GEOMETRY AND PHYSICS XIII—DERIVED GEOMETRY CENTER FOR GEOMETRY AND PHYSICS (POHANG)

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[I'm the director of the Center for Geometry and Physics. I'd like to welcome everyone especially those who traveled far from abroad. The theme of this conference fits very well the cause of this center. I'd like to thank the organizers and the staff who prepared for the successful launch of this conference. I'd like to give a brief history of the operation of this Korean government project. This is one of the 25 centers so far belonging to the organization, the Institute for Basic Science, which was created to support basic science in Korea and the world, where scientists enjoy conducting their research with the greatest freedom. This was officially started in 2012, but fully started operating only two years ago. We have emphasis within geometry and physics on symplectic and algebraic geometry, but not restricted to that. In the first year our center had its inaugural program in symplectic and contact topology and mirror symmetry. This year we have a theme year in the mathematics of quantum field theory organized by Professor Park and Calin Lazariou. This conference kicks off this year. I want to emphasize some of our programs. We have an intensive research program for small groups. If you want to spend some time with collaborators to do some intensive work, you can apply for this program. If you'd like to spend some time here, you can apply on our homepage. I'd like to ask all of you to visit our website sometime. I'd like to introduce our first lecturer, Kenji Fukaya from the Center for Geometry and Physics, also our distinguished visiting fellow.]

1. July 6: Kenji Fukaya: Floer homology for 3-manifolds with boundary I

In my generation, I'm fifty-something, Donaldson theory is something that everyone working on geometry should know. For the current generation Donaldson theory is like a classic which is respected but not so much studied.

So this is a gauge theory that is based on the Yang–Mills equation, not the Seiberg–Witten equations. Let me start with something that Floer considered.

Start with $M^3 \leftarrow E$, an $SO(3)$-vector bundle. Now $W^2(E)$ is nonzero on any connected component of M, this is an assumption, and it makes life a bit easier, because one of the most difficult things is with reducible connections, which corresponds to singular points in the moduli space. With this assumption there is no reducible connection.

We let $R(M, E)$ be the flat connections on E divided by gauge equivalence. Our bundle is not trivial, but this is kind of like a representation of the fundamental group. You count dimensions, and the dimension of this moduli space is 0. If I perturb the equation that says the curvature is 0, $F_A = 0$, then this becomes a finite set. There are many ways to perturb but I don't want to explain that. Now we consider the Floer chain complex $CF(M)$, a $\mathbb{Z}_2$-vector space with basis these points

$$CF(M) = \bigoplus_{a \in R(M,E)} \mathbb{Z}_2[a]$$

and there is a boundary operator

$$\partial a = \sum_b \langle \partial a, b \rangle b$$

with

$$\langle \partial a, b \rangle = \# \mathcal{M}(a, b)$$

where $\mathcal{M}(a,b)$ is the modul space of A, connections on $E \times \mathbb{R}$ (where t is the $\mathbb{R}$-variable) where A increases to b as t goes to $+\infty$ and A decreases to A as $t \to -\infty$ and these satisfy the anti-sel dual equation $F_A + *F_A = 0$ where $*$ is the Hodge star on $\Omega^2(M \times \mathbb{R})$, modulo gauge transformation cross translation.

Now there is a gauge choice, well, $A = A(t) = \Phi(t)dt$ where A is a connection on E and $\Phi(t)$ is a section of the adjoint bundle on E. So you can use a gauge where $\Phi \equiv 0$ where g is a gauge transformation, and $g^* A = g^{-1}Ag + g^{-1}dg + (g^{-1}\Phi g + g^{-1}\frac{dg}{dt})dt$, and you can always solve this ODE so that the dt part becomes zero and this becomes the following form,

$$\frac{\partial A}{\partial t} = *_M F_A$$

which is the ASD equations.

We consider the Chern-Simons functional A as

$$CS(A) = \int_M Tr(\frac{1}{2} A \wedge dA + \frac{1}{3} A \wedge A \wedge A)$$

and if you take the t derivative you get

$$\frac{d}{dt} CS(A) = \int_M Tr(\frac{\partial A}{\partial t} \wedge dA + \frac{\partial A}{\partial t} \wedge A \wedge A) = \langle * \frac{\partial A}{\partial t}, F_A \rangle_{L_2}$$

and so $\frac{\partial A}{\partial t} = grad_A CS$.

This is the usual Morse picture, and the Morse homology is the Floer homology I'm discussing. This is a very brief review of the gauge theory and there are no reducible connections in this special case and we can define Floer homology of a three manifold with associated bundle. Floer wrote this around twenty years ago. I want to say one more thing, about Yang–Mills Donaldson invariants. So let $\partial X^4 = M^3$ and $E_X \to X$ which restricts $E|_M = E$, then we have relative Donaldson invariants

$$q : H^2(X)^{\otimes k} \to HF(M; E)$$

and you consider connections A on X which satisfy the equation $*F_A + F_A$ up to gauge and then you do some pairing with q and get a number. So given a flat connection $HF(M,E)$, you can count solutions that have this boundary condition. So this was done twenty years ago. Something that was discussed but not done was the version for three and two dimensions.

Here you have X and some homology classes and you get numbers. The three dimensional theory you get a vector space and the 4 to 3 relative invariant is an element of this vector space. We try to generalize this to a two-dimensional story and that's what I want to explain today. In this particular case I think I can work out this kind of proposal.

So this time we have a three-manifold with boundary Σ^2 and we have $E \to M$, and I assume that $W^2(E)|_\Sigma = [\Sigma]$ in $H^2(\Sigma, \mathbb{Z}_2)$, so this means that you are nontrivial on every component of Σ. Then we can prove that $W^2(E) \cap [\Sigma] = 0$, which implies that in order that this equation holds, we will have an even number of boundary components.

The reason we need this condition, consider $R(\Sigma)$, flat connections divided by the gauge group on Σ. Under the assumption $W^2(E)|_\Sigma = [\Sigma]$, this is a smooth and symplectic manifold and

$$R(\Sigma) = \prod R(\Sigma_i)$$

and

$$\dim R(\Sigma_i) = 6g_i - 6$$

where g_i is the genus of Σ_i. This is not only symplectic but Kähler with J given by $*$ on $H^1(\Sigma, ad\ E)$, and ω is just the cup product on H^1.

This is a symplectic manifold. Now we want to consider this relative invariant. Proposed by Donaldson 23 years ago at some conference is the following.

We consider, $R(M)$, for now we leave out E, this is flat connections E divided by gauge transformations. The natural map $i : R(M) \to R(\Sigma)$ takes $A \to A|_\Sigma$. We consider $H^0(M, ad_A E)$, which is always 0 because there are no reducible connections. Then if you have $H^2(M, ad_A E) = 0$ then $R(M)$ is a smooth manifold in a neighborhood of A and i is a Lagrangian embedding in a neighborhood of A. Then $T_A R(A) = H^1(M, ad_A E)$, and the rank of H^1 is half of $H^1(\Sigma, ad\ E)$, and you can see as an exercise that the cup product restricted to this is 0. Then the restriction map (by the inverse function theorem) is an embedding locally. You might have $H^2(M, ad\ E)$ nonzero, and so you perturb $F_A = 0$ to something like $F_A = \phi(A)$ which is supported away from ∂M. After appropriate perturbations, you get this map

$$i : R(M)^\phi \to R(\Sigma)$$

which you can perturb to be a Lagrangian *immersion*. So now this is, the situation, in case you have a three-manifold whose boundary is a two-manifold, then the Segal story is about this codimension two manifold as follows. So for X^4 you get a "number" which is $H^2(X)^{\otimes n} \to \mathbb{Z}$. For M^3 you get a voctor space $HF(M, E)$. For Σ you get a category $F(\Sigma)$, and $\partial X = M$ implies that you get an element of the vector space. This is the relative Donaldson invariant $q(X)$. When $X = X_1 \sqcup_M X_2$, with $\partial X_1 = \partial X_2$ then $q(X) = \langle q(X_1), q(X_2)\rangle$. Then the three and two dimensional invariants should be related in the following way, $\partial M^3 = \Sigma^2$, then $HF(M)$ is an object of $F(\Sigma)$, and if $\partial M_1 = \Sigma = \partial M_2$, then $M = M_1 \sqcup_\Sigma M_2$, then $HF(M)$ is the morphisms from $HF(M_1)$ to $HF(M_2)$ in the category $F(\Sigma)$. You expect that when you glue, the vector space is the set of morphisms. This was proposed by Graeme Segal in the 1990s. Then for three dimenensional case you start a dimension lower, for three dimensions you get a number, and for a surface a vector space, and so on.

So the objects, Donaldson suggested, for $F(\Sigma)$, should be Lagrangian subspaces of $R(\Sigma)$ and the morphisms should be Floer homology. This is Lagrangian Floer homology, which, let me say it very briefly, let me consider a symplectic manifold Y and assume, I'll assume (as works in today's story) that it's monotone, so that $c^1(Y)$ is a positive constant times ω, so in this case $Y = R(\Sigma)$ and c is known to be 2. Maybe we should assume the minimal Chern number is

$$\inf\{\int_{S^2} \varphi^* c^1 | \varphi : S^2 \to Y, [\varphi] \nsim 0\}$$

. Here the minimal Chern number of $R(\Sigma)$ is 2. By the way, this Lagrangian Floer theory is due to Yong-Geun Oh.

We assume two embedded Lagrangian submanifolds L_1, L_2 in Y, this is already not right for us because we're immersed. So also assume monotonicity, meaning, well, you have a disk $u : (D_i, \partial) \to (F, L_i)$ and $\int u^*\omega = c\mathcal{M}(\omega)$, where $\mathcal{M}$ is the Maslov index (I don't want to explain this) and in our case I think c is again 2. So suppose $R(M)$ embeds in $R(\Sigma)$, then $R(\Sigma)$ is monotone. This case [we've been discussing] almost satisfies these conditions in general, but there is a case in which the Lagrangian is immersed, not embedded.

