This is a correspondence, and maybe I should say what a map means, a map between correspondences is a map of simplicial sets which restricts to the identity on A and B. So the correspondences with the maps between them is a subcategory of the slice category above Δ^1.

One thing that you can observe here is that the join construction $\mathrm{Set}_\Delta \times \mathrm{Set}_\Delta \xrightarrow{*} (\mathrm{Set}_\Delta)_{\Delta^1}$, the *join* of simplicial sets, the analog of the join of two spaces.

This functor is fully faithful and thus has a left adjoint. What it does, if you have $X \xrightarrow{q} \Delta^1$, you can look at the boundary and form the pullback, which picks out the inverse images of 0 and 1, and if X was the join of two simplicial sets, then the counit for this adjunction is an isomorphism. So you can look at the unit for this adjunction, which is a simplicial set with a map to $X(0) * X(1)$. So this category of correspondences has initial object $B \sqcup A$ and terminal object $B * A$. and then you can see that the category of correspondences, this sits inside $(\mathrm{Set}_\Delta)_{/B*A}$ as a reflective full subcategory. If I take $X \to B * A$ in the big category, and take the canonical map from $B \sqcup A$ into the join, and I look at the preimage, form this pullback diagram, and the value of the reflector L is the pushout of $B \sqcup A \leftarrow p^{-1}(B \sqcup A) \to X$.

So you see that in fact this category of correspondences has all limits and colimits. Of course that's

Let's suppose now that A and B are ∞-categories. I'll say that X is a *fibrant* correspondence if X to $B * A$ is a categorical fibration, exactly when the functor on homotopy categories is an isofibration, which is the same as being an inner fibration, an equivalence will lie entirely in B or entirely in A and then you can lift it back to X, and this is the same thing as asking that X is an ∞-category, so you check the lifting property against inner horns, it's obvious that this implies that X is an infinity category, and then you want the converse which is less obvious, you choose an extension of $\Lambda_{n,i}$ into X and an extension and it turns out to be compatible with the projection to $B * A$. If you suppose that X and Y are fibrant correspondences, then a map in the category of correspondences is a categorical fibration if and only if it's an inner fibration. So things become just a little simpler here.

So once you check these facts here, it's not too hard to construct a model structure for these correspondences.

Theorem 4.1 (Joyal, Lurie)**.** *If A and B are ∞-categories, then there is the structure of a left proper combinatorial model category on* $\mathrm{Corr}(A, B)$ *such that the cofibrations are monomorphisms and the weak equivalences are the categorical equivalences in the correspondences from A to B. Fibrant objects are fibrant correspondences that I've just described and the fibrations between fibrant correspondences are the inner fibrations.*

So the construction here is the edgewise subdivision, this goes $\Delta/(B^{\mathrm{op}} \times A) \to \Delta/(B * A)$, which takes $\Delta^n \xrightarrow{u} B^{\mathrm{op}}, \Delta^n \xrightarrow{v} A$ maps to $(\Delta^n)^{\mathrm{op}} * \Delta^n \xrightarrow{u^{\mathrm{op}} * v} B * A$, and so we have an adjoint triple $\sigma_! \vdash \sigma^* \vdash \sigma_*$ and we can compose the left adjoint with the reflector I wrote down to get to correspondences, and going back the other way we have an adjunction $a_! : (\mathrm{Set}_\Delta)_{/B^{op} \times A} \to \mathrm{Corr}(A, B)$ which is $L\sigma_!$ and it has an adjoint $\sigma^* i = a^*$ which has another right adjoint.

So the first choice is to say that $a_!$ takes a left anodyne map to an inner anodyne map. A condition is a kind of saturation property, so if you have a saturated class of monomorphisms and you know that it contains the initial vertex inclusions, then it contains all of these anodyne maps. So you look at maps, these are weakly saturated and satisfy a cancellation property. Then you check that the inner vertex inclusions do something nice as well, this is a little technical result that you can use, and this implies that if I take an inner fibration in the category of correspondences and hit it with a^*, this says that this should be a left fibration. So if C is an ∞-category, if you take the product of C with the interval, this is a correspondence from C to itself, this is a fibrant correspondence, and so by the observation here, you see that this map that you can write down is a left fibration, and then you get the twisted arrow category of C as $a^*(C \times \Delta^1)$. This gives another way to see that this thing is a left fibration.

Right. So then still under the assumption that A and B are ∞-categories, one can prove the following result. If one equips $(\mathrm{Set}_\Delta)_{/B^{\mathrm{op}} \times A}$ with the covariont model structure, and I've explained we have a right adjoint here, equipped with the model structure I defined earlier, then the theorem is that this is a Quillen equivalence.

One first checks that this is a Quillen adjunction. One wants to check that it takes trivial fibrations to trivial fibrations, you also want to check that it takes fibrations between fibrant objects across, and then using this fact that a^* takes inner fibrations to left fibrations, this completes the proof that it's a Quillen adjunction. One needs to prove that a^* reflects weak equivalences between fibrant objects. This follows from the fact that if I have a map between fibrant correspondences and it's a weak equivalence here, it is a pointwise homotopy equivalence, which translates into the condition that if you take $X(x, y)$ to $Y(x, y)$, then the induced map on mapping spaces is a weak equivalence. The fact that it's a map between correspondences means it's essentially surjective. You also have to check that the derived unit $X \to a^* \mathbb{R} a_! X$ is a weak equivalence, you break that down into a couple of separate problems, that a^* takes inner anodyne maps to left anodyne maps, and it turns out to be enough to show that $X \to a^* a_! X$ is a covariant equivalence and it turns out that this is left anodyne, and you can then check this when X is a simplex.

You have this furtrher adjoint a_*, and then a^* appears as a right Quillen equivalence and also a left Quillen equivalence (a^*, a_*). I should continue on, next I want to describe bifibrations in the context of simplicial sets. So back under the assumption that A and B are simplicial sets, there's a definition in Higher Topos

Theory, a map $(p,q) : X \to A \times B$ is a *bifibration* if the following three conditions are satisfied

(1) (p,q) is an inner fibration,
(2) for all $n \geq 1$ and all commutative squares

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r, "f"] & A \times B \end{tikzcd}$$

such that $\pi_B f|\Delta^{\{0,1\}}$ is degenerate in B, there is a diagonal filler.
(3) for all $n \geq 1$ and all commutative squares

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & X \arrow[d] \\ \Delta^n \arrow[r, "f"] & A \times B \end{tikzcd}$$

such that $\pi_A f|\Delta^{\{n-1,n\}}$ is degenerate in A, there is a diagonal filler.

There are a couple of examples of these. Let C be an ∞-category. If you look at maps from Δ^1 to C, this comes with a projection to $C \times C$, this is a bifibration (this is Lurie). It's also interesting to look at what happens when A and B are points. If B is a point, then $X \to A \times \{pt\}$ is a bifibration if and only if it's a left fibration. As soon as you're a left fibration you satisfy the other lifting condition as well. Then we can think of the case where A is a point, then you're a bifibration just when you're a right fibration. So it's covariant in A and contravariant in B.

In Higher Topos Theory the theory of bifibrations is not gone into detail, there's not a model structure that's described, I'll explain how this works. One way you can do this is parallel to Joyal and the covariant model structure. You might be interested in bifibrations because you want to be able to compose these, this is easier than composing distributors, essentially it's just pullback. Anyway, to describe the model structure, it's convenient to first introduce *bivariant* anodyne maps, this was chosen by Joyal to reflect the fact that they're covariant in A and contravariant in B. This is some class of maps in the slice category, generated by inner anodyne maps and all 0-horns in $A \times B$ (via (f,g)) such that $g|_{\Delta^{\{0,1\}}}$ is degenerate, and all n-horns in $A \times B$ such that $f|_{\Delta^{\{n-1,n\}}}$ is degenerate.

One can say that f is a bifibration in the slice category if and only if it has the right lifting property against B and one sees straightaway that $X \to A \times B$ is a bifibration if and only if it's a bifibration in the earlier sense.

Then there's a bunch of things that one can prove. This is going to be a list of things, but the point that I wanted to make is that this is parallel to the covariant model structure. So for instance one can prove that bifibrations are stable under exponentiation, so in other words if $X \to Y$ is a bifibration and $M \to N$ is a monomorphism in $(\mathrm{Set}_\Delta)_{A\times B}$, then the map $X^N \to X^M \times_{Y^M} Y^N$ is a bifibration in $(\mathrm{Set}_\Delta)_{A^N \times B^N}$ so if you take a bifibration, and look at $\mathrm{map}_{A\times B}(M,X)$ it's a Kan complex, which lets you define bivariant equivalence, a map $X \to Y$ is a bivariant equivalence if for any bifibration Z, the map of Kan complexes $\mathrm{map}_{A\times B}(Y,Z) \to \mathrm{map}_{A\times B}(X,Z)$ is a bifibration. Then what are some examples? It turns out that bivariant anodyne maps are trivial fibrations, and fiberwise homotopy equivalences are all bivariant equivalences.

One has the notion of bivariant equivalence, now you can get bivariant fibration, so f is a map of simplicial sets over $A \times B$, then you should have the right lifting property against all monic bivariant equivalences, and one can prove that under the assumption that X and Y are bifibrations, that f is a bivariant fibration if and only if f is a bifibration, the right lifting property against bivariant anodyne maps. Then the theorem is this.

Theorem 4.2. *Assuming A and B are simplicial sets, there is the structure of a left proper combinatorial model category on simplicial sets over $A \times B$ for which cofibrations are monomorphisms, the weak equivalences are bivariant equivalences, and the fibrations are bivariant fibrations I've described.*

What else is there to say? I won't have time to talk about Quillen equivalences to the category of correspondences, but let me say a couple of other things. It's good to be able to detect when a map is a bivariant equivalence. A map is a bivariant equivalence when it's a pointwise homotopy equivalence, this is an analog of the result for the covariant model structure.

Theorem 4.3. *A map $X \to Y$ over $A \times B$, not making any assumption that X or Y is fibrant, then it's a bivariant equivalence just when this is a homotopy equivalence:*

$$R \times_A Y \times_B L \to R \times_A Y \times_B L$$

for all right fibrations $R \to A$ and left fibrations $L \to B$. This lets you see that categorical equivalences are bivariant equivalences, so this is a left Bousfield localization of the Joyal model structure. Every bifibration is a categorical fibration. It's not so obvious to check this when the codomain is just a simplicial set. I think I should stop here. Thanks.

5. Jay Shah: aspects of the theory of real cyclotomic spectra

I'll be discussing aspects of this theory, focussing on diagrammatic relationships to [unintelligible]spectra. So the motivation is to think about structure on the topological Hochschild homology of A, either an associative or E_1 or E_∞ algebra.

I can describe the topological Hochschild homology as

$$THH(A \otimes_{A \otimes A^{\mathrm{op}}} A)$$

everything derived and done in spectra, so if A is an E_∞ algebra then this is $S^1 \odot A$, the tensor in E_∞ ring spectra is $C\,\mathrm{Alg}(\mathrm{Sp}, \otimes)$.

It's clear from this formula that S^1 acts on $THH(A)$. You have more structure which is encoded in the notion of cyclotomic spectra. There are at least two options for presenting this. The first option is due to Nikolaus and Scholze. Let's define an *NS cyclotomic spectrum* X to be a spectrum X with an S^1 action and for every prime p, a map φ_p from X to X^{tC_p} which is equivariant with respect to S^1. Recall that X^{tC_p} is the cofiber of an additive norm $X_{hC_p} \to X^{hC_p}$, a spectra version of the sum over conjugates.

This is a fairly new theory. Classically the cyclotomic structure was understood as follows. A *BHM cyclotomic spectrum* is an S^1-spectrum X that is "genuinely equivariant" with respect to all cyclic C_n inside S^1 such that by taking the geometric fixed points, that's equivalent to X for all n compatibly over n: $\Phi^{C_n} X \cong X$.

There's a lot more information, data floating around, than just X with its S^1 action. You can show that the topological Hochschild homology enhance to a BHM cyclotomic spectrum.

