BERKELEY TOPOLOGY SEMINAR, MARCH 13 2013

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1. Joey Hirsh, Derived noncommutative deformation theory

[We're happy to have Joey Hirsh visiting us from CUNY]

Thanks, Kate, Owen, for being so hospitable. I'll talk about derived deformation theory toward the end of the talk. I'm going to try a different way to give this talk, some category theory and relate that to the theorem and what I'm doing in terms of deformation theory.

The goal is to identify a complicated $\infty - 1$ or ∞ category that comes from deformation theory (basically), noncommutative deformation theory with a category which is simpler and purely algebraic.

Before, maybe I should have said this forty seconds ago. I'll give things in the land of one-categories, so you don't need to know what ∞ categories are. I hope this will be interesting to people who HATE ∞ categories.

Here's my plan.

(1) The Yoneda embedding ad universal colimits
(2) Categories by generators and relations
(3) Examples
(4) Statement of the theorem/goal

1.1. The Yoneda embedding.

Any questions about that? Recall that for a category $\mathcal{C}$, we can take the category of functors from $\mathcal{C}^{op}$ to Sets, $\mathcal{S}et^{\mathcal{C}^{op}}$. So we can embed $\mathcal{C}$ in this by taking X to $\mathcal{C}(- - -, X)$, call this functor $\hat{X}$. So we get a map from $\mathcal{C}(X, Y)$ to natural transformations $Nat(\hat{X}, \hat{Y})$. This being an isomorphism is the Yoneda lemma. This says that objects are determined by all the maps into them.

If you're not used to thinking of the Yoneda lemma, if you see a chair, our eyes collect light, the light bounces, we can take this apart photon by photon and get pixels, this is the Yoneda lemma letting us see X by bouncing whatever we have off it.

Now I'd like to say something about universal colimits. So I'll start by reminding you what colimits are. If you give me a category $\mathcal{A}$ and a small category $\mathcal{D}$ and a functor X from $\mathcal{D} \to \mathcal{A}$, and I want to tell you what the colimit of such a functor (which I'll call a diagram, I think of $\mathcal{D}$ as just dots and arrows), I look at maps out of the diagram, I want a map from each object that is coherent with the arrows of the diagram. Formally that's an object A together with an isomorphism between $\mathcal{A}(A, - - -)$ and the limit over the diagram of $\mathcal{A}(Xd, - - -)$. Let me state a fact:

(1) $\mathcal{S}et$ has all colimits. You take a disjoint union over all the objects, and then use a relation that comes from the identification of points with their images.

(2) From this, we get that any functor category landing in $\mathcal{S}et$ has all colimits, and they are computed pointwise.

So even when $\mathcal{C}$ has colimits, the Yoneda embedding $\mathcal{C} \to \mathcal{S}et^{\mathcal{C}^{op}}$ does not commute with colimits.

I'll show you an example. Let's consider topological spaces. Consider a functor X from two objects 0 and 1 with no nontrivial morphisms into spaces. The colimit is $X(0) \sqcup X(1)$. The Yoneda functor on the colimit is maps from spaces into $X(0) \sqcup X(1)$. We can instead take the colimit of maps into $X(0)$ and maps into $X(1)$. So the disjoint union of maps is not the same as maps into the disjoint union.

Not only do these not commute,

Theorem 1.1. *The category $\mathcal{S}et^{\mathcal{C}^{op}}$ is the* free *colimit-complete category on $\mathcal{C}$.*

What does that mean? In analogy, in the category of algebras, if G is a set, what is the free algebra, I'd say $\mathcal{F}(G)$ has a set map of G, and for a set map $\gamma : G \to A$ I get a realization $Re\ \gamma : \mathcal{F}(G) \to A$. Here we take an algebra, we'll take categories with colimits there. So $\mathcal{A}$ will be a category with colimits and then we get an analagous diagram given a functor $\gamma : \mathcal{C} \to \mathcal{A}$ there is a unique functor $Re\ \gamma$ making the diagram compute.

$$\begin{tikzcd} G \arrow[r, hook] \arrow[dr, "\gamma"'] & \mathcal{F}(G) \arrow[d, dotted, "Re\ \gamma"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[r, hook, "Y"] \arrow[dr, "\gamma"'] & \mathcal{S}et^{\mathcal{C}^{op}} \arrow[d, dotted, "Re\ \gamma"] \\ & \mathcal{A} \end{tikzcd}$$

As an example, let's consider Δ, whose objects are $(0, \ldots, n)$ with weakly order preserving map.

So then $\mathcal{S}et^{\Delta^{op}}$ are simplicial sets $\mathcal{SS}et$. We have a functor from Δ to topological spaces, sending n to the n-simplex. This functor induces a unique functor from $\mathcal{SS}et$ to topological spaces. This is made by gluing simplices in specified ways. We take things apart, make each piece into a simplex, and then I know how to glue them together. Topological spaces have colimits, and when I glue together I get a simplicial complex called geometric realization. That's why I'm using this language.