Under these assumptions, anyway, we may assume $L_1 \pitchfork L_2$ by perturbations, and then we take the vector space

$$CF(L_1, L_2) = \bigoplus_{p \in L_1 \cap L_2} \mathbb{Z}_2[p]$$

and the boundary is

$$\partial p = \sum_q \#\mathcal{M}(p,q)[q]$$

where $\mathcal{M}(p,q)$ counts the holomorphic disks between p and q.

Remember you have $\partial M_1 = \Sigma = -\partial M_2$ and $R(M_1)$ and $R(M_2)$ are immersed in $R(\Sigma)$; write $M = M_1 \sqcup_\Sigma M_2$.

So we have $HF(M,E)$, the Floer homology, and we have if $R(M_1)$ and $R(M_2)$ are embedded in $R(\Sigma)$ then you get $HF(R(M_1), R(M_2))$, and if these are embedded, then these two Floer homologies are isomorphic.

People tried to prove this, you have $R(M_1) \times_{R(\Sigma)} R(M_2) \cong R(M)$.

[missed a little]

Still the harder case of Lagrangian Floer theory is not defined. But if the restriction to each boundary component is nontrivial, you get rid of some difficulty. People still had trouble constructing it. They were able to compare in some cases the moduli space of instantons to the moduli space of holomorphic curves, but it wasn't totally successful. So I'll avoid the analytic problems using cobordism methods. I want to explain something about the immersed case.

Let me explain something about the immersed case of L in (Y, ω). In 2009, Akahi and Joyce generalized the Fukaya–Oh–Ohta–Ono story about the A_∞ algebra on $CF(L)$. Let me write $L = (\tilde{L}, i_L)$ where $i_L : \tilde{L} \to Y$ is a Lagrangian immersion. Then $CF(L) = H(\tilde{L} \times_Y \tilde{L})$. So $\tilde{L}$ in general is something like $\tilde{L} \to L$ in Y. We have several self-intersection points. Let's assume that they are transversal at each intersection. Then the fiber product $\tilde{L} \times_Y \tilde{L}$ as $\tilde{L} \sqcup$ finitely many points, where these are $\{(p,q) \in \tilde{L} \times \tilde{L} | p \neq q \text{ but } i_L(p) = i_L(q)\}$. Note that $(p,q) \neq (q,p)$.

Then $CF(L) H(\tilde{L} \times_Y \tilde{L}) = H(\tilde{L}) + \oplus_{p,q} \Lambda_0[p,q]$ where Λ_0 is a Novikov ring

$$\Lambda_0 = \{\sum a_i T_i^\lambda | a_i \in \mathbb{Z}_2, \lambda_i \geq 0, \lambda_i \uparrow \infty\}.$$

Then

$$CF(L_1, L_1) = H(\tilde{L}_1 \times_Y \tilde{L}_2)$$

and

$$m_k : CF(L_0, L_1) \otimes \cdots \otimes CF(L_{k-1}, L_k) \to CF(L_0, L_k)$$

which is given by counting polygons.

So I'll talk about this more tomorrow, but

$$\sum m_{k_1}(\cdots m_{k_2}(\cdots)\cdots) = 0$$

and these are the A_∞ relations. In general $m_0 \neq 0$ so $m_1 m_1 \neq 0$ and so we want to define $HF = \frac{ker\ m}{im\ m}$ but that won't be defined in this case. But we say $b \in CF^1(L)$

is a bounding chain if $\sum m_k(b,\dots,b) = 0$ and $b \equiv 0 \pmod{T}^\epsilon, \epsilon > 0$. Then with bounding chains, $HF(L_1,b_1),(L_2,b_2)) = \frac{ker\ d^{b_1,b_2}}{im\ d^{b_1,b_2}}$ is defined where

$$d^{b_1,b_2}(x) = \sum m_{k_1+k_2+1}(b_1^{k_1},x,b_2^{k_2}$$

which allows $d^{b_1,b_2}d^{b_1,b_2} = 0$.

Theorem 1.1. (1) *Let $\partial M^3 = \Sigma$ then there exists a bounding cochain $b_M \in CF^1(R(M))$ with $\sum m_t(b,\dots b) = 0$*
(2) *If $\partial M_1 = \Sigma = -\partial M_2$ and $M = M_1 \sqcup_\Sigma M_2$, then $HF(M,E) = HF((R(M_1),b_1),(R(M_2),b_2))$*
(3) *$R(M) \hookrightarrow R(\Sigma)$ implies $b_M = 0$. and b_M can be created in an invariant way.*

2. Ezra Getzler: The derived Maurer–Cartan locus

[I'm happy to introduce Ezra Getzler from Northwestern University, who will talk about the derived Maurer–Cartan locus] Forgive me if I'm confused about which continent I'm on, I arrived at 1AM.

The Maurer–Cartan equation is the equation for flat connections. If L^* is a differential graded Lie algebra, with grading unbounded above and below. For instance we could take differential forms on a manifold with coefficients in endomorphisms of a flat vector bundle $\Omega^*(M,End(E))$, and then we can look at the equation for $x \in L^1$,

$$dx + \frac{1}{2}[x,x]$$

, this goes from $L^1 \to L^2$ and the vanishing locus is called the Maurer–Cartan locus.

Now all of this can be generalized, many of you will have seen L_∞ algebras, except there's nothing particular in my talk that needs Lie algebras. For L_∞ algebras, you get all affine varieties, for a Lie algebra you get quadratic polynomials but then for L_∞ you can get higher degree.

There has been interested in the *derived* version of this back all the way to Tate and then the work of Batalin–Vilkovisky in the 80s, and then [unintelligible]of [unintelligible]and Kapranov. Let me give an impressionistic picture from Batalin–Vilkovisky.

Let's think of a graded manifold with degrees shifted by 1. The underlying manifold, the part we want to think of as the manifold is L^1 and then L^0, L^{-1} and so on; paired with these we have L^2, L^3, L^4, and so on. We're specifically interested in Lie algebras with a pairing between L^i and L^{3-i}. You'll have the fields ϕ as coordinates on L^1, the ghosts c, coordinates on L^0, the ghosts of ghosts γ on L^{-1}, and so on. There are theories in this formalism where you get a whole tower of ghosts of ghosts and so on. Then on the other side, in L^2 you have the antifields ϕ^*, c^*, γ^*; this two minutes of my talk would have been the whole talk of Toën. So c has to do with stacks, γ with two-stacks, and so on. The antifields are higher and higher levels of derived geometry.

My subject today is, how do I describe this in algebraic geometry. I'll talk about two formalisms and an equivalence between them. First I'll review the traditional Maurer–Cartan locus and give a review of derived geometry.

I took the word derived too seriously in the title of the conference, and someone has to tell you what a derived scheme is. I'll be talking about affine schemes, so I don't have to glue them together, which simplifies the story. Many of the basic ingredients of derived geometry are handed to you in this language. Maybe it's too strong a statement, but almost all the derived schemes one sees in practice are derived Maurer–Cartan loci.

So what's going on? The basic idea of a derived scheme is that it doesn't have positive degree coordinates, but it has 0 and negative degree coordinates.

So we have M, and then $\mathcal{O}(M)$ is a differential graded commutative algebra concentrated in degrees ≤ 0. [missed question about characteristic p]. I should say, this is very special, I think that in the literature one assumes that the *cohomology* is concentrated in negative degrees. I'm a geometer and I don't know how to handle the coordinates in higher degrees if I'm trying to do geometry.

I'm going to give examples of such guys, namely the Chevalley Eilenberg complex of a differential graded Lie algebra concentrated in degrees ≥ 1. Let me very quickly review this construction. It works very nicely for differential graded Lie algebras.

We suppose we have a homogeneous basis z_i of our differential graded Lie algebra so that the differential $\partial z_i = a_i^j z_j$ and I have a tensor $[z_i, z_j] = c_{ij}^k z_k$. I had a student in complex geometry, I shouldn't say this, I wrote something like this on the board and he asked if we were using coordinates and I said definitely and he left and never returned. So if you want to go...

So $C^*_{CE}(L)$ has coordinates ζ^i, the graded symmetric algebra with the degree of ζ^i equal to $1 - deg(z_i)$. The differential has two terms, the linear term $a^i_j \zeta^j$ and the quadratic term $\frac{1}{2} c^i_{jk} z^j z^k$. The differential squares to zero if and only if you have a differential graded Lie algebra.

So $Spec(H^0(\mathcal{O}(M))) \hookrightarrow M$, this is $t_0 M$, if we take $Spec(C^*_{CE}(L))$, with $L^i = 0$ for $i \leq 0$, then $t_0(M) = MC(L)$. There's a very natural derived scheme which envelops any given Maurer–Cartan locus. This is a nonlinear analogue of homological algebra. We want some condition in terms of the algbera that is a cofibrancy condition or in terms of the scheme a fibrancy condition. Whatever notion of cofibrancy you have, this has it because it's free when you forget the differential.

Let me put my cards on the table. I'll want to understand differential graded Lie algebras in Banach spaces. I want derived geometry for differential graded Banach Lie algebras. If I want Kuranishi theory, I don't want a Frèchet version, I want a Banach version. I could try very hard to understand what the Chevalley–Eilenberg complex is for a Banach Lie algebra, but intsead I'll use a model that is cosimplicial. This is a more plausible explicit way of handling things for me in the infinite dimensional case. You have to decide, for instance, on this side, what tensor product you're using. We won't have those problems.