What's the comparison result? Nikolaus and Scholze show that these definitions coincide if X is bounded below, meaning that the homotopy groups vanish for sufficiently small n.

I'll revisit this equivalence in the talk and take into account the action of an involution. What about *real cyclotomic spectra*. What if C_2 acts on A via an *anti-involution* (if A is associative).

Then the C_2 action propagates to give me a C_2 action on $THH(A)$ as a "Borel" C_2-spectrum, which refines to a genuine C_2 spectrum, this is Hesselholt–Madsen–Dotto–Hogenhaen, $THH(A)$ refines to $THR(A)$ a genuine C_2 spectrum.

That doesn't tell us how the cyclotomic structure refines, which is what I want to address in this talk. The main question for us is how should I refine cyclotomic structure to take into account this involution on THH?

Let's look at option two, you should think about the S_1 action intertwining with the C_2 action to give the orthogonal group, $1 \to S^1 \to O(2) \to C_2 \to 1$. This gives an $O(2)$-action on $THH(A)$. Let's look at a fixed prime p, and look at the dihedral group D_{2p^∞} iside $O(2)$. This is like looking at $C_{p^\infty} \to D_{2p^\infty} \to C_2$ inside what I said before.

You can take $\mathrm{Sp}^{D_{2p^\infty}}$ as $\lim \mathrm{Sp}^{D_{2p^n}}$. Then the definition due to Dotto-Moi–[unintelligible]-Reeh in a different context is that a *real p-cyclotomic spectrum* is X in $\mathrm{Sp}^{D_{2p^\infty}}$ so that $X \cong \Phi^{C_p} X$ in $\mathrm{Sp}^{D_{2p^\infty}}$.

What about refining Nikolaus–Scholze? Fix a prime p, a real cylotomic spectrum consists of X with a D_{2p^∞} action, taking X to $X^{t_{C_2} C_p}$, D_{2p^∞}-equivariantly, and the goal of this talk is to explain this.

Fix a finite group G. Define a *Borel G-spectrum* Sp^{hG} to be the functor category $\mathrm{Fun}(BG, \mathrm{Sp})$. This is the most naive version. The more sophisticated version is defined as follows:

(1) Sp^G is Top^G with real equivariant [unintelligible]inverted, or
(2) Sp^G is functors preserving direct sums from spans of finite G-sets (denoted $\mathrm{Span}(\mathbb{F}_G)$)into spectra, where the objects are finite G-sets and the morphisms are correspondences $X \leftarrow Y \to Z$. The second definition is equivalent to the first, and it's getting closer to, this is a diagrammatic presentation of G-spectra in terms of spectra. I should send each orbit to a spectrum. Each one of these has an action, and there are maps in the span category. Or finally,
(3) there's a third way which is really the way I want to use in what follows, which requires a few notions from lax category theory. The second version is due to [unintelligible]and Barwick. This version is due to Ayala–Mazel-Gee–Rozenblyum, and for this you can write down a lax functor from posets of subgroups of G modulo conjugacy to Cat_∞, sending H to $\mathrm{Sp}^h(NGH/H)$ and the map is a generalized Tate functor and the diagram doesn't commute but you have a transformation from the Tate construction for the composition to the composition of Tate constructions. The oplax limit of this is equivalent to Sp^G. Now I didn't tell you what the generalized Tate construction is, but for domain 1, you get $\mathrm{Sp}^{hG(-)^{TG}} \to \mathrm{Sp}$ where $(-)^{TG}$ is initial omong functors with natural transformation $(-)^{hG} \to (-)^{TG}$ that kills G/H for $H \subsetneq G$ (here P_G is the subgroup poset of G mod conjugacy).

So this means precisely, if I write down a locally coCartesian fibration $\widetilde{\mathrm{Sp}^G} \to P_G$, then the oplax limit is $\mathrm{Fun}^{\mathrm{coCart}}_{/P_G}(\mathrm{sd}(P_G), \widetilde{\mathrm{Sp}^G})$, where $\mathrm{sd}(P_G)$ is strings in P_G with the max functor to P_G. And the result of Ayala–Mazel-Gee–Rozenblyum is that this oplax limit is equivalent to Sp^G.

So examples, if I take C_p, then P_{C_p} is Δ^1, and I think that I have only $1 \to C_p$.

The map from $P_G \to \mathrm{Cat}_\infty$ takes $1 \to C_p$ to $\mathrm{Sp}^{hC_p} \to \mathrm{Sp}$ by Tate. So then we can express Sp^{C_p} as a pullback of Sp_{hC_p} and Sp^{Δ^1} along spectra. So I have as my objects $(X^u, X^{\phi C_p}, \alpha)$ where X^u is a Borel C_p-spectrum and $\alpha : X^{\phi C_p} \to (X^u)^{tC_p}$.

Let's do the example of $G = C_{p^2}$. What's the poset? $1 \to C_p \to C_{p^2}$ and the lax functor to categories, I have $\mathrm{Sp}^{hC_{p^2}} \xrightarrow{(-)^{tC_p}} \mathrm{Sp}^{hC_p} \xrightarrow{(-)^{tC_p}} \mathrm{Sp}$ and the long composite is the mysterious TC_{p^2}.

No the data is that I have X^u with a C_{p^2} action and $(X^{\phi C_p})$ with a C_p action and $X^{\phi C_{p^2}}$, and I have maps

$$X^{\phi C_p} \xrightarrow{\alpha} (X^u)^{tC_p}$$

$$X^{\phi C_{p^2}} \xrightarrow{\alpha} (X^{\phi C_p})^{tC_p}$$

$$X^{\phi C_{p^2}} \xrightarrow{\alpha} (X^u)^{tC_{p^2}}$$

plus a homotopy. So you can think that this is really a lot of data.

How do we reduce this to the smaller model of Nikolaus and Scholze?

Lemma 5.1 (Scholze).

$$(X_{hC_p})^{t(C_p^2/C_p)} \cong 0$$

if X is bounded below.

This matters to us because

$$X^{TC_{p^2}} \cong (X^{hC_p})^{t(C_{p^2}/C_p)} \to (X^{tC_p})^{tC_p}.$$

So for X bounded below in $\mathrm{Sp}^{C_{p^2}}$ you get just $(X^u, X^{\phi C_p}, X^{\phi C_{p^2}})$ along with α and β maps.

That's enough of that theory for my purposes, so I'll move on toward real phenomena.

There are two things to understand. The first thing to understand are what the actions are, the Borel actions, and the second thing is what are the analagous decompositions of the dihedral spectra?

For actions, the base object is now a spectrum which is an equivariant C_2-spectrum. This is more complicated than functors from a groupoid to C_2-spectra.

Definition 5.1. A C_2-*category* is a functor $O_{C_2}^{\mathrm{op}} \to \mathrm{Cat}$ which I'll thnk about as a coCartesian fibration $C \to O_{C_2}^{\mathrm{op}}$. So for example, we can take $\mathcal{C}$ to be the following, since $O_{C_2}^{\mathrm{op}}$ is $[C_2/C_2 \to C_2/1]$, so the example is

$$C = [\mathrm{Sp}^{C_2} \xrightarrow{\mathrm{res}} \mathrm{Sp}]$$

where the codomain has the trivial action of C_2. I'll call this $\underline{\mathrm{Sp}}^{C_2}$.

So as an example take as the C_2-space $BS^1 \cong \mathbb{CP}^\infty$ with complex conjugation. Then I get OC_2^{op} to spaces as giving me $\mathbb{RP}^\infty$ inside $\mathbb{CP}^\infty$ equipped with complex conjugation. Then an "$O(2)$" action means a C_2-functor $B_{C_2}^T S^1 \to \underline{\mathrm{Sp}}^{C_2}$ over $O_{C_2^{\mathrm{op}}}$, where the domain here is the Grothendieck construction. So here this forgets to

$BO(2) \to \mathrm{Sp}$, the $O(2)$-action here is on the underlying spectrum, so this refines the Borel action to give the extra information of the genuine action.

The second step is to understand how to decompose dihedral spectra. We have, let m just recall, we have a decomposition of C_p-spectra, if I take Sp^{C_p}, then Borel C_p-spectra, another category given by spectra, I get

$$\mathrm{Sp}^{hC_p} \xrightarrow{j_*} \mathrm{Sp}^{C_p} \xrightarrow{i^*} \mathrm{Sp}$$

as $(-)^{tC_p}$ and having this decomposition of Sp^{C_p} tells me how to give a *definition* of the C_p-Tate construction.

So now for D_{2p}-spectra, I have $1 \to C_p \to C_p \rtimes C_2 \to C_2 \to 1$, and you have $\mathrm{Sp}^{D_{2p}} \xrightarrow{\Phi^{C_p}} \mathrm{Sp}^{C_2}$. So I can take $\mathrm{Sp}^{D_{2p}}$ sitting over $P_{D_{2p}}$ sitting over Δ^1 and I can break this up taking the oplax limits over 0 and 1. So I can take an oplax Kan extension along π to obtain a [unintelligible]of $\mathrm{Sp}^{D_{2p}}$.

Let me try looking at D_6, so I have $1 \to C_3 \to D_6 \to D_2$, I pick a splitting, and I get a lattice

$$\begin{tikzcd} & D_6 \arrow[dl] \arrow[dr] & \\ C_2 \arrow[dr] & & C_3 \arrow[dl] \\ & 1 & \end{tikzcd}$$

and if I split off D_6 and C_3 then I get $\mathrm{Sp}^{D_6} \xrightarrow{\Phi^{C_3}} \mathrm{Sp}^{C_2}$ and then the magic (I won't be able to explain this) is that the complement $\mathrm{Fun}_{C_2}(B^T_{C_2} C_3, \mathrm{Sp}^{C_2})$ where $B^T_{C_2} C_3$ is the C_2-space given by C_2 acting on BC_3 acting by inversion.

If you believe this story works for any prime then I can restate things for real cyclotomic spectra.

Definition 5.2. Take X a real cyclotomic spectrum which consists of the following data. I enhance X to have an action of the dihedral group in the C_2-parameterized sense

$$\mathrm{Fun}_{C_2}(B^T_{C_2} C_{p^\infty}, \mathrm{Sp}^{C_2})$$

where $B^T_{C_2} C_{p^\infty}$ is the C_2-space given by the conjugation action on BC_{p^∞} and a D_{2p^∞}-equivariant map $\varphi_p : X \to X^{t_{C_2} C_p}$.

Finally I'd like to point out the connection between the generalizations. We have the following parameterized Tate orbit lemma. If X is in $\mathrm{Fun}_{C_2}(B^+_{C_2} C_{p^2}, \underline{Sp}^{C_2})$, and if X is slice bounded below, then

$$(X_{h_{C_2} C_p})^{t_{C_2}(C_{p^2}/C_p)} \simeq 0$$

and this yields the theorem due to myself and Quigly that the two definitions of real cyclotomic spectra coincide, you have to play the same game, which I didn't explain in the non-real setting. The crux is the Tate orbit lemma along with the diagrammatic description of equivariant spectra.

6. Michael Ching: Tangent ∞-categories and Goodwillie calculus

This is joint with K. Bauer and M. Burke, both at Calgary. I don't know how many of these notions you have encountered before, but this is about two notions of "tangent category".

Let me briefly describe them. The one that is probably more familiar to this audience is the notion coming from deformation theory, where you essentially have the tangent category (I should add some ∞s in various places) to an ($\infty-$)category which should capture infinitesimal deformations. This is meant to be an analog to the tangent bundle to a manifold. The other notion, less familiar, comes from abstract category theory, an axiomatization of the tangent bundle functor, which is defined purely in category theory, where you can think of the functor as going from the category of manifolds to itself. That is by making this analogy more precise, connecting these notions.

I want to show that number one is an example of number two, so that the tangent-∞ category to an ∞-category is in a precise way the same kind of thing as the tangent bundle to a manifold.

I want to spend the first part of the talk telling you about these axiomatizations of the tangent bundle.