1.2. **Generators.** If A is an algebra, we call a subset $G \overset{i}{\hookrightarrow} A$ a generaing set if $Re\ i : \mathcal{F}(G) \to A$ is surjective.

If $\mathcal{A}$ has colimits, by analogy, we call $\mathcal{C} \xrightarrow{i} \mathcal{A}$ generating if $Re\ i : \mathcal{S}et^{\mathcal{C}^{op}} \to \mathcal{A}$ is surjective. I could say more by using a little more category theory but maybe I don't want to.

1.3. **Examples.** So now we're at the examples.

The first one is kind of easy. Take fAb, which is finitely presented Abelian groups and their morphisms. Then take the inclusions into Ab. The only thing to realize is that in an arbitrary Abelian group you get lots of generators, but every relation can be realized using only finitely many generators.

So that's one example. I thought I'd go through a few others. There's the category $\mathcal{E}uc$, with objects natural numbers and morphisms $n \to m$ the smooth maps $\mathbb{R}^n \to \mathbb{R}^m$. There's a functor from $\mathcal{E}uc$ to smooth manifolds, taking n to $\mathbb{R}^n$, and I claim this is a generating subcategory for smooth manifolds.

Essentially what needs to be checked is that we can check everything about smooth manifolds by looking at maps from $\mathbb{R}^n$. I think we all believe that.

We have the category $Herm$ which has objects the natural numbers and morphisms holomorphic maps from $\mathbb{C}^n \to \mathbb{C}^m$. This generates complex manifolds.

These examples are illegal because they don't have colimits, but there is a way to finesse that, but here's another example. Let's take commutative algebras over $\mathbb{R}$, these have algebra maps between them. One of the fundamental ideas of algebraic geometry is that to consider algebraic manifolds, you do that locally by considering colimits of commutative algebras. So this category generates algebraic $\mathbb{R}$-manifolds, or what are called schemes over $\mathbb{R}$.

So all of this works for ∞-categories. A category has objects and morphisms between them, and morphisms have morphisms between them. I can't draw a three category. I can't draw in perspective so this is all you get. If you allow everything all the way up, you get an ∞-category. What is the $(\infty - 1)$ category? After stage one, everything is pseudo-invertible, all the higher morphisms are invertible. That's one way to describe what's happening. The first number is how high morphisms go and the second number is above this you have invertibility. I think what this is supposed to give you is more along the lines of $\infty -$ (1 category) meaning a 1-category up to homotopy.

For any $(\infty - 1)$ category $\mathcal{A}$ wiht homotopy colimits, for any functor $\mathcal{C} \to \mathcal{A}$ you get a realization functor $\mathcal{SS}et^{\mathcal{C}^{op}}$ into $\mathcal{A}$ preserving homotopy colimits, essentially unique. Good references are Dan Dugger, "Universal homotopy theories" or "Combinatorial model categories have presentations."

I planned to tell you this but not after another lie. I let you believe that in category theory, you can do generators and relations. I think of an algebra as being intrinsic and generators and relations are something that you can work with. In category theory, you might have generators and relations and that's really hard to work with. So in deformation theory we have the following picture.

In commutative deformation theory, well, I told you that in algebraic geometry we were talking about functors into $\mathcal{S}et$. Schemes over k is generated by commutative algebras over k. So if I'm doing algebraic geometry, I'd write down a functor from commutative algebras to $\mathcal{S}et$. So here we restrict to finite dimensional nilpotent algebras. Then we get an interpretation of the restriction which is geometric, this functor is covered by infinitely small commutative algebras.

In derived deformation theory, we'd want to land in spaces and study those.

I guess I'm ready to state the theorem.

1.4. **Theorem.**

Theorem 1.2. *The generators and relations of commutative deformation theory, which is* $Fun(CAlg^{fd,nil}, \mathcal{SS}et)$ *gives a presentation for the homotopy theory of differential graded Lie algebras.*

We consider functors from such things into spaces, they are Lie algebras. The way you get these things is Koszul duality.

I'm basically done. So the only thing I'd like to say about this is if commutative deformation theory describes functors from commutative algebras to spaces by Lie algebras. So what kind of algebraic structure controls functors from a different kind of commutative algebras.

So if you extend that duality, you have a guess for what the theorem should be. Algebraic structures turn out to come in dual pairs. An operad is a thing that controls algebraic structures, and these adjectives like associative, Lie, commutative, and so on, come in pairs $(\mathcal{O}, \mathcal{O}^!)$.

Theorem 1.3. *(H.)*
The generators and relations of $\mathcal{O}$-deformation theory, $Fun(\mathcal{O}Alg^{fd,nil}, \mathcal{S}Set)$ gives a presentation fro the homotopy theory of differential graded $\mathcal{O}^!$-algebras via the duality that goes from $\mathcal{O}$-algebras to $\mathcal{O}^!$-algebras.