So I'm going to give a different perspective but you have a first definition.

Definition 2.1. The *derived Maurer–Cartan locus of a differential graded Lie algebra* is the derived affine scheme associated to the stupid truncation $\sigma_{\geq 1} L$, where $(\sigma_{\geq 1} L)^i = L^i$ if $i \geq 1$ and 0 if $i < 1$.

So this is not a sophistiacted construction. The sophistication is the construction of appropriate differential graded Lie algebras. The only hard part is the projective geometry they do.

I want to change the story now. I'm, someone here will know much better than me when the idea of using cosimplicial schemes arose, or simplicial rings instead of differential graded rings. The original introduction of simplicial commutative rings into algebraic geometry is certainly due to Quillen. My point in this talk, even though the Chevalley–Eilenberg construction is familiar, if you go instead into simplicial commutative rings, the story becomes even easier, that's what I'm going to be trying to explain.

I can't assume you know what simplicial objects or the Dold–Kan correspondence are. Let me spend 15 minutes on this, this would be a few lectures in a course on algebraic topology.

This is a mysterious thing, I've never seen a convincing a priori description of why this should happen.

So simplicial objects in a category $\mathcal{C}$, which you can think of as an Abelian category or category of rings, is a contravariant functor from some category Δ to $\mathcal{C}$. And what's Δ, there's one way which is hyperelegant but really mysterious, is that it's the category of well-ordered, finite non-empty sets. Actually, any well-ordered finite non-empty set is isomorphic to a standard number. The arrows are functions that preserve the ordering. So there's a skeleton, where the objects are $[n]$ where $n \geq 0$ and this is the set $0 \leq 1 \leq n]$. The arrows are functions between these sets that preserve the ordering. Let's look at the bottom of the story. We've got $[0]$, $[1]$, $[2]$, and let me write some maps between them.

$$\begin{tikzcd}
{[0]} \arrow[r, shift left=4, "d^0"] \arrow[r, shift right=4, "d^1"'] & {[1]} \arrow[l, "s^0"'] \arrow[r, shift left=8, "d^0"] \arrow[r, shift left=4, "d^1"] \arrow[r, "d^2"] & {[2]} \arrow[l, shift left=4, "s^0"] \arrow[l, shift left=8, "s^1"]
\end{tikzcd}$$

Now you get the maps going in the other way in the simplicial object.

$$\begin{tikzcd}
A_0 \arrow[r, "\sigma^0"] & A_1 \arrow[l, shift right=4, "\partial^1"'] \arrow[l, shift left=4, "\partial^0"] \arrow[r, shift right=4, "\sigma^1"] \arrow[r, shift right=8, "\sigma^0"] & A_2 \arrow[l, shift right=8, "\partial^2"'] \arrow[l, shift right=4, "\partial^1"'] \arrow[l, "\partial^0"']
\end{tikzcd}$$

now there is a normalized chain complex of a simplicial object in an Abelian category $\mathcal{A}$, $A_* \mapsto N_*(A)$ where $N_n(A)$ is $A(n)/\sum_{i=0}^{n-1} im(\sigma_i : A_{n-1} \to A_n)$. In an additive category we can add these and then take a kernel since we're in an Abelian category.

This is the same as
$$\bigcap_{i=1}^{n} \ker(\partial_i A_n \to A_{n-1}).$$

In the first representation, the boundary map is $\sum(-1)^i \partial_i$ and in the second case just ∂_0.

You see that the first one is a colimit construction and the second one is a limit construction.

Here's the punchline. The theorem of Dold and Kan says that this functor is an equivalence of catgeories between simplicial objects and connective ((-1)-connected) chain complexes. This is $V_i = 0$ for $i < 0$ which is the same thing as $V^i = 0$ for $i > 0$. The $V_i = 0$ for $i < 0$ is very natural for topological things; in cohomological grading we're in the world of derived geometry. So simplicial objects are pretty much the same as chain complexes. We use both categories as the same thing, but there are functors on one side that look extremely unnatural on the other side. So we're going to see what happens on the other side with the Lie algebras.

I should remark that the right adjoint of the functor N gives you the equivalence. It was introduced by Eilenberg–MacLane in 1953, and this is $K.$, the Eilenberg–MacLane space of a connective chain complex. The n-simplices of V_* have to be $Hom_{chain}(N_*(\mathbb{Z}\Delta^n), V_*)$. This is convenient, because the n-simplex, the convex hull of the numbers 0 to n, its combinatorial structure, there are an infinite number of simplices but once you normalize there's only a finite number yet. When you unravel what this is, you get one of the two descriptions of K. The punchline of this talk is that this is also the *left* adjoint of N. But no one works this out in practice. That means there's a really funny formula for that adjunction. I don't have time to give it now, but it's related to introducing *cosimplices*, I've said that simplicial objects are contravariant functors, I should say that cospimplicial objects are covariant functors. They're very different. We had this duality between positive and negative degrees. It feels like simplicial and cosimplicial handles the two halves. So going to the cosimplicial world, right and left adjoints are switched and so you get an elegant formalism for the left adjoint. It was Kaledin who explained to me this way of understanding Dold–Kan for cosimplicial objects.

So on the one hand you have N^*, the normalized cochains of a cosimplicial Abelian group. I can show you the left adjoint in the cosimplicial world by giving you a right adjoint, $Hom_{cochain}((N^*\mathbb{Z}\Delta_n), V^*)$. The nth cosimplex is this bizarre thing where $(\Delta_n)^m$ is $(\Delta^m)_n$. I can calculate this guy and it's the dual of an exterior algebra, which works over the integers, with the shuffle product, and I can delinearize this whole theory, and $Hom_{cochain}$ is close to the linear Maurer–Cartan equation.

Finally with only a few minutes, let me jump and give my formula. Let $Lambda^n$ be a differential graded commutative algebra, and it's actually the dual of $N^*(\mathbb{Z}\Delta_n)$. It's the exterior algebra on $\xi_0, \ldots, x_n$, where these guys have degree -1 and $d\xi_i = 1$. This is just some new object, cosimplicial in n, the cosimplicial structure is to reindex according to your cosimplicial map. Now if I have a differential graded Lie algebra, I take its tensor product with the differential graded commutative algebra, $\Lambda^n \otimes L^*$, so I take $n \mapsto MC(\Lambda^n \otimes L^*)$, and that's my derived Maurer–Cartan locus. So I have $MC^0(L) \cong L_1$ mapping by d^0 and d^1 to $MC^1(L) \cong L^1 \times L^2$. Here $d^0(x) = (x, \delta x + \frac{1}{2}[x,x])$ and $d^1(x) = (x, 0)$, so the equalizer is the Maurer–Cartan locus.

Theorem 2.1. *$\mathcal{O}(MC^*(L))$ is a simplicial commutative ring, so there's an annex to Dold–Kan, part of Eilenberg–Zilber, which says that N_* takes this to a commutative differential graded algebra in degrees 0 and below, and the theorem is that this is quasi-isomorphic to $C^*_{CE}(\sigma_{\geq 1} L)$.*

There's more. The Milnor–Moore theorem says that if you have a commutative Hopf algebra, it's a symmetric algebra (something like that). And by a theorem of Willwacher that no one noticed, this is a commutative Hopf algebra. [missed a little] I'd better stop there.

[what about replacing Λ^n with Ω^n?]

The talk I've been giving for the last ten years is to use $C_*(\Delta^n)$ wich is $N^*((\mathbb{Z}\Delta^n)^\vee)$, which is a finite dimensional guy which is similar to but different to differential forms. So I'd think about $\Lambda^n \otimes C_m \otimes A^*$, putting an associative algebra there. This is my realization of the derived stack of [unintelligible]. Illusie says that simplicial cosimplicial Abelian groups stands in for unbounded complexes. This is like simplicial cosimplicial [unintelligible], like unbounded manifolds. So algebraic geometry can discuss directly nilpotent Lie groups. But the exponential map is only polynomial for nilpotent Lie algebras. So you can make this work for a more general situation by putting a Lie algebra there (or by working with polynomial differential forms, as you suggest).

3. Mikhail Kapranov: Combinatorial approach fo Fukaya categories of surfaces I

[It's my great honor to introduce Mikhail Kapranov from IPMU]

Thank you very much. This could be called "very elementary aspects of Fukaya categories." Suppose someone says "I don't know anything about symplectic manifolds, and I don't want to know anything about them." What do you say to that person? That they're already doing it without knowing it, as soon as they're doing homological algebra.

I want to start today in the foundation of triangulated categories and see how some of the structures are there already.

3.1. **Triangulated categories from the symplectic point of view.** Triangulated categories have several meanings and different versions foreground different qualities. Let me start with the "classical" approach of Grothendieck and Verdier.

Triangulated categories are an axiomitization of $D^b(\mathcal{A})$ for $\mathcal{A}$ an Abelian category, which is complexes in $\mathcal{A}$ localized at quasi-isomorphisms, $C^b(\mathcal{A})[qis^{-1}]$, or the homotopy category of $C(\mathcal{A})$, where morphisms are the homotopy classes of morphisms of $\mathcal{A}$.

For each actual morphism $A \xrightarrow{f} B$ you get a triangle with $Cone(f)$ where the map $Cone(f) \to A$ has degree +1.

A *triangulated category* is an additive category $\mathcal{V}$ with a "shift functor" $\Sigma : A \mapsto A[1]$ and a class of triangles

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, "+1"] & \end{tikzcd}$$

which satisfy some axioms. I won't say all of them but will highlight the most geometric ones:

(1) rotation invariance:

$$\begin{tikzcd} C \arrow[rr] & & A[1] \arrow[dl] \\ & B[1] \arrow[ul, "+1"] & \end{tikzcd}$$

is also a distinguished triangle.