I'll try to avoid using the phrase tangent category because I want to distinguish between the two notions. So I'll call this *tangent structures on a category.*

So the motivating example is manifolds and smooth maps, and there's a functor from this category to itself which is taking the tangent bundle. So you might ask about the categorical structure of this functor. This was first addressed, I think, by Rosizký in 1984 and reinvented by Cockett and Crottwell in 2014.

If you have a category X then a *tangent structure* on X is an endofunctor $T : X \to X$. What other structure do you have? You have a projection map, so a natural transformation $T \to \mathrm{Id}$, a zero section $0 : \mathrm{Id} \to T$, and I could go on for a while, and there are a lot of conditions, lots of diagrams commute.

That's not a very convenient description to infinitify. This is the hands-on definition. So let me switch to the definition, this is due to Leung, I want to write down the free tangent structure, this will be a representing object, I write down a category Weil whose objects are Weil algebras, augmented commutative $\mathbb{N}$-algebras (i.e., rigs), rings without additive inverses, my algebras should be presentable with a presentation of the form $\mathbb{N}[a_1, \ldots, a_k]/(a_i a_j | i \sim_R j)$ for an equivalence relation R on my finite generators. So the example is $\mathbb{N}[a]/(a^2)$ which I'll call W. If I have two generators, I could have $\mathbb{N}[a,b]/(a^2, ab, b^2)$ or $\mathbb{N}[a,b](a^2, b^2)$. This latter is $W \otimes W$, which will be the coproduct, and the other one is the product of W with itself. The morphisms are just the morphisms of augmented $\mathbb{N}$-algebras.

This has a monoidal structure given by coproduct, and Weil is a strict monoidal category under $\otimes$ with unit $\mathbb{N}$. Now I can define a *tangent structure on X* is a strict monoidal functor $T^\bullet$ from $(\mathsf{Weil}, \otimes, \mathbb{N})$ to endofunctors of X with composition. This captures all of the data that Cockett and Crottwell came up with, and there are a couple of extra conditions, which are that $T^\bullet$ preserves the following pullbacks

$$\begin{tikzcd}
A \otimes W^{m+n} \arrow[r] \arrow[d] & A \otimes W^n \arrow[d] \\
A \otimes W^m \arrow[r] & A
\end{tikzcd}$$

and the next one which is the key:

$$\begin{tikzcd} W^2 \arrow[r, "a\mapsto ab, b\mapsto b"] \arrow[d, "\epsilon"] & W \otimes W \arrow[d, "a\mapsto a, b\mapsto 0"] \\ \mathbb{N} \arrow[r, "\eta"] & W \end{tikzcd}$$

Let me say what this looks like in the category of manifolds. I'm only seriously going to talk about the standard example, X is manifolds and smooth maps, and if I have W, that's the one that captures the usual tangent bundle functor. Then I have the augmentation, $\mathbb{N}$, $T^{\mathbb{N}}$ is the identity functor, and this $W \to \mathbb{N}$ is sent to projection. The unit map is the zero section. You can keep working out what structure maps you get. There's a map from $W^2 \to W$ sending a and b to a. We said T preserves the product, so $T^{W^2}M$ is $TM \times_M TM$ and this is the additive structure on the bundle.

As I said, to get to grips with these tangent structures, you need to understand these Weil algebras, and you'll see that they correspond to things that are familiar.

So what is the key lemma, the fact that the tangent bundle functor on manifolds preserves this one pullback? What does it say? It says that for each manifold M there is a pullback in manifolds of the following form.

$$\begin{tikzcd} TM \times_M TM \arrow[r] \arrow[d, "p"] & T(TM) \arrow[d, "T(p)"] \\ M \arrow[r, "0"] & TM \end{tikzcd}$$

I won't write down precisely what the map along the top is, it's telling you about $T(TM)$, some property of it, which is that if you look at the fiber of this over some point in M, and I get the fiber of the tangent bundle crossed with itself, so $T_xM \times T_xM \cong T(T_xM)$, and this is coherent in x.

That's the main (motivating) example of tangent structures on categories, if you take X to be schemes over a field, then the Zariski tangent scheme satisfies this, and you know that this has to do with the dual numbers object W.

Let me infinitify things, which is going to be very easy. This is now *tangent structures on an ∞-category.* I want to give a more general version that you might want here. The greatest generality in which this makes sense, let $\mathcal{X}$ be an object in an $(\infty, 2)$-category, that's the kind of thing where each object has an endomorphism object that's a (monoidal) $(\infty, 1)$-category. If you're interested in tangent structures on an ∞-category, you want $\mathcal{X}$ to be an ∞-category. In that case, the endomorphisms of $\mathcal{X}$ is just $\mathrm{Fun}(\mathcal{X}, \mathcal{X})$. Then we can copy the definition from above, a *tangent structure* on $\mathcal{X}$ is a monoidal $(\infty, 1)$-functor from $\mathsf{Weil}^{\otimes} \xrightarrow{T^{\bullet}} \mathrm{End}(\mathcal{X})^{\circ}$ such that T preserves those same pullbacks. Now I mean that they are preserved in the sense of $(\infty, 1)$-categories. That's the definition now.

How should you think of this if you're not familiar with monoidal functors between monoidal (∞-categories), you have lax maps which are equivalences, with higher coherences.

One thing that makes this relatively easy to write down is that Weil and endofunctors both have strict structures, which makes writing down the structure a little easier.

That's a definition of tangent structures on an ∞-category. Now let me turn to the other notion of tangent category, which I won't call a tangent structure. I'll call these *tangent bundles to an ∞-category*. This is the definition of Lurie (7.3 in Higher Algebra). So the setup is that I have $\mathcal{C}$ a presentable ∞-category, and we want to say what the tangent space to $\mathcal{C}$ at the object c is. It will be the ∞-category (capturing infinitesimal deformations)

$$T_c\mathcal{C} = \mathrm{Sp}(\mathcal{C}_c)$$

where $\mathcal{C}_c$ is the slice both over and under c. (Sp is the stabilization, this is the category of spectra over this). This is a universal type definition, and there's also a nice model for this, a subcategory of the excisive functors from $\mathrm{Top}_*^{\mathrm{fin}}$ to $\mathcal{C}$, excisive meaning that they take pushouts to pullbacks, and this subcategory should take the point to c. This also shows how to fit these together, this was the tangent space over c, the tangent bundle of $\mathcal{C}$ is the ∞-category $T\mathcal{C}$ is all excisive functors from finite pointed spaces to $\mathcal{C}$. You have a projection to $\mathcal{C}$ given by evaluating at the point.

If $\mathcal{C}$ is the category of spaces, you have spaces over and under c, so parameterized spectra. Then $T\mathcal{C}$ is all parameterized spectra.

I want to turn T into a functor so I should say what it does to morphisms of presentable ∞-categories, so $F : \mathcal{C} \to \mathcal{D}$ is a functor preserving filtered colimits. I want to define $TF : T\mathcal{C} \to T\mathcal{D}$, which takes L an excisive functor, the obvious thing to do is take it to $F \circ L$ which is no longer likely to be excisive, so I can take the excisive approximation $P_1(F \circ L)$. This is how you define the total derivative functor for a map between manifolds. You have a curve in one manifold. You compose that curve with F, and then take the tangent vector that curve generates.

So what I claim then is that this construction can be made into a functor T from presentable infinity category (with morphisms preserving filtered colimits), to [interrupted by questions].

Let me at least state the main theorem, which is as advertised, this structure of the tangent bundle is part of a tangent structure on

Theorem 6.1 (BBC)**.** *There is a tangent structure on $\mathrm{Cat}_\infty^{\mathrm{pr}}$ such that, for $A = \mathbb{N}[a_1, \ldots, a_k]/(a_ia_j | i \sim_R j)$, that $T^A\mathcal{C}$, if $A = W$, I should get Lurie's T. We'll generalize that to all Weil algebras, so $T^A\mathcal{C} \cong \mathrm{Exc}^A((\mathrm{Top}_*^{\mathrm{fin}})^k, \mathcal{C})$, where A-excisive functors are excisive in each variable separately (multi-excisive) and then each relation between two different generators I'll have a joint reduced condition, that $L(\ldots, X_i, \ldots, X_j, \ldots)$, well I could plug in the one point space into either entry or both and this is a pullback square (for $i \sim j$):*

$$\begin{array}{ccc} L(\ldots, X_i, \ldots, X_j, \ldots) & \longrightarrow & L(\ldots, *, \ldots, X_j, \ldots) \\ \downarrow & & \downarrow \\ L(\ldots, X_i, \ldots, *, \ldots) & \longrightarrow & L(\ldots, *, \ldots, *, \ldots) \end{array}$$

So what does this say? $T^W\mathcal{C}$ is just $T\mathcal{C}$, excisive. What about $T^{W\otimes W}\mathcal{C}$, this has two generators with no extra relation, this is $\mathbb{N}[a,b]/(a^2, b^2)$, so this is excisive functors of two variables.

The slightly harder one to work out is, what is $T^{W^2}\mathcal{C}$? This is L functors of two variables excisive in each such that the condition says that $T(X,Y) =$

$L(X,*)\times_{L(*,*)} L(*,Y)$, so this says that L is the product of two excisive functors which have the same value at a point, pairs of excisive functors which agree on the point, which is $T\mathcal{C}\times_{\mathcal{C}} T\mathcal{C}$, verifying that this is exactly the thing you want it.

I'll end by writing that key lemma,

$$\begin{tikzcd} T\mathcal{C}\times_{\mathcal{C}} T\mathcal{C} \arrow[r] \arrow[d, "\mathrm{ev}_*"] & T(T\mathcal{C}) \arrow[d, "T(\mathrm{ev}_*)"] \\ \mathcal{C} \arrow[r, "\mathrm{const}"] & T\mathcal{C} \end{tikzcd}$$

and again in words, just a little bit of Goodwillie calculus goes into working out what this is, functors of two variables that are excisive in each variable and reduced in one versus pairs of excisive functors. So this proof is basically a splitting result.

7. Gijs Heuts: Tate coalgebras and spaces

Let me pick a rough goal for this talk, which is to discuss some coalgebraic or algebraic models for the homotopy theory of pointed spaces $\mathcal{S}_*$.

The archetypical result in this direction is always Quillen's rational homotopy theory. Quillen proved that if you take rational simply connected pointed spaces, this actually admits two different models, this ∞-categroy is equivalent to Lie algebras in chain complexes over $\mathbb{Q}$, dg Lie algebras over the rational numbers, connected, with some degree shift hidden, sending a space to a dg Lie algebra whose homology is the rational homotopy groups of the space. He proved that this is equivalent to cocommutative algebras in chain complexes over $\mathbb{Q}$, simply connected. I'll abbreviate cocommutative by commutative.

The idea is that you'd like to generalize this. Can you generalize this to other localizations of the homotopy theory of spaces, or all of $\mathcal{S}_*$? There are many directions in which you can take this, let me list a few, Mandell gave a model for p-adic homotopy theory, which relates something like p-complete spaces to some version of E_∞ algebras, we can also investigate coalgebraic structures on $C_*(X,\mathbb{Z})$, also some kind of E_∞ coalgebra and ask what we need to do to an E_∞ coalgebra to make it like chains of a space. There's also a story related to chromatic homotopy theory. Rational is the first or zeroth localization coming out of a chromatic story, so you could talk about v_n-periodic localizations, where rational corresponds to $n=0$, something like the localization at nth Morava K-theory. Now there's a model in Lie algebras in spectra. This is some version of Lie algebras constructed by Michael Ching and Salvatore. The goal for today is not algebraic in the sense of differential graded algebra but in the ∞-category in spectra, so the theory of Tate coalgebras, a certain refinement of the theory of commutative coalgebras in the ∞-category Sp of spectra. It turns out that you can get a more or less complete model.

The starting point, if you want to compare spaces to coalgebras in spectra, any space is a coalgebra so if you take the suspension spectrum you get a coalgebra and you can ask how good that functor is. I'll use the following notation for coalgebras in spectra: $\mathrm{co}\,\mathrm{CAlg}^{\nu}(\mathrm{Sp})$, the ∞-category of commutative coalgebras in spectra without counits, this is just the opposite category of non-unital commutative algebras in the opposite category of spectra.