(2) the octohedron, for every one of these:

$$\begin{tikzcd}
A \arrow[rr, "+1"] & & A_2 \arrow[dl] \arrow[dd] \\
& A_{12} \arrow[ul] \arrow[dr, "+1"] & \\
A_{13} \arrow[uu] \arrow[ur] & & A_3 \arrow[ll]
\end{tikzcd}$$

with the upper and lower triangle exact and the right and left commutative, there is one of these

$$\begin{tikzcd}
A \arrow[rr, "+1"] \arrow[dr, "+1"] & & {} \arrow[dd] \\
& A_{23} \arrow[ur] \arrow[dl] & \\
A_{13} \arrow[uu] & & A_3 \arrow[ll] \arrow[ul]
\end{tikzcd}$$

with the right and left triangles exact and the top and bottom commutative; and vice versa. This is hard to think of, and also to memorize. One way to memorize it, draw a simplex and put the middle points of the edges of the simplex. Write A_{ij} on the edge between i and j for $i, j \neq 0$ and A_j on the edge between j and 0. The triangles come one the side of the simplex and the others on the transversal of the simplex. That's the difference between the commutative and exact triangles.

Let's look at this in more detail.

3.2. **Objects on edges systematically.** Take a triangle and put some numbering on it like for the standard simplex:

$$\begin{tikzcd}
& 1 \arrow[dr, dash, "C"] & \\
0 \arrow[ur, dash, "A"] \arrow[rr, dash, "B"'] & & 2
\end{tikzcd}$$

and I imagine I have $f : A \to B$, $g : B \to C$, and h, a degree 1 map $C \to A$. Then we can rewrite the octahedron suppressing the commutativity. Then I get

$$\begin{tikzcd}
{} \arrow[r, "A_1 3"] \arrow[d, "A_3"'] & {} \\
{} \arrow[r, "A_2"'] \arrow[ur, "A_{12}"] & {} \arrow[u, "A_1"']
\end{tikzcd}$$

and I can pass to this picture:

$$
\begin{tikzcd}
{} \arrow[r, "A_13"] \arrow[d, "A_3"'] \arrow[dr, "A_{23}"] & {} \\
{} \arrow[r, "A_2"'] & {} \arrow[u, "A_1"']
\end{tikzcd}
$$

and this corresponnds to the switch of a triangulation. This suggests something which is invariant of a triangulation.

Let me recall that in combinatorial topology, we consider manifolds which are triangulated, and pass between manifolds that are triangulated in different ways by elementary moves, called *Pachner moves*. In dimension d we have one of type (p, q) for $p + q = d + 2$. For $d = 2$ we can break a triangle into three triangles or vice versa, this is $(1, 3)$ and $(3, 1)$. For $(2, 2)$ it's the picture above.

For the Euclidean model, take $d + 2$ points in $\mathbb{R}^d$ in general position. The convex hull has exactly two triangulations. There is a middle type in any dimension, $p = [d/2]$ and $q = [d/2 + 1]$ for an appropriate understanding of the integer part. So for Street, you have $\partial_{odd}(\Delta^{d+2})$ and $\partial_{even}(\Delta^{d+1})$.

Anyway, in dimension two, this is all you have. It's convenient and pleasant. The octahedron has something to do with invariants of a manifold which might have different triangulations.

Let me give another example, Postnikov systems (that is, towers of fibrations). Typically this starts with an object A_1, which is covered by A_{12}, and A_2 is the fiber. This is covered by A_{13} with fiber A_3, and so on, where the fiber of A_{1n} over $A_{1,n-1}$ is A_n. This is written as a triangulation of a polygon. [picture] So polygons are very special manifolds. The easiest situation in which we can work is when the triangulated category is

3.3. **2-periodic triangulated categories.** Here $\Sigma^2 = \mathrm{id}$ or $A[2] = A$. An example would be the category of 2-periodic complexes. Many version of the Fukaya category are like this. In this case, we can improve the geometric language. If we put an oriented edge on an object A, then change of orientation with correspond to the shift. Change twice and we get the original object back. So to any choice of orientation we get an object. To the opposite orientation we get the shifted object.

We can work with oriented but not ordered triangles. We can assemble these into more complicated shapes that are not necessarily polygons.

Now let's generalize a little bit.

Definition 3.1. A *surface Postnikov system* in a 2-periodic trinagulated category $\mathcal{V}$ is

(1) An oriented surface S possibly with boundary.
(2) A curved triangulation $\mathcal{T}$ of S.
(3) An assignment to every oriented edge e of a triangulation an object $A(e)$ so that $A(e^*) = A(e)[1]$ and a morphism for every corner of every triangle so that
(4) every triangle becomes exact.

Then we learn that we can pass from a Postikov system depending on one triangulation to a Postnikov tower depending on another. [picture]

One known fact, one can pass from one triangulation to another with the same vertices, from *Teichmüller theory*, is such that if $M \subset S$ vertices is such that $S - M$ is hyperbolic, then $\mathcal{T}$ and $\mathcal{T}'$ with vertices M are connected by $(2,2)$-froms.

Corollary 3.1. *To pass between Postnikov systems of different type on the same surface. Denote by $Post_{\mathcal{T}}(\mathcal{V})$ is the "collection" (later category) of $\mathcal{T}$-Postnikov systems, which depends (up to equivalence, late) only on (S, M). In particular, it is acted on by the Teichmüller group.*

Just from playing the octahdral axiom, we moved into something geometric. For one example, the surface is a torus with only one point. Then we have the same picture we drew before. [picture]

So $Cone(f) \cong Cone(f')$.

Let's try to analyze some simple versions. Let $\mathcal{V}$ be $\mathcal{D}^{(2)} Vect_{\mathbf{k}}$. So A and B are complexes. This is a representation of the Kronecker quiver. As such it gives an object of the 2-periodic dervied category of coherent sheaves over $\mathbb{P}^1$. So this is like $A \otimes \mathcal{O}_{P^1} \mapsto B^* \otimes \mathcal{O}_{P^1}(1)\}$.

So $Cone(f)$ is the fibre of E^* at 0 in P^1 and the cone on f' is the fiber as X escapes to ∞. S then I get a nodal cubic curve, and can think of that as a perfect complex on X. So $SL_2(\mathbb{Z})$ acts on $Perf(X)$. This is a known example. Let me explain briefly how it goes. Because it acts on triangulations of $(\mathbb{T}^2, 0)$, well, each T can be flipped. It's also known, if it has a triangulation T_i and this contains $[\alpha], [\beta], [\gamma]$ in $H_1(\prod^2 \mathbb{Z})$.

[More pictures]

I think it's a good time to stop I showed how by doing nothing other than the axioms, we arrive in the topology of elliptic curves and Teichmüller theory. Thank you very much.

4. July 7: Herman Verlinde: Conformal Bootstrap, Hyperbolic Quantum Geometry and Holography I

Good morning, again, it's a pleasure to be at this conference and school on derived geometry. Now I assumed there would also be "and physics" so I'm the "and physics part." Two weeks ago I was at a conference that was of a similar nature, about mathematics but inspired by physics. I gave a similar talk and said some things about number theory. I'll give my talk but basically use my own interpretation of "derived."

It's the 100th birthday of general relativity. The questions we as physicists have now are mostly related now to the quantum. There is a tension I like between figuring out how to use math in quantum physics (for me it's mostly a tool). I will not make much effort to tell you what space objects will live in. Maybe it will be a group effort to make sure that the objects I'm talking about have a space associated to it. The other part of the tension is, math gives us useful tools but that often, well here the motivation works backwards. I'd normal begin a talk with the motivation, but here I'll end the talk with the motivation.

Now when I was in graduate school one of the interesting subjects that has returned these days is conformal field theory. This is a kind of quantum field theory, which is hard to define in general. Graeme Segal and others have given a more precise definition mathematically for 2D conformal field theory. Even there the examples are limited. There is the ADS-[unintelligible]correspondence between a field theory (in any dimension) and a field theory in the "bulk" one dimension higher where this conformal field theory lives on the boundary. That's what we'll do on Thursday and that goes by the name *holography.* In my case that will be in $2+1$ dimensions. There's a theory that I should call quantum gravity that has certain rules, we don't know what they are, but we try to use a relationship between this theory in one dimension less and the theory in one more dimension.

A key point in all of this is that "conformal" means that the field theory has conformal invariance, so that the field theory we're talking about in quantum gravity is hyperbolic geometry. There are very fundamental things that people are disagreeing about right now, the "firewall" debate which would mean that this hyperbolic geometry would break down at special locations which correspond to black hole horizons. Again, this is for Thursday.

The key point that I'm trying to hopefully get across is that while the mathematics here doesn't seem to be very well controlled, there's something here that is related to a natural mathematical object which I'll call *quantum Teichmüller theory*, so the Teichmüller space of a two dimensional surface with marked points. It's natural for a physicist to look at this at a quantized level because it's Kähler. There's a very direct connection between 2D conformal field theory and quantum Teichmüller space and also between Teichmüller space and this hyperbolic geometry.

I won't always be careful with references. There's an early paper of mine that put down some of these connections, but the actual mathematical versions are those of Tescher and others, Kashaev, TZ, ZZ, others. In particular Tescher and Ponsot cracked the main mathematical problem of this Teichmüller theory and showed a mathematical relationship with something, I don't know whether to call it a tensor category, a modular functor, but there's a Hopf algebra, a quantum symmetry, $U_q(SL_2)$, and if you know how to take tensor products here then the structure starts to fall in place.

2D field theory has played a role in the earlier, sort of, one of the versions of topological field theory, 2+1 Chern–Simons theory. Let me distinguish two kinds of field theories in two dimensions. There are "rational" conformal field theories. I'll focus later on the central charge, and if $c < 1$ we have a rational conformal field theory, the geometry is compact and finite dimensional and you can relate it to Chern–Simons theory in 2+1 dimensions, and that leads to things like the Jones polynomial, looking at correlations which can be used to calculate the expectations of Wilson lines that trace out a particular knot.

This technology allows us to solve and compute these invariants exactly. The regime that I'm going to be interested in is an orthogonal regime, where $c \gg 1$, and where the rational case is compact, this case is noncompact in a certain sense, and the SL_2 symmetry is a noncompact group, and this will be a noncompact quantum group. I'll start today with the quantum Teichmüller theory, tomorrow I'll discuss 2D conformal field theory, and then on Thursday quantum gravity.

[What is noncompact?]

For the Jones polynomial it's flat gauge fields (using the Hamiltonian formalism), flat connections on a Riemann surface with certain curvature singularities.

Here we'll have constant curvature metrics, so a flat $SL_2(\mathbb{R})$ gauge fields, $SL_2(\mathbb{R})$-connections, so this is noncompact, so not just the symmetries but the phase space is non-compact. The spectrum of [unintelligible]is infinite dimensional and probably even continuous. So it's the same as $SL_2(\mathbb{R})$, which has continuous and discrete [unintelligible].

To start I'll consider an example which has all the technical ingredients to give the whole program here. So I'll look at the sphere with four marked points and look at the corresponding Teichmüller space $T_{0,4}$. You could with a bit more work do $T_{g,n}$. If I talk about the sphere and am interested in the Teichmüller space, I'm interested in the constant curvature metrics I can put here with singularities at these four points. So I can write $ds^2 = \frac{dzd\bar{z}}{(z-\bar{z})^2}$ but not uniformly because of these points. This has the familiar $SL_2(\mathbb{C})$ symmetry $z \mapsto \frac{az+b}{cz+d}$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$. So there's one complex parameter for this Teichmüller space because I can put three of the points at 0, 1, and ∞. I can specify, I have a transition function going around one of the points, and I get an $SL(2,\mathbb{C})$ element going around each point.

If I'm going to describe Teichmüller space, I'll describe it in terms of an $SL(2,\mathbb{R})$ action, and then my matrix entries will be real, and I can describe

$$T_{0,4} = \{g_1, g_2, g_3, g_4 \in SL(2,\mathbb{R})^4 | tr(g_i) = L_i, g_1g_2g_3g_4 = 1\}/SL(2,\mathbb{R}).$$

This is a two dimensional space, keeping L fixed, and just counting the number of dimensions left.

[this is noncompact?]

This is not compact but you'd want to do it by the usual Deligne–Mumford thing.

If $L_i < 2$ you call this elliptic and then you have a conical defect. If the group element $L_i > 2$ you call it hyperbolic and locally this thing will look like, well, if it's hyperbolic, the puncture, it means it's asymptotically a hyperbolic region like that. Let's allow this to have infinite area for now although we'll eventually run into trouble and have to regulate that.

I'll look at scattering of particles in the background of the black hole. Two of these punctures will be particles and two will be asymptotic regions of the black hole. So we'll have two elliptic and two hyperbolic ones. But you could do something more general.

The symplectic form I'll use is the Weil–Petersen symplectic form. Another way to get this is to start with a Chern–Simons action in $2+1$ dimensions,

$$CS = \hbar \int d^2xdtTr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)$$

which is again an $SL(2,\mathbb{R})$ gauge theory and the phase space is the same.

For the elliptic conjugacy class I'll write this as

$$2\cos(\theta_i/2)$$

and for the hyperbolic case

$$2\cosh(\ell_i/2)$$

where θ_i is the *deficit angle* and ℓ_i the geodesic length around the marked point.

I have free conjugacy classes I didn't specify, like this one [picture], called L_α [separates one hyperbolic and one elliptic from the other two] and L_β [the same but with the other choice]. These intersect. I can take those conjugacy classes and compute the corresponding geodesic length. These operators will not commute with each other because these curves intersect. So $[\hat{\ell}_\alpha, \hat{\ell}_\beta] \neq 0$. I'll eventually give a Hilbert space on which these operators will act. I could define a state I could call $|\alpha\rangle$ where $\hat{\ell}_\alpha$ will have an eigenvalue ℓ_α, and the same for $|\beta\rangle$, and the object I want to compute is the overlap if these states are normalized, it's essentially an R-matrix, the overlap of α and β, $R_{\alpha\beta} = \langle\alpha|\beta\rangle$. The answer is actually very beautiful.

I'll tell you more about the R-matrix and its relationship with knots and conformal field theory. This is a very natural object. I can ask about the basis transformation to go from one to another with these noncommuting operators.

Let me see how fast I can do this calculation, let me give just a few ingredients and then write down the answer for what R is.

Let me write down L_α and L_β in terms of the group elements. So $L_\alpha = tr(g_1 g_2)$ and $L_\beta = tr(g_3 g_4)$. I could have also introduced $L_\sigma = tr(g_1 g_3)$ and $L_\tau = tr(g_3 g_4)$. I believe it was Goldman who showed that $\{L_\alpha, L_\beta\}_{WP}$, the usual Poisson bracket, is $L_\sigma - L_\tau$, and the basic rule that goes into this is if the two curves intersect, then in the commutator, you essentially have to break it open at the intersection point, and take the minus sign. The fact that this happens locally like that is the consequence that my [unintelligible]comes from a local Lagrangian. The commutator between the Wilson lines can be figured out by doing something local between the [unintelligible].

The whole point is that L_σ, L_α, L_β and so on are not independent quantities, $L_\sigma + L_\tau = L_1 L_3 + L_2 L_4 - L_\alpha L_\beta$. You can also write $L_\sigma L_\tau$ as a polynomial in terms of these things of higher order. This is a local expression of the relation but you *could* write it globally using L_1 and so on.

There's another result by Wolpert that one can rewrite the Weil–Petersen form in this case $\Omega_W P = d\ell_\alpha \wedge d\tau_\alpha$ where this is the length-twist parameterization of Teichmüller space. You cut open the space and twist, this τ is the angle by which you twist, and there's also a dual variable, you could say this is $d\ell_\beta \wedge d\tau_\beta$, and here you can write $\{\ell_\alpha, \tau_\alpha\}_{WP} = 1$ and $\{\ell_\beta, \tau_\beta\}_{WP} = 1$. So now we're changing bases between these two. I can write down an equation you'd have to solve to do this translation. These twist variables are also coordinates on Teichmüller space. So it's possible to express ℓ_β as $\ell_\beta(\ell_\alpha, \tau_\alpha)$, $\tau_\alpha = \tau_\alpha(\ell_\alpha, \ell_\beta)$, and vice versa. The semiclassical answer is that

$$R_{\alpha\beta} = e^{\frac{i}{\hbar} S(\ell_\alpha, \ell_\beta)}.$$

The relationship for the generating function is that

$$\frac{dS(\ell_\alpha, \ell_\beta)}{d\beta} = \tau_\beta; \quad \frac{dS(\ell_\alpha, \ell_\beta)}{d\alpha} = -\tau_\alpha.$$

If you work hard enough you can solve this and find S, which you can write in terms of classical dilogarithms. Even more nicely, the complicated expression has an elementary geometric interpretation. Can anyone make a guess as to what this thing is?

Maybe to, let me give you two hints. It depends an six lengths. I have L_1 through L_4 and then the ℓ_α and ℓ_β.

So

$$S(\ell_\alpha, \ell_\beta) = Vol(Tetra\begin{pmatrix} 1 & 3 & \alpha \\ 2 & 4 & \beta \end{pmatrix}).$$

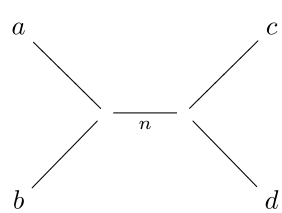
These are lengths or maybe dihedral angles. Can someone now guess what the R is? So Wigner knew, there are objects with tetrahedral symmetry. You may know the Wigner $6j$-symbol, with the semiclassical limit for large spin, he already knew that was related to the volume. For us with $SL(2,\mathbb{R})$, the quantum theory, this turns out to be the $6j$-symbol of $U_q(SL_2)$. That turns out to be the quantum answer to this question.

Let me give a little hint as to how this comes about. Let me give a schematic derivation of the fact that a quantum $6j$ symbol is naturally related to the tetrahedron.