You can look at the following functor Σ^∞ as a functor from S_* to spectra that factors through coalgebras via the diagonal, $X\mapsto X\wedge X$, this functor has a problem, which is that it's far from being fully faithful. If we try to compare the mapping space between two spaces, these look not very much alike. The basic solution is to identify the extra structure that suspension spectra have. A different way to parse or interpret what I'm going to tell you, I'll tell you one way to recognize suspension spectra. In order to identify this extra structure I'll have to go on a little digression involving Tate constructions and the Tate diagonal. These featured heavily in Jay's talk yesterday. I'll be interested in finite groups, so let's say G is a finite group and E is a spectrum with an action of G, and here I mean in the naive sense, I'm not thinking about equivariant homotopy theory, so $E \in \mathrm{Fun}(BG, \mathrm{Sp})$. Then there are two evident spectra you can construct, homotopy orbits and homotopy invariants. So you can form E_{hG} and E^{hG} and there's a norm map

$$E_{hG} \to E^{hG}$$

and as Jay wrote this is $[x]$ mapping to $\sum_{g\in G} gx$ and the cofiber here is the Tate construction measuring the difference between the orbits and the fixed points. One characterization of the norm map is as follows, I won't define it but here is a characterization. It takes as input a spectrum with G action and outputs another spectrum $E \mapsto E^{tG}$ which is initial among functors with a natural transformation from the homotopy fixed points and satisfies the following two properties, first of all it's an exact functor, meaning it preserves cofiber sequences, and the real crucial property is that it's identitically zero on *induced* G-objects, meaning objects of the form $G_+ \wedge X$ for some spectrum X. There are many different ways to package this characterization, in terms of Verdier quotients for example.

Maybe I should give you some quick examples.

(1) The first example is maybe the reason for the name. Here if E is $H\mathbb{F}_p$ with trivial G-action, what does this look like? So orbits are like the homology of G with coefficients in $\mathbb{F}_p$ and the fixed points are the cohomology, and so by splicing these together you get the *Tate cohomology*

$$\hat{H}^{-*}(G, \mathbb{F}_p).$$

(2) Another example, if E is the sphere spectrum, G is a cyclic group of order p with trivial action, then $\mathbb{S}^{tC_p} \cong \mathbb{S}_p^\wedge$, the p-completion of the sphere. This is a consequence of the Segal conjecture.

The first example has things spread out in all dimensions but here you get a connective spectrum in the second case. These won't play a major role in the talk.

Let me try to explain why the Tate construction is relevant and say what we're trying to do here. So the point is, let's look at the following maps, say I had a suspension spectrum, then $\Sigma^\infty X$, so X has a diagonal and I get a map to $(\Sigma^\infty X \wedge \Sigma^\infty X)^{h\Sigma_2}$ and I can compose this with the canonical map to the Tate construction:

$$\Sigma^\infty X \xrightarrow{\delta_2} (\Sigma^\infty X \wedge \Sigma^\infty X)^{h\Sigma_2} \to (\Sigma^\infty X \wedge \Sigma^\infty X)^{t\Sigma_2}$$

So some facts:

- first, δ_2 does not extend to a natural map $E \to (E \wedge E)^{h\Sigma_2}$ for general spectra E. So there's no general diagonal.
- the composite $\Sigma^\infty X \to (\Sigma^\infty X \wedge \Sigma^\infty X)^{t\Sigma_2}$ *does* extend to a natural map $E \xrightarrow{\tau_2} (E \wedge E)^{t\Sigma_2}$ called the *Tate diagonal.* The reason is that $E \mapsto (E^{\wedge p})^{tC_p}$ is exact, if you specify something on the sphere it uniquely extends if it's

supposed to be exact. You can build finite spectra from fiber and cofiber sequences from the sphere spectrum.

Let me indicate the proof. Let me check that this preserves direct sums. The pth power of $E \oplus F$, you get

$$E^{\wedge p} \oplus F^{\wedge p} \oplus \bigoplus_{1 \le i \le p-1} \binom{p}{i} E^{\wedge i} \wedge F^{\wedge p-i}$$

and all of these cross terms are induced and the summand disappears and I just get $E^{\wedge p} \oplus F^{\wedge p}$.

What about the general case? You can reduce the general case to this one by a simple filtration argument. For a general cofiber sequence $E_1 \to E_2 \to E_3$, you can look at the pth power of the middle term, this admits an equivariant filtration with associated graded looking like the direct sum, $(E_1 \oplus E_3)^{\wedge p}$, and then you do the same thing.

So that's the proof of this lemma.

So spectra have this weaker Tate diagonal. So if E is a coalgebra in spectra and E is equivalent to the coalgebra arising to some suspension spectrum then the following commutes up to homotopy.

$$\begin{tikzcd} & (E \wedge E)^{h\Sigma_2} \arrow[d] \\ E \arrow[ur, "\delta_2"] \arrow[r, "\tau_2"] & (E \wedge E)^{t\Sigma_2}. \end{tikzcd}$$

This is something that suspension spectra have that a general coalgebra spectrum need not have.

This observation suggests an attempt to approximate spaces using something slightly better than coalgebras which takes this extra structure into account. I'll define an ∞-category which I'll call $\mathcal{P}_2\mathcal{S}_*$, let me just describe the objects. They can be described by the following pieces of structure. First of all we should have a spectrum E, and second of all it should be equipped by something that looks like a comultiplication, i.e., a map δ_2 as above into these homotopy fixed points $\delta_2 : E \to (E \wedge E)^{h\Sigma_2}$, it's just the very beginning of a commutative coalgebra structure. We could call this a 2-truncated commutative coalgebra, and a homotopy between the composite

$$E \xrightarrow{\delta_2} (E \wedge E)^{h\Sigma_2} \to (E \wedge E)^{t\Sigma_2}$$

and τ_2. This accepts a functor from the category of spaces by taking the diagonal coalgebra.

So we get adjunctions as follows.

$$\begin{tikzcd} & \mathcal{P}_2\mathcal{S}_* \arrow[dl, shift left, "\Omega_2^\infty"] \arrow[d, shift right, "\text{forget}"'] \\ \mathcal{S}_* \arrow[ur, shift left, "\Sigma_2^\infty"] \arrow[r, shift left, "\Sigma^\infty"] & \mathrm{Sp} \arrow[l, shift left, "\Omega^\infty"] \arrow[u, shift right] \end{tikzcd}$$

and some facts, $X \to \Omega^\infty\Sigma^\infty X$ is $(2\,\mathrm{conn}(X) - 1)$-connected (this is Freudenthal) and $X \to \Omega_2^\infty\Sigma_2^\infty X$ is $(3\,\mathrm{conn}(X) - 2)$-connected, here $\Omega_2^\infty\Sigma_2^\infty$ is [unintelligible], Goodwillie's 2-excisive approximation.

So this is the beginning of the Goodwillie tower

$$\begin{tikzcd}
 & \mathcal{P}_3\mathcal{S}_* \arrow[d] \\
 & \mathcal{P}_2\mathcal{S}_* \arrow[d] \\
\mathcal{S}_* \arrow[uur, "\Sigma_3^\infty"] \arrow[ur, "\Sigma_2^\infty"'] \arrow[r, "\Sigma^\infty"'] & \mathrm{Sp}
\end{tikzcd}$$

I'll discuss Σ_3^∞ because it has most of the features of the general case. Let me just record for those of you who don't know Goodwillie calculus well, we always have $\Omega_n^\infty \Sigma_n^\infty \cong \mathcal{P}_n[\textit{unintelligible}]$.

So what does $\mathcal{P}_3\mathcal{S}_*$ look like? In general it'll be inductive, the objects will be the following pieces of data. The first thing to specify is a 2-truncated Tate coalgebra E, and now I have to upgrade it to $\mathcal{P}_3\mathcal{S}_*$. First of all I have to upgrade it to a 3-truncated coalgebra. So first I should specify a map $\delta_3 : E \to (E^{\wedge}3)^{h\Sigma_3}$, compatible with δ_2 in the following sense. If I already have δ_2 I can build maps to the cube in a lot of ways. I could go

$$E \xrightarrow{(\delta_2 \wedge 1)\circ\delta_2} (E \wedge E) \wedge E$$

and so for each cyclic permutation of bracketings I get a different way of doing this, and this eventually factors through the homotopy fixed points of $E^{\wedge 3}$.

That's a three-truncated coalgebra, some sort of coassociativity of δ_2 there, and then there's still the Tate structure, what I'd call a 3-truncated Tate coalgebra. Any E in $\mathcal{P}_2\mathcal{S}_*$ has a natural map $E \to (E^{\wedge 3})^{t\Sigma_3}$ which will exist inductively because the codomain $(E^{\wedge 3})^{t\Sigma_3}$ is a 2-excisiv functor of E and $\mathcal{P}_2\mathcal{S}_*$ has a universal property with respect to 2-excisive functors. This is again similar to how I constructed τ_2. So I have such a natural map and I need to encode a compatibility, let me draw that in a diagram, I need homotopies making the following commute,

$$\begin{tikzcd}
E \arrow[dr, "\delta_3"] \arrow[drr, bend left, "\tau_3"] \arrow[ddr, bend right] & & \\
 & (E^{\wedge 3})^{h\Sigma_3} \arrow[r] \arrow[d] & (E^{\wedge}3)^{t\Sigma_3} \arrow[d] \\
 & ((E\wedge E)\wedge E + \cdots)^{h\Sigma_3} \arrow[r] & ((E\wedge E)\wedge E + \cdots)^{t\Sigma_3}
\end{tikzcd}$$

For general n it's very similar but the expression $(E \wedge E) \wedge E$ and its permutations gets replaced by a limit over a diagram of trees with n labeled leaves and at least one internal edge. In $n = 3$ this is discrete, you get three points. [pictures for $n = 4$]

The Tate coalgebra is the limit over n of $\mathcal{P}_n\mathcal{S}_*$, and if you put those together then you get the theorem

Theorem 7.1. $\mathcal{S}_*^{\geq 2} \cong \mathrm{coalg}^{\mathrm{Tate}}(\mathrm{Sp})^{\geq 2}$.

You can think of these Tate coalgebras as a way of making explicit what it means to be a coalgebra for the $\Sigma^\infty\Omega^\infty$ comonad. The coalgebras for this comonad model spaces. The problem with that is that being a coalgebra for that is not very explicit and this makes things more explicit, arguably.

The other remark is that nothing is particular to spaces, all of this works for general ∞-categories $\mathcal{C}$ which are pointed and compactly generated. By all of this I mean you construct these and set up Tate coalgebras, the only thing that might break down is the equivalence, which will depend on the convergence properties of the Goodwillie tower of the identity. You just have to check for each $\mathcal{C}$ individually.

8. Joost Nuiten: Deformation problems from Koszul duality

Thanks to the organizers for the invitation, it's been a great workshop so far. What I want to talk about is joint work with Damien Calaque and Ricardo Campos. Some of the goal of my talk is to explain a variant of the relationship between dg Lie algebras and deformation problems. Throughout my talk I'll work over a field with characteristic zero, and I'll replace Lie algebras with some algebras with other structure.

Let me explain some of the classical picture from deformation theory. If you think about this in terms of geometry you have some moduli space and a rational point in this moduli space, and you want to study the formal neighborhood around that point, $\mathcal{M}_x^\wedge$. One way to describe this object is with a functor of points approach, a functor from Artinian local algebras, for more refined moduli spaces, you want this to take values in groupoids or spaces. This is, informally, if you take an Artinian ring, if you take the spectrum of it, a space with the point inside it, you probe $\mathcal{M}$ with these and that's functorial in A.