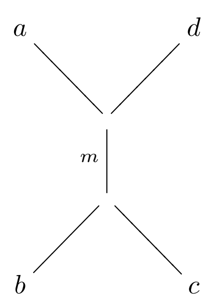
So one way of thinking about this is to go to the rational conformal field theory case. Then I could start with a gauge theory with a different gauge group and played the same game and I could have also looked at the 4-punctured sphere, looking at the flat guage fields with curvature singularities at four points

$$F(A) = \sum_{i=1}^{4} \tau_a d(z - z_a),$$

and quantized the space and looked at wavefunctions which are conformal blocks of WZW conformal field theory for $\mathcal{G}$, and this corresponds invariant tensors of the quantum group $\mathcal{G}$. This is a curvature singularity of the gauge field, but it also specifies a representation in which this thing transforms. I can think of the four-point function, I can ask about the representation associated with cutting the four-punctured sphere in two halves, with two punctures on either side, and I could take the tensor product of the two representations corresponding to the two punctures on either side. I can say, look at $(V_a \otimes V_b)$ and later take the tensor with V_c; I could also take the tensor product $V_a \otimes (V_b \otimes V_c)$, and this should be the associator isomorphism. We associate a diagram to $(V_a \otimes V_b) \otimes V_c$, we project to a representation, I want to specify d that sits between a and b, let me call it n, and then I tensor with c and get d, and there should be a basis transformation between this way and the way where I take b and c first and later put in a

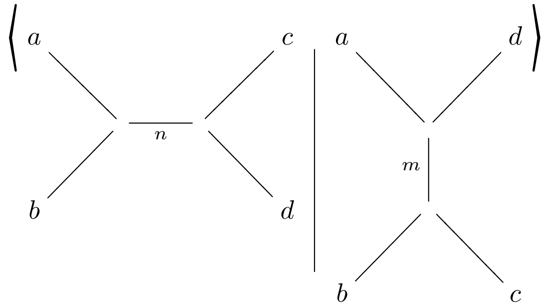


and



and so I can associate a state by putting this Wilson line in my time evolution in a surface, and I can do either one of these things. When I take the inner product, by gluing, when I look a little carefully this is a tetrahedron. The inner product gives me a tetrahedron shaped Wilson line. Once you quantize this theory you get the quantum $6j$ symbol of the corresponding quantum group:

$$\left\{\begin{matrix} a & c & m \\ b & d & n \end{matrix}\right\} =$$



We're doing a gauge symmetry or Kac–Moody symmetry. But I'll have just one symmetry, for S^1, I'll talk about $Diff S^1$ and the Virasoro algebra and not another gauge group. So I'll have the Virasoro algebra and not the Kac–Moody algebra will be what turns out to be the salient thing.

5. Alexander Polishchuk: Semiorthogonal decompositions of the derived category of W-equivariant sheaves

I'd like to thank the organizers for inviting me. I'm going to talk about joint work with Michel Van den Bergh. So the setup is, I'll first explain the general setup. I'm interested in the situation where you have a finite group G acting on an algebraic variety X and then I want to consider the category of equivariant coherent sheaves $Coh_G(X)$, so coherent sheaves with an action of G, so in the simplest case of a vector a bundle, a lifting of the action of G to an action on the total space of the vector bundle, and there's a version of this for sheaves, and I'll write for $D^b(Coh_G(X))$ just $D_G(X)$. When X is an affine variety, it is described by some algebra of functions, $X = \operatorname{spec} A$, then $Coh_G A$ you can think of finitely generated modules over the algebra $A \rtimes G$, so linear combinations of elements of G with coefficients in A and when you want to swap coefficients you act by A.

In the case when X is a quasiprojective variety, there is a decomposition by conjugacy classes in G. Consider for example Hochschild homology. There is a decomposition $HH_*(Coh_G(X)) = \bigoplus_{g \in G/\sim} HH_*(X^g)^{C(g)}$, where $C(g)$ is the centralizer of g. The characteristic should probably be zero, I'll work always over $\mathbb{C}$. When X is affine, it's a statement about this first algebra $A \rtimes G$ but it's easy to rewrite this, it's an algebraic result.

We can rewrite this.

Proposition 5.1. *If Y is a smooth quasiprojective variety with an action of G such that Y/G is smooth, the geometric quotient. So G is still finite. Then the Hochschild homology of Y, the G-invariants of that, is the same as the Hochschild homology of the quotient, $HH_*(Y)^G \cong HH_*(Y/G)$. So this reduces to Brion's theorem. You use Hochschild–Kostant–Rosenberg, and in this case when the quotient is smooth you reduce to Y/G.*

[Is there a Hocschild cohomology invariant?]

I think so but I'm interested in additive invariants. If all the quotients, if $X^g/C(g)$ are all smooth, for all $g \in G$, then we can rewrite the right hand side as

$$HH_*(Coh_G(X)) = \bigoplus_{g \in G/\sim} HH_*(X^g/C(g))$$

and so the problem we pose is, can this decomposition be lifted to the level of categories? The left hand side has to do with coherent sheaves on X and the right hand side with thes quotients.

There's one situation where you can assert that the Hochschild homology decomposes, and that's when you have a semiorthogonal decomposition. Let me remind you what this is. When you have a triangulated category $\mathcal{T}$ with two full subcategories $\mathcal{A}$ and $\mathcal{B}$, triangulated subcategories, then we say there is a semiorthogonal decomposition of T into $\mathcal{A}$ and $\mathcal{B}$, and write $\mathcal{T} = \langle \mathcal{A}, \mathcal{B} \rangle$ if

- for $b \in \mathcal{B}$ and $a \in \mathcal{A}$ we have $Hom(b, a) = 0$ and
- for all $x \in T$ there is a unique exact triangle $b \to x \to a$ where $b \in \mathcal{B}$ and $\mathcal{A} \in A$, and the choice of b and a are functors adjoint to the inclusions of $\mathcal{A}$ and $\mathcal{B}$.

Then we can recursively write a definition for $\mathcal{T} = \langle A_1, \ldots, A_n \rangle$. In the presence of a semiorthogonal decomposition, additive invariants decompose so you get

$$HH_*(\mathcal{T}) = \bigoplus HH_*(\mathcal{A}_i).$$

There is the condition that all the quotients are smooth.

Conjecture 5.1. (there already may be a counterexample) If $X^g/C(g)$ are all smooth then there exists a semiorthogonal decomposition of $D_G(X)$ into subcategories which are equivalent to $D(X^g/C(g))$.

This may need sharpening; I've heard in the last week that there's a counterexample but I don't know what it is. I'll focus on positive results.

So a simple example. Let $G = \mathbb{Z}/2$, and then let X be a smooth variety with an involution τ. Wanting the quotient to be smooth is the same as asking that X^τ is a smooth divisor $D \subset X$. Then the decomposition should be two pieces, and it's easy to see that, well, we have a quotient π from $X \to Q \coloneqq X/\tau$, and we have the inclusion $D \overset{i}{\hookrightarrow} X$ and so

$$D_{\mathbb{Z}/2}(X) = \langle \pi^* D(Q), i_* D(D) \rangle.$$

This second term would not normally be fully faithful but the group action does something that means that higher Ext groups don't come into play.

I'll restrict first to a very special case, when X is a vector space and the action is linear, and furthermore I'll assume that the quotient of X modulo the G-action is smooth, and we get a complex reflection group on the complex vector space $V/\mathbb{C}$. In this case you can often check this condition, that the quotients are smooth.

Proposition 5.2. *$V^g/C(g)$ are all smooth in the following cases.*

- *If you take a Weyl group acting on a maximal torus for types A, B, or C, or F_2 or G_4,*
- *for real reflection groups of rank at most 3 (rank is the dimension of V here)*
- *for the complex reflection group $G(m, 1, n)$, the cyclic group of order m acting by roots of unity to the power of n, semidirect product with S_n, $(\mu_m)^n \rtimes S_n$ acting on $\mathbb{C}^n$.*

I know any D or E type Weyl group they won't all be smooth, and that even for small complex reflection groups the same is true, something like $G(4,4,5)$ or something.

Our main result is that in some of these cases we indeed have a semiorthogonal decomposition

Theorem 5.1. *There is a semiorthogonal decomposition as desired for the Weyl groups above and for $G(m,1,n)$.*

I'm going to shift gears and change notation. What I called G before I'll call W because it will be a Weyl group. I need G to denote a connective reductive group.

The construction of the semiorthogonal decomposition goes through Springer correspondence.

Somehow the basic idea is that semiorthogonal decompositions appear naturally in topology. We can use the picture with nilpotent orbits to get certain semiorthogonal decomposition for constructible sheaves. Then there's a remarkable result of Lusztig that lets us upgrade this to coherent sheaves in smoe cases.

So let G be a connective reductive group and $\mathfrak{g}$ the Lie algebra of G and $\mathcal{N} \subset \mathfrak{g}$ the nilpotent cone. So we take a resolution $\widetilde{\mathcal{N}}$ of the nilpotent cone of the form $\{(x,B) | x \in B\}$, which sits inside $\mathcal{N} \times X$, where $X = G/B$ is the flag variety. Then π is the projection $\widetilde{\mathcal{N}} \to \mathcal{N}$. The main kind of character in this correspondence is the Springer sheaf, the pushforward of the constant sheaf on this resolution, $\mathcal{A} = R\pi_* \mathcal{C}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}]$. There's some additional data. This has a G action, this is a G-equivariant constructible sheaf now, and furthermore, it's actually a perverse sheaf. I have no time to explain but this is a nice (Abelian) subcategory, and the fact that the sheaf is perverse has to do with π being semismall. The most remarkable thing is that $\mathcal{A}$ has a W-action, which comes, actually, from a more general Grothendieck resolution, and you use Goresky–MacPherson, it's a quite nontrivial construction. It acts trivially, though, on $\mathcal{N}$, so the Springer sheaf has a decomposition

$$\mathcal{A} = \bigoplus_{\chi \in Irr(W)} \chi \otimes j_{a!} \mathcal{L}_\xi [\dim O_a]$$

where $G_a = O_a \overset{j_a}{\hookrightarrow} \mathcal{N}$ and $\mathcal{L}_\xi$ is a local system with $\xi \in Irr(G_a/G_a^o)$.

So you get a correspondence between χ and (O_a, ξ). In the case of the group GL_n, all possible, there is no ξ, then you just get that the partitions are the same on the left and right. Not all pairs on the right may actually appear.

Anyway, the main point is the following crucial fact which follows from the work of Lusztig, although we learned of it through [unintelligible]. There is a canonical identification

$$Ext^*_G(\mathcal{A}, \mathcal{A}) \cong H^*_G(X) \rtimes W$$

since this is a pushforward of the constant sheaf on $\widetilde{N}$, which is the cotangent space of the flag variety, you get an embedding from functoriality from right to left and it turns out to be an isomorphism. Lusztig considers the C_*-action and you get a deformation on one side. For us this is the main isomorphism, we can easily see that $H^*_G(X) = H^*_G(G/B) = H^*_B(pt) = H^*_T(pt)$, which is $S(t^*)$, functions on t, so this is $S(t^*) \rtimes W$.