Whenever your moduli space arises like this it satisfies two conditions, one is kind of obvious. If you look at the value on $\mathbf{k}$ itself, then $\mathcal{M}_x(\mathbf{k})$ is contractible. And there's a second condition that this arises from geometry, and that's the Schlessinger condition, if you have two surjective maps of Artinian rings, $A_1 \to A_0 \leftarrow A_2$, then this functor sends the pullback to the pullback of spaces,

$$\mathcal{M}_x^\wedge(A_1 \times_{A_0} A_2) \xrightarrow{\sim} \mathcal{M}_x^\wedge(A_1) \times_{\mathcal{M}_x^\wedge(A_0)} \mathcal{M}_x^\wedge(A_2).$$

I'll call this a *formal moduli problem.*

The point of view going back to Kodaira–Spencer is that this should be governed by a differential graded Lie algebra, but this is not ever going to be unique. So then instead of looking at any Artinian algebras, you look at $\mathrm{Art}^{\geq 0}$, the augmented commutative algebras in connected chain complexes, where the homology is finite and π_0 is Artinian in the classical sense. You also need to upgrade the Schlessinger condition, asking for it for maps that are surjective on π_0 instead.

So then you get a deformation theory that lets you describe things in terms of square zero extensions and the maps between them. A way to make this precise is the following result of Pridham and Lurie, which says that there is an equivalence of ∞-categories between the formal moduli problems and the category of differential graded Lie algebras, i.e., Lie algebras in the category of chain complexes. This basically takes a formal moduli problem and, if you think of this as encoding, you send it to the tangent space $X \mapsto \Omega TX$, the tangent space of the loop space informally, the loop space is a group so the tangent space has a Lie bracket. You can realize ΩTX as a spectrum, it's a collection of spaces, it's the value of X on square zero extensinos, $\Omega X(\mathbf{K}[\epsilon_n])$, where ϵ_n squares to zero and is of degree n.

There's some kind of description in the other direction, and this is more or less the space of solutions of the Maurer–Cartan equation in $\mathfrak{g}$, it sends A to the "space of Maurer–Cartan elements of $\mathfrak{g}\otimes\mathfrak{m}_A$", and this requires a choice of models.

So let me maybe give one example to see how this theorem comes into action, if you study deformations of algebras over operads. Fix P some differential graded operad, and B a connective P-algebra. Then there's a canonical formal moduli problem that you can write down, you write the functor from Artinian algebras to spaces which sends A to $P\text{–alg}(\mathrm{Mod}_A)\times_{P\text{–alg}(\mathrm{Mod}_{\mathbb{K}})}\{B\}$, and this is a formal moduli problem, and so there is a Lie algebra that classifies this, and it's classified by the complex of derivations of B as a P-algebra, $\mathrm{Der}^h_P(B,B)$ (properly derived), and derivations have an obvious bracket, the commutator bracket. There are explicit ways to compute this object, so for example, you can open Loday–Vallette's book and see this, for example if B is an associative algebra, then you get maps from B to itself:

$$\mathrm{Hom}(B,B)\to\mathrm{Hom}(B^{\otimes 2},B)\to\cdots$$

and there's also an explicit formula for the Lie bracket. In this case the explicit formula is

$$[\alpha,\beta]=\alpha\circ\beta-\beta\circ\alpha$$

where $\alpha\circ\beta$ is the sum over all possible ways of composing α with β.

I want to make one observation about things of this form. From this kind of description you can see that if all operations of arity greater than or equal to two are zero, then you don't just have a Lie algebra structure, but actually the complex of derivations has a preLie structure, maning it comes with an operation $\circ$, and it satisfies a funny equation,

$$\alpha\circ(\beta\circ\gamma)-(\alpha\circ\beta)\circ\gamma=\alpha\circ(\gamma\circ\beta)-(\alpha\circ\gamma)\circ\beta.$$

You can open Loday–Vallette for the general picture. So we wanted to understand this preLie structure at the level of formal moduli problems. In particular, it's maybe good to emphasize this, you do some point set model and you see that there's a structure, it's not clear that it's homotopy invariant. Even if you don't care too much about preLie algebras, you get similar things in other contexts, like in Deligne's conjecture where you say that the derivations are a Gerstenhaber algebra, and now this is a theorem with two proofs, one by Kontsevich–Tamarkin and one by Francis and Lurie, and you can ask if those are the same thing, for which purpose it would be a nice thing to get a clean description.

To address questions of this sort, you want to interpret some algebraic structure in terms of formal moduli problems, you need a version of Pridham–Lurie that works for other kinds of algebras.

Let me start by fixing, on the formal moduli side what is the commutative operad.

Fix P an augmented one-colored operad, and furthermore let me assume throughout that all $P(n)$ are connective and also $P(0)=0$. This assumption is not really necessary but it makes things a bit easier. This operad P is supposed to play the role of the commutative operad in the older story.

Let me start by telling you how to define formal moduli problems parameterized by algebras over such an operad.

Definition 8.1. Say that a P-algebra A is Artin if it satisfies two conditions:

(1) $\pi_*(A)$ is finite dimensional for all $*$ and vanishes for $*<0$ and $*\gg 0$.

(2) For every i, there is an action of $\pi_0(A)$ on $\pi_i(A)$, and you ask that it acts nilpotently, so

$$\pi_0(P(n)) \otimes \pi_0(A)^{\otimes(n-1)} \otimes \pi_1(A) \to \pi_1(A)$$

So that's the analog to what you do in the commutative case. So then you can anticipate the definition of a formal moduli problem.

Definition 8.2. A *formal moduli problem* is a functor

$$\mathrm{Art}_P \xrightarrow{X} \mathcal{S}$$

such that

(1) $X(0) \cong *$ and
(2) X preserves pullbacks along maps which induce a surjection on π_0.

This explains the left hand side of the Pridham–Lurie theorem. The right hand side, the role is played by the dual of the operad P.

Let me recall, we saw this in previous talks, if you have an augmented operad, you can take its bar construction, which is, one way to define it is as the relative circle product $1 \circ^h_P 1$, and you can describe this as decorated trees where each vertex is decorated by the kernel of this map (and there's a degree shift) and this has the structure of a cooperad by cutting trees along edges. In particular if you take the linear dual you get an operad $P_!$.

Then the theorem that we prove is the following, if P satisfies this list of conditions, and satisfies one other condition, one way of saying it is that P has a free resolution with the property that, the homological degree of the generators in arity n, and you let n go to ∞, thn the degrees go to ∞. So in each homological degree you only have generators in finitely many arities.

then formal moduli problems over P are equivalent to algebras over $P^!$ and the functor is basically the same, taking X to the tangent space of X. If you want a spectrum model, evaluate this on trivial algebras.

It's a good time to say that people have thought about this theorem before, maybe the thing was that it wasn't quite clear what this extra condition should be, it seems like this works. It's really, here, about the asymptotic behavior, what happens as arity goes to ∞ and not in fixed arity.

What are some examples of opreads that satisfy this condition? You can take P to be any associative algebra (connective). In this case it reduces to algebras, i.e., left modules over $B(P(1))^\vee$, are equivalent to reduced excisive functors from $\langle \mathbb{K} \rangle^{\geq 0}$ to $\mathcal{S}$, the domain the thick subcategory generated by $\mathbb{K}$. This is also $\mathrm{Ind}(\langle \mathbb{K} \rangle^{\mathrm{op}})$.

If P is quadratic Koszul and has finitely many generators, then $P^!$ is the quadratic dual up to a shift. So commutative and Lie is an example, and a more interesting example is the Poisson operad which appears in Kontsevich–Tamarkin, and E_n, with dual E_n up to a shift by n, either by Fresse or by [unintelligible].

In particular you can open Loday–Vallette's book, pick your favorite quadratic operad, and then [unintelligible].

Let me not say anything about how the proof works, instead let me talk for the rest of the time about the baby example, trying to understand the preLie structure on derivations, this is a kind of funny example.

Recall that if you have an algebra and all operations in arity at least two are zero then you get a preLie structure, so you need to understand the Koszul dual, which is the permutative operad, a Perm-algebra is a non-unital associative algebra, such that

$$a(bc) = a(cb),$$

it's symmetric as soon as you multiply on the left by some element. It's a theorem of Chapoton–Livernet that the dual is $\mathsf{Perm}^! = \mathsf{Pre}\,\mathrm{Lie}\{\pm 1\}$.

Okay, so this is supposed to be encoded by a formal moduli problem over permutative algebras, and so you can wonder what that formal moduli problem is, if A is a permutative alegbra, then denote by A^+ the associative algebra obtained by adding a unit to it. Then associated to this is a dg operad whose colors are the right modules over A^+, and maybe I should assume cofibrant and connective. The morphisms are all given by multilinear maps, $A^+ \to V_0$ or $V_1 \to V_0$, except for that in the case that the arity is at least two you ask for the map from $V_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_n$ to land in $V_0 \otimes_{A^+} A \subset V_0$.

If you apply this to the zero permutative algebra you get the category of chain complexes.

and then you can say that if $A \to B$ then $V \mapsto V \otimes_{A^+} B^+$ lets you go from the operad M_A to M_B and this requires you to be permutative.

Okay then the lemma is that the map

$$\mathrm{Art}_{\mathsf{Perm}} \xrightarrow{A \mapsto M_A} \mathsf{Operads}$$

preserves pullbacks along π_0-surjections.

The corollary is that for any $B : P \to M_0$ (which is chain complexes over k), is the same thing as a P-algebra in chain complexes where all arity at least two operations are zero. So if you fix this thing, then there's a functor from Artinian permutative algebra to spaces, sending A to the space of lifts from $P \to M_0$ to $P \to M_A$, and by the lemma this is a formal moduli problem, and furthermore it is classified by a preLie algebra structure on this complex of derivations, $\mathrm{Der}^h_P(B, B)$, and you can prove that this is precisely the thing that people found using bar cobar in the classical case.

I have five minutes left. Are there questions about this? Let me spend the last five minutes mentioning that we have a version for colored operads. In the colored setting there is something else that you can do. Instead of taking augmentations over the base field you can augment over any dg category. Then the entire story can be done over this category, and there's a funny, there's an example of this, one example is the case where $\mathcal{C}$ is finite sets and bijections, linearized, and P is the operad for non-unital one-colored operads. In this case it turns out that if you compute the dual relative to finite sets and bijections, it's self-dual up to a shift. So now you find that the category of one-colored operads has an interpretation as a moduli problem over basically nilpotent operads. You can interpret any algebra over operads as [unintelligible]—this is a bit of a formal exercise, but it's kind of funny that you can apply this in the case of operads themselves.

9. Inbar Klang: Twisted Calabi–Yau algebras and duality

Thank you to the organizers for inviting me and giving me the opportunity to visit Korea for the first time. This is joint with Ralph Cohen. I'll start by talking about how I think about Calabi–Yau algebras, which will go through field theories and factorization homology.

This is a thing that has input an n-manifold M, maybe you want to require this to be framed, parallelizable, and an E_n-algebra A, which can be in spaces, spectra, chain complexes, other things, and the output will be $\int_M A$, the factorization homology of M with coefficients in A, and the output will be wherever A lives.

Some properties:

(1) The first thing I want to say is that $\int_{\mathbb{R}^n} A \cong A$, an analog of the dimension axiom for ordinary homology
(2) this is a "homology theory for manifolds", functoriality in the algebra and the manifold, an excision axiom,
(3) it generalizes Hochschild homology: $\int_{S^1} A$ is the Hochschild chains on A if A is a differential graded algebra, the $THH(A)$ if A is a ring spectrum, if you take an E_∞ algebra A, then this generalizes higher Hochschild homology.

So maybe I'll draw a picture of how I like to think about this, I learned this picture from Jeremy Miller, so here A is a topological Abelian group, so a very specific kind of E_n algebra in the category of spaces. You have M, and you're going to take a configuration space of points in M labeled by A, you're going to see what happens when points get close, so informally when points collide you multiply them. So configurations in M with labels in A, and when points collide you multiply the labels. A remark, what does it mean when points collide multiply the labels? As a set this is a disjoint union of configurations for all n. When you have points close together you say you're approaching the point where your configuration has one fewer point and it's labeled by their product.

There's also the Dold–Thom theorem, saying that $\pi_* \int_M A \cong H_*(M, A)$ for A discrete.