So this is unfortunately still not enough to transfer facts from the constructible to the coherent world and the reason is that if I now consider the subcategory $\langle \mathcal{A} \rangle \subset D_{G,c}(\mathcal{N})$, then this is actually described not by this Ext algebra but by a certain dg-algebra that has this as its cohomology. So there is a dg-enhancement A_W^{dg} with cohomology A_W. Then $\langle\mathcal{A}\rangle$ is equivalent to the derived category of dg modules over A_W^{dg}.

So the additional fact that I need is

Theorem 5.2. *This algebra A_W^{dg} is formal, quasi-isomorphic to its cohomology, so I can replace the algebra by A_W. The proof is based on, it uses the reduction to a finite field and Frobenius weights.*

There is an old trick of Deligne proving formality using Frobenius weights, in a commutative algebra, but here it's noncommutative. Nevertheless we can still reduce. The basic difficulty is to reduce to the case where you hae a Frobenius action that is locally finite at the level of cochains, so if you can do this in the homotopy category of dg algebras where everything is locally finite. The Frobenius thing acts with pure weights. There is a part of this trick which is standard, and getting to the setup there are some technicalities.

So you use these together and get that

Corollary 5.1. *The subcategory $\langle\mathcal{A}\rangle$ is equivalent to $D(dg-mod-A_W)$.*

This is still different than the category I want but if I have orthogonality in one I get orthoganality in the other. There are things that I'm hiding under the rug.

Now we replace $j_{a!*}\mathcal{L}_\xi$ by $j_!\mathcal{L}_\xi$, and these already have semiorthogonality when one thing is not in the orbit of another. Then you take $RHom(j_{a!}\mathcal{L}_\xi,\mathcal{A})$, and here there is one more technical point due to Lusztig. Not all pairs (O_a,ξ) appear in the Springer correspondence. He asked if there's a more general correspondence where all pairs appear, and there is, it's the generalized Springer correspondence. You need to know that you actually appear in the Springer correspondence, and this is also a deep fact.

Maybe the easiest example is in type A, you finish after the first step; the nilpotent orbits are numbered by partitions λ, partitions of n and for each partition you get the corresponding module M_λ, a module over A_W, and these modules form a semiorthogonal decomposition. You order them in a certain way, using the dominance order on partitions.

For other types, even for types when this theorem is not true, you do something with nilpotent orbits, and then you have to describe which irreducibles appear, and you get a similar type of problem, and repeat this multiple times to see what kind of pieces come out.

Maybe I'll give one example, S^3 acting on the two-dimensional representation. So we have $D_{S_3}(\mathbb{C}^2)$. There should be three pieces in our semiorthogonal decomposition corresponding to $3,0$, to $2,1$, and to $1,1,1$. There's a model corresponding to the full space, so this should just have the symmetric alegbra $C[x,y]$, so $A_W = C[x,y] \rtimes S^3$, so the module corresponding to $1,1,1$ is $C[x,y]$, the one corresponding to $3,0$ is just $\mathbb{C}$, and for $2,1$ there should be something having to do with lines. It's not just a structure sheaf of [picture]. You take three lines through the origin in space. This is a different variety than three lines in the plane. This turns out to be the right module, these three modules generate a semiorthogonal decomposition.

Finally, let me say, there is a global version of this. The global version of the problem, taking nonlinear actions. You take a smooth curve, so for type A you take a smooth curve, you have S_n acting on C^n, and you can consider $D_{S_n}(C^n)$, and the theorem is

Theorem 5.3. *$D_{S_n}(C^n)$ has a semiorthogonal decomposition of the required type.*

You identify these things as Springer fibers, and use Goresky–MacPherson to get to linear arrangements, and this globalizes to this case. For a partition λ you get $C[\lambda] \subset C^n$, where in each part of λ the coordinates are equal and then you consider inside of $C[\lambda] \times C^n$, you consider the union over $w \in S_n$ of the graphs of $w : C[\lambda] \to C^n$, call this Z_λ, and then you take the quotient by the induced W_λ action and get $\overline{Z}_\lambda$, which maps on one hand to $C[\lambda]/W_\lambda$ and on the other to $[C^n/S_n]$. So you use this correspondence to build the functor.

6. Kenji Fukaya: Floer homology for 3-manifolds with boundary II

So I want to, today, to talk about the geometric and algebraic part of the proof I talked about, and next time I'll talk more about the analysis. We have M^3 and an $SO(3)$-bundle E over it, and the boundary of M^3 is Σ and $W^2(E|_\Sigma) = [\Sigma]$, these are the assumptions. Then we have the moduli spaces of flat connections $R(M)$ and $R(\Sigma)$ and we assume that the map $R(M) \to R(\Sigma)$ is an immersion, which we can always do by perturbing. Then this $R(\Sigma)$ is symplectic and $R(M)$ is a Lagrangian. Then we consider $CF(R(M)) = H^*(R(M) \times_{R(\Sigma)} R(M), \Lambda_0^{\mathbb{Z}_2})$, and we have $m_k : CF(R(M))^{\otimes k} \to CF(R(M))$, and I won't define this because it will take up the whole talk, but it basically involves counting polygons.

Theorem 6.1. *There exists a canonical choice of b_M in $CF(R(M))$ which is 0 $(\mathrm{mod}\ T)^{\epsilon>0}$ which satisfies the Maurer–Cartan equation $\sum m_k(b_M, \dots b_M) = 0$. This lets us perturb so that Floer homology is defined; in general we have no hope because m_0 is nonzero.*

Theorem 6.2. *So then $HF(M_1 \sqcup_\Sigma M_2, E) = HF((R(M_1), b_{M_1}), (R(M_2), b_{M_2}))$.*

I'm mainly going to talk about the first theorem, which happens in a couple of steps. Let me recall the Yoneda embedding. Say C is an A_∞-category with objects and a space of morphisms $C(c_1, c_2)$, and compositions m_k. Then there is a Yoneda embedding $C \to Func(C^{op}, ch)$, where ch is the dg category with objects chain complexes. The opposite category has objects the same, but $C^{op}(C_1, C_2) = C(C_2, C_1)$. There are signs on m_k but I'm using $\mathbb{Z}_2$ coefficients today.

So D is an object of this functor category, this means that for any c in $Ob(\mathcal{C})$, then $D(c)$ is a chain complex. If $c_0, \dots, c_k$ are objects, then we have a map $n_k : D(c_0) \otimes \bigotimes_{i=1}^k \mathcal{C}(c_{i-1}, c_i) \to D(c_k)$, and we have the relations

$$\sum n_{k_2}(n_{k_1}(y, x_1, \dots,) \dots, x_k) + \sum n_{k_1}(y, \dots m_{k_2}(\cdots)) = 0$$

and this is the definition of being an A_∞-right module. So for c and c' in $\mathcal{C}$ we can take $\mathcal{C}(c, c')$, and you can take m to n, and you can prove that this A_∞ functor is an equivalence [unintelligible]its image.

So we are in the situation that $\partial M = \Sigma$ and $R(\Sigma)$ is symplectic. So we consider the A_∞-category $F(\Sigma)$ whose objects are Lagrangian immersions $\widetilde{L} \xrightarrow{i} R(\Sigma)$ along with $b \in CF(\tau)$ such that $\sum m_k(b, \dots, b) = 0$.

We have $F(\Sigma)$ mapping into $Func(F(\Sigma)^{op}, ch)$. First we'll construct $\mathcal{HF}_M$ here in the functor category, $F(\Sigma)^{op} \to ch$, and then the second step is to show that there is b_M so that $(R(M), b_M)$ which represents this functor.

One step is to cook up the A_∞ functor and the second step is to show that it's representable. Once we cook up this functor, then b_M is well-defined. The first step, I was writing some papers in 1997 or something like that and I claimed to have this functor. I couldn't prove this next step so I stopped. What do I mean by this functor? One is, given (L, b_L), we should find $HF(M, (L, b_L))$. Then given (L_i, b_i), we should find

$$CF(M(L_0, b_0)) \otimes \bigotimes CF((L_{i-1}, b_{i-1}), (L_i, b_i)) \to CF(M, (L_k, b_k)).$$

In 2000 or something, Salamon–Werhein did this first thing for objects for L monotone. The second step was sort of outlined in 1997 but never published. The representability I didn't know how to do 20 years ago. Now I've found a way to go through.

The idea is the following. You consider the ASD equations on $M \times \mathbb{R}$, and the boundary is something like $\Sigma \times \mathbb{R}$. You also have kind of ends at $\pm\infty$. If M is empty, you have asymptotic boundary conditions, but here you also have [unintelligible]. The idea is to use L as a boundary condition.

We consider that $M\backslash\text{cpt}$ is $\Sigma \times (-1, 1)$ where $\Sigma \times 1 = \partial M$ with the direct product metric $ds^2 + g_\Sigma$. So for $\mathcal{A}$ a connection on $M \times \mathbb{R}$, we have $*F_{\mathcal{A}} + F_{\mathcal{A}} = 0$. What happens on $\mathcal{A}|_{(\Sigma\times(1))\times\mathbb{R}}$. You have some boundary conditions. Their result, they proved the first option. Their method to deal with compositions has some trouble, which is why I wnated to do something. I stopped for 17 years after my paper of 1998 in GAFA. So take this function χ, this function $[-1, 1] \to [0, 1]$, [picture], you start at 1 and becomes 0 after 0, so $\chi(s) > 0$ for $s < 0$ and $\chi = 1$ in a neighborhood of $s = -1$, so something like $\chi(s) = e^{\frac{1}{s}}$ around 0. I use this χ and this metric, $\mathfrak{g}$, which is g on a compact subset of M but then is $ds^2 + \chi(s)^2 g_\Sigma$ for $\Sigma \times [-1, 1]$. This degenerates in the fiber direction for positive s.