For A an E_n-algebra, this puts my drawing skills to the test, instead of having points that collide you can have disks in your manifold labeled by A, and disks inside disks, and we'll take this to disks in M labeled by A. Over on the left side this is [pictures].

That was factorization homology or maybe the way that I like to think about it, maybe the next thing I want to talk about is topological field theories.

Definition 9.1. An *n-dimensional topological field theory* is a functor Z from the category of $(n, n-1)$-dimensional cobordisms to $\mathcal{C}$, symmetric monoidal so it sends disjoint unions of manifolds to the monoidal product $\otimes$ of $\mathcal{C}$. For an $n-1$-manifold you get $Z(M)$ in $\mathcal{C}$ and for a cobordism W^n you get a morphism from $Z(M)$ to $Z(N)$ and the disjoint union of manifolds goes to the monoidal product.

So for example for $\mathcal{C}$ chain complexes and $n = 2$, then you can look at the cobordism which is the pair of pants from two circles to one circle, and this is going to give you a multiplication map from $Z(S^1) \otimes Z(S^1) \to Z(S^1)$. This will tell you that $Z(S^1)$ is an algebra and it'll tell you other things as well.

What about *fully extended TFTs*? This is going to be super-informal because I'm kind of afraid of (∞, n)-categories, so

Definition 9.2. A *fully-extended TFT* is a functor Z from Cob_n to $(\mathcal{C}, \otimes)$ has 0-manifolds, one dimensional cobordisms between them, two dimensional cobordisms between those, up to n, and $\mathcal{C}$ will be a monoidal $(\infty-)n$-category.

This will sent M^0 to an object in $\mathcal{C}$, it'll send a 1-dimensional manifold to a 1-morphism, and so on, up until n-dimensional manifolds go to n-morphisms. This will also be symmetric monoidal, sending disjoint unions to $\otimes$.

I should probably remark about variants of this, you could talk about *framed* or *oriented* manifolds, and just change your cobordism category.

An example, we can take $\mathcal{C}$ to be the Morita 2-category $\mathrm{Alg}_{E_1}(\mathbf{k})$, where the objects are algebras, the 1-morphisms between A and B are (A, B)-bimodules, and the 2-morphisms are maps of bimodules. Then a two dimensional topological field theory will take $*$ to an algebra A, [pictures].

Theorem 9.1 (Lurie)**.** *A fully extended two-dimensional TFT Z from $\mathrm{Cob}_2^{\mathrm{fr}} \to \mathrm{Alg}_{E_1}(\mathbf{k})$ is determined by its value on $Z(*)$.*

Once you know what it does on a point, then you know what it does generally. One thing you could ask is what conditions A should satisfy to be one of these. Maybe I'll talk about that in a minute.

So another example of a fully extended topological field theory (an n-dimensional one). So an E_n algebra gives

$$\int_{-} A : \mathrm{Cob}_n^{\mathrm{fr}} \to \mathrm{Alg}_{E_n}(\mathbf{k}).$$

this is valued in the Mortia $(n+1)$-category of E_n-algebras considered as an n-category.

Now I'll veer away from the n-dimensional case and go back to the lower dimensional case, I'll go back to $n = 1$. so a point goes to A, and your interval goes to A as an (A, A)-bimodule, and S^1 goes to the Hochschild chains on A. It's good to know that you have this. But maybe you want to extend this to a two-dimensional field theory.

To extend this to a two-dimensional topological field theory, A has to be perfect as $\mathbf{k}$-module and as an $A \otimes A^{\mathrm{op}}$-module. This is called "compact" and "smooth" sometimes. Those are pretty restrictive conditions. You can only really do this completely if you have these restrictive finiteness conditions.

One way around this is to relax hypotheses. One solution is to restrict the kinds of two dimensional cobordisms that you allow. Today I'll talk about $\mathrm{Cob}_2^{\mathrm{or,nc}}$, so nc means non-compact, meaning only some cobordisms are allowed.

Two options for this that I like,

(1) cobordisms have non-empty ingoing boundary
(2) cobordisms have non-empty outgoing boundary

A bunch of people have written about this, Costello, Lurie, Kontsevich, with a bunch of collaborators, and these give two different dualizability conditions on an algebra.

The first condition gives you compact Calabi–Yau, which is for A perfect over $\mathbf{k}$, and the second condition gives you smooth Calabi–Yau, perfect over $A \otimes A^{\mathrm{op}}$.

We've turned these conditions into, you'll get some kind of field theory but it won't work for all cobordisms. Heuristically, a compact Calabi–Yau is something like "Frobenius", meaning you have A is a shift of its linear dual in a specific way. The second one is that smooth is something like "$CH_*(A)$ is some shift of $CH^*(A)$ in a specific way." I like to think of these as incarnations of Poincaré duality, in the first case between A and its linear dual and in the second case between its Hochschild homology and cohomology.

Definition 9.3. A *smooth Calabi–Yau algebra* is a pair (A, σ) where A is an algebra, perfect over $A \otimes A^{\mathrm{op}}$, and σ is some kind of fundamental class $\Sigma^m \mathbf{k} \to CH_*(A)$, which factors through the homotopy S^1 fixed points of the Hochschild chains. The dualizability condition is that, here you get

$$\Sigma^m k \otimes R\hom_{A\otimes A^{\mathrm{op}}}(A, A\otimes A^{\mathrm{op}}) \xrightarrow{\mathrm{ev}_\sigma} A$$

is an equivalence.

Here ev_σ, well, you get from σ a map to

$$(A \otimes^L_{A\otimes A^{\mathrm{op}}} A) \otimes R\hom(A, A\otimes A^{\mathrm{op}}).$$

For example, if M is a closed manifold or a Poincaré duality space then chains on the basd loop space of M is a smooth Calabi–Yau. Maybe you want M to be oriented to be precise.

Then you can ask, this algebra is smooth Calabi–Yau, what is the field theory you get from this? The corresponding two-dimensional TFT gives string topology operations on the homology of the free loop space of M, which is the Hochschild homology of $C_*(\Omega M)$, so you have everyone's favorite cobordism, the pair of pants, which gives the Chas–Sullivan loop product.

Maybe I'll just *say* a compact Calabi–Yau thing, then you can say cochains on M.

So right. Now I'll talk about our joint work with Ralph, about twisted Calabi–Yau algebras, extending to the category of spectra.

Definition 9.4. Basically we're going to mimic this definition but you can't always expect things like spectra to be Calabi–Yau because you might have twists. A *twisted smooth Calabi–Yau* ring spectrum is a triple (A, P, σ) with A an algebra perfect over $A \wedge A^{\mathrm{op}}$, P is an A-bimodule, such that there exists a ring spectrum E so that $P \wedge E \cong A \wedge E$ as $E \wedge A \wedge A^{\mathrm{op}}$-modules. Basically you want P to be twisted in a way that lets you untwist in some way. Finally $\sigma : S^m \to THH(A, P)$ is the fundamental class, it is such that ev_σ is an equivalence:

$$S^m \wedge R\hom_{A\wedge A^{\mathrm{op}}}(A, A\wedge A^{\mathrm{op}}) \to P.$$

This means that A is its own bimodule dual up to a twist, whatever P is. Where is S^1? That's a pretty necessary thing when you talk about Calabi–Yau conditions. There's also a notion of orientation for a smooth Calabi–Yau ring spectrum. I won't write it down completely, but what it means basically is a ring spectrum E such that this condition "untwists"; after this, things factor through homotopy S^1 fixed points,

$$THH(A\wedge E, A\wedge A)^{hS^1}.$$

For example,

(1) the one I should definitely mention is the suspension spectrum of the based loop space has this structure, so M is a closed manifold or Poincaré duality space, $\Sigma^\infty_+ \Omega M$,
(2) and you can also talk about twisting this a little bit further, you have the Thom spectrum $\Omega M^{\Omega f}$, the Thom spectrum of a loop map, $\Omega M \to BGL(\mathbb{S})$ which has this structure,

maybe I'll verbally say some things about orientation, to get an orientation on $\Sigma^\infty_+\Omega M$ you need an orientation on M, to get one on $\Omega M^{\Omega f}$ you need an orientation that trivializes the Hopf map, this also shows up in symplectic geometry of cotangent bundles, this is my nemesis or maybe my frenemy.

Maybe before I end I want to say two things, you also have the notion of a compact Calabi–Yau ring spectrum, the Spanier–Whitehead dual will be one of these as you might expect.

Some related questions:

- does higher order string topology come from such a thing, i.e., on the mapping space from S^n to M?
- If so, you could probably formulate twisted E_n-Calabi–Yau conditions. Evidence for this that came up in my thesis is this kind of twisted duality between S^n factorization homology of $\Omega^n M^{\Omega^n f}$ and topological Hochschild cohomology of $\Omega^n M^{\Omega^n f}$. Does this come from an E_n-Calabi–Yau condition? What does this give in terms of field theory?

10. David Carchedi: the Universal Property of Derived Manifolds

Thanks very much, and for the invitation, it's a pleasure to be here, derived manifolds are near and dear to my heart, and I feel like we're finally leaving the wild west, so everything I'm talking about is joint work with Steffens (a student of Calaque) and you can find all of this on the arXiv, posted last week.

Let me give a little introduction, motivation, the motivation for derived geometry comes out of transversality. Every time I say manifold, I mean smooth manifold.

So suppose I have smooth maps $f: M \to L$ and $g: N \to L$, then we say that f is transverse to g, written $f \pitchfork g$, if for all (m,n) in $M \times N$ such that $f(m) = g(n)$, that $f_*(T_*M) + g_*(T_nN)$, I get T_pL, the full tangent space. In particular, this buys us that $M \times_L N$ is a closed submanifold in $M \times N$ and is a pullback in the category of manifolds. If we don't have transversality this can fail, and as badly (epicly) as possible. For example, suppose I have M a manifold and C an arbitrary closed subset. There exists a smooth map f from M to $\mathbb{R}$ such that C is the zero set of f, you could get a Cantor set, no hope of it being a manifold. Having a lack of transversality obstructs many things you might like to do. For example, it obstructs the smoothness of certain moduli spaces. A recipe for many moduli spaces, start with a base space with a vector bundle over it and a section, and the moduli space you're after is the zeros of your section, the intersection of the section with the zero section. You could think of σ as defining a differential equation, and if σ is not transverse to the zero section then $\mathcal{M}$ the intersection could be quite singular.

Some famous examples of moduli spaces coming up this way include moduli spaces of J-holomorphic curves or instantons or if you ignore finite dimensionality, the solutions of the equations of motion for some action functional. In fact Owen Gwilliam and I are trying to model these using derived geometry in an infinite dimensional setting. Already, though, at the level of algebraic topology, you can see a lot. So suppose X is a manifold, and I can look at the unoriented cobordism ring of X, and this can be presented $MO^*(X)$, and this is presentable as proper smooth maps $f: M \to X$ (call this $\widetilde{MO}(X)$) quotiented by cobordism.

This is an Abelian group where addition corresponds to coproduct, which forces zero to correspond to the map from the empty manifold. There's a ring structure I'll mention in a moment, but first let me say something. As I said if I don't have transversality I could get something singular, but maybe by accident f is not transverse to g but the fiber product exists. This won't be cohomologically correct, it won't behave well with respect to cohomology, or cobordism theory. To explain that I'll have to say what the ring structure is.

But first let me say a stupid example. The point and the point in S^1 intersects in a point, which is certainly not transverse, those tangent spaces are zero and their sum is certainly not S^1.

Let's define the product on the cobordism ring. What you do is, if you have a class of $f_1 : M_1 \to X$ and $f_2 : M_2 \to X$, and I'll take the fiber product:

$$[f_1] \smile [f_2] = [M_1 \times_X M_2 \to X]$$

when f_1 is transverse to f_2, and otherwise replace f_2 with a smoothly homotopic f_2' which is transverse to f_1. The naive answer, should it exist, might not agree with this recipe. So in the smooth cobordisms of S^1, if a is the point in S^1. Then on the one hand, $[a] \smile [a]$, I'd replace the second a with another transversal copy, but then this forces it to be empty. So the empty manifold corresponds to zero so $[a] \smile [a]$ is zero. But if we had the formula without transversality I'd have $[a] \smile [a] = [a]$.

Moreover, you can already see a problem, if we think of $\widetilde{MO}(X)$ as a functor, well, there's a problem because we can't pull back classes.

One goal of derived manifolds is to make this problem go away, so that everything works out.

Let's first start with a vague idea, and eventually I'll tell you a precise property. Let's start for now at a categorical level.

The idea is, without transversality there should always exist a derived pullback, whatever that means, call it $M \times_L^\infty N$ which is a derived manifold (I haven't told you what that is), and if $f \pitchfork g$ then the derived pullback should be equivalent to the ordinary one. We also want that our formula for $\smile$ holds without transversality if we use derived pullbacks.

Let's make this a little bit more precise. There should exist a category $D\,\mathrm{Mfd}$ of derived manifolds and we should have a fully faithful inclusion of manifolds into the category of derived manifolds which should preserve transverse pullbacks and the terminal object, and $M \times_L^\infty N = i(M) \times_{i(L)} i(N)$, and we want our $\smile$ equation to hold.

A problem is this: there cannot exist such a category of derived manifolds, and here is the proof which is a few lines. Look at $i(*)$ going to $i(S^1)$, since i preserves the terminal object we get the terminal object for $i(*)$, and then for any M, I get unique maps to $i(*)$ which agree by the definition of the square so if the $\smile$ equation holds we get a unique map to the pullback which is then terminal.

What went wrong? We're in a one-category. If we're in spaces, the ∞-category, when I take the pullback of the terminal object in X I get the based loops in X, so then $D\,\mathrm{Mfd}$ has a chance to exist (it will) as a *higher* category.

Enough mucking around, let's give a precise definition.

The first universal property, motivated by work of Andrew Macpherson, who's in the audience, we proposed the following universal property: we'll say that derived manifolds is an idempotent complete ∞-category with finite limits and a functor from Mfd preserving transverse pullbacks and the terminal object, universal with respect to this property, meaning that for all idempotent complete ∞-categories $\mathcal{C}$ with finite limits, compositon with i produces an equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{lex}}(D\,\mathrm{Mfd}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}^{\pitchfork}(\mathrm{Mfd}, \mathcal{C})$$

where the superscripts are preserving finite limits and preserving transversal pullbacks and the terminal object.

The idempotent completion is an ∞-category thing, if I have finite limits and am an n-category then I'm already idempotent complete.

Here's one easy consequence, we have the underlying space functor from Mfd to Top, and now I have an induced functor $D\,\mathrm{Mfd} \to \mathrm{Top}$ preserving finite limits. This was built into some other models of derived manifolds but comes for free from the universal property.

So I haven't proved that this exists, it would be an initial object in some category but we don't know that this initial object exists. So we'll show it exists by giving another description of the universal property.

So for this we'll talk about C^∞ rings. A C^∞ ring is a commutative $\mathbb{R}$-algebra together with n-ary operations $A(f) : A^n \to A$ for all smooth maps $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ plus a natural compatibility. Let me give an example and then a more precise definition. So the prototypical example, is $C^\infty(M)$, for $f : \mathbb{R}^n \to \mathbb{R}$. Suppose g_1 through g_n are smooth functions, then you can do $f(g_1, \ldots, g_n)$.

Let $C^\infty \subset \mathrm{Mfd}$ be the full subcategory on $\mathbb{R}^n$ for all n.

Definition 10.1. A C^∞ ring is a finite product-preserving functor from C^∞ to sets.

Note that $\mathbb{R}$ is a ring, and addition and multiplication are smooth maps, which tells me that there's a ring structure on $A(R)$, with extra structure, I get a commutative ring and the extra structure comes from the fact that this is a functor.

Now C^∞ rings act a lot like rings, if I quotient [unintelligible]by an ideal there's a natural structure of a [unintelligible]ring so you can really start to do commutative algebra with these guys.

Also C^∞-rings have a coproduct, which we'll denote by $\otimes_\infty$, and we know that $C^\infty(M) \otimes_\infty C^\infty(N) \cong C^\infty(M \times N)$. In fact there's a theorem:

Theorem 10.1 (Moerdijk–Reyes)**.** *The functor $C^\infty : \mathrm{Mfd} \to (C^\infty - ring)^{\mathrm{op}}$ taking M to $C^\infty(M)$ is fully faithful and preserves transverse pullbacks and the terminal object*

Definition 10.2. If $\mathcal{C}$ is an ∞-category with finite products, then a C^∞ ring in $\mathcal{C}$ is a finite product preserving functor $C^\infty \xrightarrow{A} \mathcal{C}$, and the infinity category of these I denote by $C^\infty(\mathcal{C})$

So for example $C^\infty(\mathrm{Set})$ is C^∞-rings. But $C^\infty(\mathrm{Spc})$ is (equivalent to) the category of simplicial C^∞-rings

$$(\text{simplicial } C^\infty\text{-rings})^\circ_{\mathrm{proj}}$$

with the projective model structure.

There's a version of Dold–Kan saying that this is equivalent to dg C^∞-algebras.

There's a theorem due to myself and Steffas saying that $C^\infty : \mathrm{Mfd} \to C^\infty(\mathrm{Spc})^{\mathrm{op}}$ is fully faithful and preserves $\pitchfork$-pullbacks and the terminal object. We have this theorem here, it takes us ten pages but anyway.

Let's return to universal properties, we have

$$C^\infty \overset{q}{\hookrightarrow} \mathrm{Mfd} \overset{i}{\hookrightarrow} D\,\mathrm{Mfd}$$

which is a C^∞-ring in $D\,\mathrm{Mfd}$ so $\mathcal{O}_{D\,\mathrm{Mfd}} : D\,\mathrm{Mfd}^{\mathrm{op}} \to C^\infty(\mathrm{Spc})$ which takes M to $\mathrm{Map}(M, i \circ q)$, this looks like a sheaf sort of, so we say a collection of maps

$$(f_\alpha : T_\alpha \to M_\alpha)_\alpha$$

in $D\,\mathrm{Mfd}$ is a *cover* if $(UT_\alpha \to UM_\alpha)_\alpha$ is an open cover in Top. This gives $J_{D\,\mathrm{Mfd}}$ a Grothendieck topology and it turns out that $\mathcal{O}_{D\,\mathrm{Mfd}}$ is a sheaf of C_∞-rings on $D\,\mathrm{Mfd}$ with respect to this topology.

Definition 10.3. An *algebraic theory* is an ∞-category $\mathbb{T}$ with finite products and a $\mathbb{T}$-algebra in $\mathcal{C}$ is a finite product preserving functor from $\mathbb{T}$ to $\mathcal{C}$, and we'll denote the ∞-category of such as $\mathrm{Alg}_{\mathbb{T}}(\mathcal{C})$.

For example, taking $\mathbb{T}$ to be C^∞, then $\mathrm{Alg}_{\mathbb{T}}$ is C^∞-rings.

We'll say a $\mathbb{T}$-algebra A in spaces is *finitely presented* if it is a compact object; if I look at maps from A to blank, this preserves filtered colimits.

Note that for any object t in $\mathbb{T}$, if we write $j(t)$ is maps from t to blank, as a functor from $\mathbb{T}$ to spaces, this preserves finite products, this is just the free $\mathbb{T}$-algebra on t. So this will be the free algebra, in our example, on n generators, this is $C^\infty(\mathbb{R}^n)$.

So we get get a functor j from $\mathbb{T}$ to $(\mathrm{Alg}_{\mathbb{T}}(\mathrm{Spc})^{\mathrm{fp}})^{\mathrm{op}}$, and this functor preserves finite objects, so it's a $\mathbb{T}$-algebra, a canonical $\mathbb{T}$-algebra in finitely presented $\mathbb{T}$-algebras, op.

Here's a theorem.

Theorem 10.2. *If $\mathcal{C}$ is an idempotent complete ∞-category with finite limits, then composition with j induces an equivalence between left exact functors from the opposite category of finitely presented $\mathbb{T}$-algebras in $\mathcal{C}$ and just $\mathbb{T}$-algebras in $\mathcal{C}$.*

Maybe the following is useful, you want to be a retract of a finite colimit of free algebras to be compact, concretely, so things are determined completely by what happens to free algebras. So this is classical for algebras in sets, and we just mimic the proof.

Why am I saying this, let me just tell you another thing to be explicit. This equivalence is composition with j, but how do we go backward? Given a functor from $\mathbb{T}$ to $\mathcal{C}$ that preserves finite limits, the universal way to get a functor from finitely presented $\mathbb{T}$-algebras is to take the right Kan extension $\mathrm{Ran}_j(A)$.

So here's a second universal property, provided we have a category of derived manifolds,

Theorem 10.3. *The first universal property is equivalent to the following: for all idempotent-complete ∞-categories $\mathcal{C}$ with finite limits, left exact functors from derived manifolds to $\mathcal{C}$ is equivalent to $C^\infty(\mathcal{C})$.*

I'll tell you why this is true in a moment, since this is the same property as before, this shows that $D\,\mathrm{Mfd}$ is $(\mathrm{Alg}_{C^\infty}(\mathrm{Spc})^{\mathrm{fp}})^{\mathrm{op}}$. So every derived manifold is affine.

So why is this true? The lemma says that for a functor $C^\infty \to \mathcal{C}$ (as above), the following are equivalent:

1. F preserves finite products
2. $\mathrm{Ran}_q F$ (where q, remember, is the inclusion of C^∞ into all manifolds) exists and preserves transversal pullbacks and the terminal object.

What you need for this is first of all the ∞-version of the theorem of Moerdijk and Reyes, and that every manifold is a retract of a transverse pullback of $\mathbb{R}^n$s. Every open subset can be realized in this way, this uses tubular neighborhoods, so there's geometry there. Then some abstract category theory with Kan extensions.

A corollary, I gave you this corollary, that these are the same, but let me add, the sheaf $\mathcal{O}^{\mathrm{op}}_{D\,\mathrm{Mfd}} \to \mathrm{Alg}_{C^\infty}(\mathrm{Spc})^{\mathrm{op}}$ lands in finitely presented algebras, realizing the equivalence. You can unwind that.

I have two minutes left, I want to mention Spivak's model and how you recover it. So Spivak's model, well you hav Loc_{C^∞} the ∞-category of spaces locally ringed in simplicial C^∞ rings. So Spivak defines derived manifolds as a subcategory here. First of all we have an inclusion of manifolds into Loc_{C^∞} taking M to M equipped with smooth functions. Then

$$\begin{tikzcd} Rf^{-1}(0) \arrow[r] \arrow[d] & \mathbb{R}^0 \\ \mathbb{R}^n \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

is a pullback in Loc_{C^∞}, and $(X, \mathcal{O}_X)$ locally of this form is a quasi-smooth derived manifold. Then you can say that $d\,\mathrm{Man}_{\mathrm{Spivak}}$ is the smallest subcategory of Loc_{C^∞} containing Mfd and closed under finite limits and retracts.

$\mathbb{R}$ is a C^∞-ring in Loc_{C^∞} so $\varphi : D\,\mathrm{Mfd} \to \mathrm{Loc}_{C^\infty}$ is left exact.

Theorem 10.4 (C.–Steffans). *This φ is fully faithful and its essential image is $d\,\mathrm{Man}_{\mathrm{Spivak}}$.*

I'll stop there but I can tell you more if you ask me.

11. Imma Gálvez Carrillo: Decomposition spaces and objective models of symmetric functions

[I do not take notes at slide talks]

12. Joachim Kock: Operadic categories and 2-Segal spaces

Thanks for the conference, it has been really nice, all of the talks have been good up to now. This is going to be very elementary, all that I want to do is explain a definition, that of operadic categories of Batanin–Markl. I want to relate it to 2-Segal spaces that now you all know what they are. People coming from Dyckerhoff and Kapranov say "2-Segal spaces" and this goes to all higher numbers. Decomposition spaces is really for the combinatorics of decomposition.

So this formalism is for cyclic operads, colored operads, it fits into a diagram with these other concepts, hopefully I can get to it at the end. Batanin and Markl used this to prove the duoidal Deligne conjecture. You all know the regular Deligne conjecture, about the center of a monoid being a 2-monoid. If you want to do enriched monoids, if $\mathcal{V}$ is a braided monoidal category then it makes sense to talk about about monoidal categories enriched in $\mathcal{V}$. You can do more general duoidal categories, this has two different monoidal structures compatible by means of a lax distributive law. In a braided monoidal category it's invertible. So if you have a duoidal category you can enrich. With a duoidal enrichment you get a kind of center, a sort of Hochschild cohomology, and that has an action of a 2-operad.

So they introduce the notion of an operadic category. An operadic category is a category $\mathcal{C}$ with three pieces of structure:

(1) chosen local terminal objects
(2) a cardinality functor $|\cdot|$ from $\mathcal{C}$ to $\mathbb{F}$ (a skeleton of finite sets and maps)

(3) a fiber functor, for each $X \in \mathcal{C}$ and for each i in the cardinality $|X|$ of X, a fiber functor $\varphi_{X,i} : \mathcal{C}_{/X} \to \mathcal{C}$ which takes f to $f^{-1}(i)$.

plus nine axioms, I don't want to list the axioms, I want to give examples and then a new approach.

You should have in mind $\mathbb{F}$ so that the cardinality is the identity and the fibers are the actual fibers.

The motivation for this definition is that an *operad* for $\mathcal{C}$ in $\mathcal{V}$ is a lax functor from $\mathcal{C}$ to $\Sigma\mathcal{V}$ (a bicategory with one object, the objects of $\mathcal{V}$ as one-cells, and the morphisms of $\mathcal{V}$ as two-cells) so that $P(Y \xrightarrow{f} X)$ is isomorphic to

$$\bigotimes_{i \in |X|} P(f^{-1}(i)).$$

For $\mathbb{F}$ you get the usual notion of a symmetric operad.

The algebraist's Δ (finite ordered sets including the empty set and monotone maps), operads recover non-symmetric operads, this is Day–Street 2001. Let's see how that works to get a feeling for how this kind of structure encodes an operad. A lax functor into $\Sigma\mathcal{V}$, Day–Street formulate this a little bit differently, in this case there's actually a monoidal structure on Δ_{alg} and $\otimes$ on $\Sigma\mathcal{V}$. In this case there should be a strictly monoidal lax functor, for Day and Street. Since it's strictly monoidal, something like $n \to k$ is a sum of a lot of maps like $n_1 \to 1 + \cdots + n_k \to 1$, and so you're forced to send this to the tensor product $\bigotimes_{i=1}^k V_i$ where V_i is $P(i \to 1)$. So define V_i in this way. Then let's see how the lax conditions on P translate into the operad axioms.

Suppose I have a composition $n \to k \to 1$, then the lax condition says there should be a map from a certain tensor product somewhere, this is

$$V_{n_1} \otimes \cdots \otimes V_{n_k} \otimes V_k \to V_n.$$

So the lax structure give us this map and the axioms for a lax functor tell us that this is an operad.

Batanin–Markl formulate the monoidal condition differently in terms of fibers, because there's nothing saying the whole map is the product of the fibers, you have to put this tensor condition instead. This is a $\mathcal{C}$-operad in $\mathcal{V}$.

One example is that you could have true fibers, an operator category in the sense of Barwick, this is a special case where the category has a terminal object and such that pullbacks with the terminal object always exist, and these fibers are physical fibers. Any operator category is [unintelligible]operadic category.

I should warn you that this is not stable under equivalence because you had to choose local terminal objects.

I don't want to list the axioms, partly because I don't remember them and partially because I don't want you to remember them. They take two pages and are a little bit subtle. You'd like to understand this better, and one reinterpretation was given by Lack, and he says they're certain skew-monoidal categories, I mention this because it's a nice repackaging of the whole concept, it has nice connections to quantum algebra, this is Szlachányi, a categorical approach to weak Hopf algebras. I don't want to say more about that.

Let me start the new approach to operadic categories, with two questions you might have, why chosen terminals and why artificial fibers?

The category of finite sets and surjections tells you why you need artificial fibers that aren't even subobjects. From our point of view this can be seen in the following example, not from Batanin–Markl. So take $\mathcal{C}$ any category, consider $D\mathcal{C}$, the decollage $\mathrm{Dec}_T\,\mathcal{C}$. Take the nerve, shift everything down, and remove the top face and degeneracy map, this is $\sum_{x\in\mathcal{C}}\mathcal{C}_{/x}$. A nice starting point for decollage is Danny's paper.

So the point is that $\mathrm{Dec}_T\,\mathcal{C}$ is an operadic category, with chosen terminals $\mathrm{id} : x \to x$ and where all objects have cardinality 1, and finally the fibers, a map is a triangle $z \to y$ over x, a map g from fg to f. I only need to indicate one fiber. That should be just g. It does not live in the same component of the category. You check the axioms and see that it works and it has a different flavor from the examples given by Batanin and Markl. Since all objects have cardinality 1, the fiber functors are just $\varphi_x : \mathcal{D}_{/x} \to \mathcal{D}$, and you can assemble them into a map from $D\mathcal{D}$ to $\mathcal{D}$.

I want to say that an operadic category is unary if all objects have cardinality one. In this case the fiber functors have cardinality 1. It turns out that you can do a lot of interesting stuff with unary operadic categories.

Lemma 12.1. *Categories with chosen terminals is the same thing as D-coalgebras.*

For that we first need to remember that D is a comonad, the best way to see that is to step up to simplicial sets for a moment, Δ are finite non-empty linear orders, and Δ^t has top elements and top element preserving maps, and the forgetful map has +1, add a top element, and this monad induces a comonad on presheaves which restricts to the comonad on categories.

So $\mathcal{C}$ goes to $\mathcal{C}$ itself. Then x in $\mathcal{C}$ you need to give, this is supposed to be a coalgebra $\tau : \mathcal{C} \to D\mathcal{C}$, and you take x and take it to the unique map to the chosen terminal of the component of x. Now you should use the coalgebra axioms to see that this is an equivalence but first let's see how to go back. If I have a D-coalgebra, a category $\mathcal{C}$ together with $\tau : \mathcal{C} \to D\mathcal{C}$, then I can send x to τx, and first you use the counital condition here to figure out that the domain of the map τx is x itself. Using coassociativity you figure out that the target is terminal. I don't want to do more calculation, it would be confusing, but if you do it yourself it's easy.

The next step is, now you've given chosen terminals, the cardinality is always one, and you just need to reinterpret the fiber functor, which goes from D to the identity. This looks like it would like to be an algebra, but D is a comonad. If we write $D-\mathrm{coalg}$ as Cat^D, then it's a standard fact that the forgetful functor from $\mathrm{Cat}^D \to \mathrm{Cat}$ has a right adjoint, which produces a monad $\widetilde{D}$ on the category of coalgebras. If you take the axioms I didn't write and implement them in the unary case, you find out that unary operadic categories is the same thing as $\widetilde{D}$-algebras on Cat^D.

First we use D as a comonad to encode the terminal object structure, and then you produce the other part with the monad structure. I don't want to do proofs, this is just an unpacking.

Now I want to, this is a preliminary result for unary operadic categories, this already gives some hints about interesting structure. I erased that D was given by presheaves on the "add one" monad. This shows that D-coalgebras have the following shape. So D-coalgebra, let me write down $X = N\mathcal{C}$, and the fact that we're in Δ^t, we have extra top degeneracy maps, which are the terminal object structure, an extra top degeneracy on the nerve of the category. Now if you're optimistic, a $\widetilde{D}$-algebra will precisely give you, in addition to τ, the terminal object structure, an extra top face map. If you work through the associativity of $\widetilde{D}$-algebras, you get the face map relations. The structure of a unary operadic category on $\mathcal{C}$ is to add τ and φ, extra degeneracy and face maps. Now you can add the colimit, and get a unique extension, adding X_{-1}.

Proposition 12.1. *A unary operadic structure on $\mathcal{C}$ is an undecking of $N\mathcal{C}$.*

In particular, it means to find a simplicial set X such that DX is $N\mathcal{C}$. Since $\mathcal{C}$ is a category, it satsifies a Segal condition. Then X is lower 2-Segal. Imma told you the 2-Segal condition, and there were two pullbacks, and this one is lower 2-Segal and the other one is upper 2-Segal. For this [[unintelligible]] you need to know that every 2-Segal space is unitary.

Now you can ask about upper Segal.

Definition 12.1. $\mathcal{C}$ an operadic category is *regular* if $\varphi_{x,i}$ are left fibrations.

We defined this just because it corresponds to upper Segal. So regular unary operadic categories are the same thing as discrete 2-Segal spaces.

So we put this in to get to 2-Segal. It actually turns out that all examples that appear in practice are regular. So what we want to do now is incorporate the multi aspects.

We have to go back and see when we put unary, that was saying all objects have cardinality 1, now we really have $\varphi_{x,i} : \mathcal{C}_{/x} \to \mathcal{C}$. The part with the terminal objects is fine, that doesn't change. You can try to assemble these maps into a single map, first you define $\check{D} : \mathrm{Cat}^D \to \mathrm{Cat}^D$, where you assemble objects $\mathcal{C}$ into $\sum_{x,i\in|x|} \mathcal{C}_{/x}$. That's $\check{D}$, and now you want φ to be a structure map $\check{D}\mathcal{C} \to \mathcal{C}$. Is this an algebra map? unfortunately $\mathrm{op}\,\mathrm{Cat} \to \mathrm{Cat}^D$ is not monadic. In fact, this isn't even well-defined because you don't have a cardinality. So what I should do is take the slice of (Cat^D) over $\mathbb{F}$, and then there's a canonical choice, returning identities, of a cardinality map. Now it actually works.

Theorem 12.1. *Operadic categories are the same as $\check{D}$-algebras on $\mathrm{Cat}^D_{/\mathbb{F}}$.*

So we were very happy with the unary picture, where you could call them D-bialgebras, the monad changes so it's not strictly speaking the same, but if you just make the definition, it looks good. So we'd very much like to produce $\check{D}$ as the algebra from a coalgebra.

Let me just say that it works. A modified decollage on the arrow category of Cat works slicewise and you have a lot of sums floating around. That decollage is a comonad and restricts to the lax slice of Cat over $\mathbb{F}$ and then you can take coalgebras in there, and then you can take algebras in that monad but you only get the notion of lax operadic categories, which I don't think anyone is interested in that so far. So there's one trick we're not completely happy with where you enforce a Cartesian condition.

I wanted to outline briefly another approach that works in groupoids but not in sets, there should be a higher version which admits the following nice description.

The idea (which needs groupoids or higher) is that $\varphi_{x,i} : \mathcal{C}_{/x} \to \mathcal{C}$ should be assembled into $\varphi : DX \to SX$ (returning a family of fibers). Here S is the symmetric monoidal category monad. Now you could say that an *abstract* operadic category, is a D-bialgbra in the Kleisli category of S (meaning S appears in the codomain of the functor).

You need to go to groupoids because [unintelligible]is not Cartesian and this quickly gets you into all kinds of trouble, a general symptom of this problem in classical combinatorics, that multisets is not Cartesian.

In this context undecking still works. You start with your operadic category, and just from the terminal objects structure you get the extra structure, the face maps, the return maps are only Kleisli maps, you get a family of objects instead. This is what you see in the two sided bar construction of an operad, the top face map returns a family of objects. This is to work in the Kleisli category. Then again you can complete this in a unique way to a complete undecking in $\mathrm{Kl}(S)$, and only the top part is a Kleisli map, the new degeneracy is an old map. So that's how it should look in higher category theory.

Just one thing, I'll put down the chalk. You have a map from operads to operadic categories, that's the white part on the board, undecking is the two-sided bar construction. Only one fiber, that is like a kernel, so you get a unary operadic category if you have kernels, and then undecking is the $S_\bullet$ construction. You also get that here too, and these are two big classes of decomposition spaces. Let me stop here.