So $\mathcal{A}$ is a connection on $M \times \mathbb{R}$, and the equation is $F_{\mathcal{A}} + *_{\mathfrak{g}} F_{\mathcal{A}} = 0$, that it's anti-self dual. It makes sense on the compact part and when s is negative. But what about when s is positive?

Let me consider $[1, 1) \times \mathbb{R} \times \Sigma$ and write $\mathcal{A} = A + \Phi ds + \Psi dt$. This is a two-parameter family of connections on Σ. Then the anti-self dual equation is equivalent to the following two equations:

$$\frac{\partial}{A}\partial t - d_A \Psi = *_{g_\Sigma}(\frac{\partial A}{\partial S} - d_A \Phi)$$

and

$$\chi(s)^2(\frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi, \Psi]) + *_{g_\Sigma} F_A = 0.$$

These equations work even for $\chi = 0$. Then the second equation is just $F_A = 0$. So somehow you have a domain [picture], and on one part of it you have a family of flat connections parameterized by s and t. What is the first equation? You know that $\frac{\partial A}{\partial t} - d_A \Psi$ and $\frac{\partial A}{\partial s} - d_A \Phi$ is d_A-closed. and the second is the $*$ of the first, so then they are d_A-harmonic. This means that $\varphi(s, t) = [A] \in R(\Sigma)$. Then $\frac{\partial \varphi}{\partial s} \in H^1(\Sigma, A)$ is the harmonic forms $\mathcal{D}^1(\Sigma, A)$. Then the second equation is $\frac{\partial \varphi}{\partial t} = *\frac{\partial \varphi}{\partial s}$ and this is [missed], I'll talk about this equation more on Thursday. We use a package, something like Yang–Mills theory or Gromov–Witten theory. Today I'll pretend everything works and [unintelligible].

Okay, so now we consider the moduli space. One thing i can say is the following. You can put the boundary value L at $s = 1$ very naturally. When t goes to $\pm\infty$ you need the following asymptotic boundary conditions. Take $R(M)\times_{R(\Sigma)}\widetilde{L}$, $\widetilde{L}\to R(\Sigma)$, and this is a natural thing to put at the boundary. So let's assume that this fiber product is $\coprod R_i$, a disjoint union of smooth manifolds (that is, that the intersection is *clean*). So then $\mathcal{M}(L,R_i,R_j,E)$ is the set of connections $\mathcal{A}$ on $M\times\mathbb{R}$ satisfying these equations $*F_{\mathcal{A}}+F_{\mathcal{A}}=0$. Then $A(t,s=1)$ has gauge equivalence class on L. Then as $t\to-\infty$, the element of R_i, as $t\to+\infty$, elements of R_j, and you want to divide by gauge equivalence classes times $\mathbb{R}$, you have

$$E=\int_{s\le 0}\|F_{\mathcal{A}}\|^2+\int\varphi^*\omega$$

So you have R_i and R_j at the various ∞s and then lies on L on the other part of the boundary.

Now $\mathcal{M}(L;R_i,R_j,E)$ maps to R_i and R_j by taking $t\to\pm$ limits. So one can make this a smooth manifold but for safety it's probably better to do this a different way.

So then we can cook up something $\Omega(R_i)\to\Omega(R_j)$. So $\int f(E)(\alpha)\wedge\beta=\int_{\mathcal{M}(L,R_i,R_j,E)}ev_-^*\alpha\wedge ev_+^*\beta$. Maybe it's better to write this, well, write $n_{0,E}$ for the coefficient, and it's $\sum_E T^E n_{0,E}$

In general $n_0\circ n_0$ is nonzero. You have this

$$\partial\mathcal{M}(L,R_1,R_2,E)=\coprod\mathcal{M}(L,R_1,R,E)\times_R\mathcal{M}(L,R,R_1,E_2)$$

and this implies that $n_0\circ n_0=0$.

The problem here is that, there's another boundary at this level, a disk bubble. You have solutions that show up as boundary values like this [picture]. We can handle them as we do in regular Floer theory. We have this chain complex $CF(M,L)\otimes CF(L)^{\otimes k}\to CF(M,L)$ which makes this a right $CF(L)$-model.

So now we have this structure, we put marked points down and this gives m_k. Now I explained the case with just one manifold. This is the same for right A_∞-modules (over categories) in this way. In particular, a bounding chain in $CF(L)^{\otimes k}$, well $n_k(b(\eta,b,\cdots b)=d^b$ and $d^bd^b=0$. Then we can define $HF(M,(L,b))$ as $\frac{kerd^b}{imd^b}$.

Then the key new thing that came this year is that this is unobstructed. Suppose that D is a right A_∞ module over $\mathcal{C}$ and I want $\mathbf{1}\in D$ to be cyclic. We have a map $\mathcal{C}\to\mathcal{D}$ which sends $x\mapsto n_1(\mathbf{1},x)$, and the first assumption is that this is an isomorphism. Secondly, $n_0(\mathbf{1})$ is equivalent to 0 modulo T^ϵ for $\epsilon>0$. Then there exists a unique $b\in C$ which is $0\pmod{T}^\epsilon$ such that $\sum m_k(b\cdots b)=0$ and $d^b(\mathbf{1})=0$.

Let me explain the proof very quickly. To prove this Lemma, we solve the equation $d^b(\mathbf{1})$ by expanding by a power series in d. Then we find a unique solution.

Let me consider now a geometric situation. Suppose $\partial M=\Sigma$ and we have the immersion and the A_∞-functor $CF(M,R(M))$, the right $CF(L)$-module. So now we take $L=R(M)$. This $CF(M,L)$ is a right $CF(R(M))$ module. This is a Λ_0-module.

[missed a little]

The element we construct satisfies the construction and we can apply the lemma to get $b(m)$. I want to explain why this is correct. We consider n_1 modulo T^ϵ. How do you define it? You have $\mathcal{A}$ and you have $*F_{\mathcal{A}}+F_{\mathcal{A}}=0$. You only consider energy 0, that's the trivial case, so $\mathcal{A}$ is globally a flat conncetion. Then you have trivial flat connections, everywhere the same. You can just see this in the first part.

[missed a little of the explanation]

Then we can use this algebra to obtain b_M. We're solving the equations above inductively on energy. So now if we just see the proof, we define this inductively, [picture]. These are highly nontrivial equations and getting an explicit answer is basically impossible so this is really more of an existence theorem.

7. Si Li: Deforming holomorphic Chern–Simons at large N

Thanks very much for the opportunity to talk here. I'm going to talk about joint work with Kevin Costello. This work is motivated by a rather deep idea in physics, about gauge gravity duality. The most famous version is AdS-CFT, which says that gravity on the bulk is a gauge theory on the boundary. So you might have SUGRA on the bulk and SYM on the boundary. We try to understand this correspondence from [unintelligible]point of view and also from open string duality. My goal is to understand open closed theory in the topological B-model and this has some relation with this gauge-gravity duality.

For this open-closed business, also related to homological mirror symmetry, and one essential ingredient, suppose I work with a free algebra $A = \mathbb{C}[x_i]$, and for example, I can compute $HH(A)$ as $\bigwedge Der(A)$. This acts like a connecting map between open and closed. I'll be focusing on algebras on [unintelligible]. I want to focus on the following message. First of all, this is a connecting map, and on the other hand, plays the role of anomaly cancellation, when you try to [unintelligible]. So let me try to explain this a little bit.

I'll start with the gravity side. The kind I want to discuss is called Kodaira–Spencer gravity. This is related to deformation theory on Calabi–Yaus, so I'll be working with X a Calabi–Yau 3-fold. I'll work with the simplest case $X = \mathbb{C}^3$. Obviously the quintic $\{z_1^5 + \cdots + z_5^5 = 0\} \subset \mathbb{P}^4$ is more interesting.

Let X be compact Calabi–Yau. First I want to put some gravity on this background. I want to start with Einstein gravity. We can, for example, have a Ricci flat metric, $Ric_g = 0$. By the way, I should say I will fix a Kähler class. This is related to the famous Calabi conjecture, Yau theorem. A complex structure in this Kähler class, there's a unique flat metric. This is described by a Maurer–Cartan equation $\bar{\partial}\mu + \frac{1}{2}\{\mu,\mu\} = 0$. These have relations but it's very deep to understand how they're connected.

This kind of Maurer–Cartan data can be realized on a Calabi–Yau 3-fold as the equations of motion of a so-called Kodaira–Spencer gauge theory. This is called BCOV theory, and I'll call this KS gravity. The original BCOV proposal works for a Calabi–Yau surface, and we extend the structure to any Calabi–Yau with Costello. At the end of the day, the full content is given by fields, polyvector fields on X, with a formal variable, $PV(X)[[t]]$, and by the way, I really mean $PV(X) = \mathcal{A}^{0,*}(X, \bigwedge^* TX)$. The classical equations of motion for this extended theory is somehow equivalent to this kind of equation $\partial\mu + \frac{1}{2}\{\mu,\mu\}$, where $Q = \bar{\partial} + t\partial$ where ∂ is the divergence with respect to the Calabi–Yau volume form Ω_X.

This is the closed string field theory in the topological B-model.

Now let me move to open string on the gauge side.

The kind of gauge theory I want to discuss is holomorphic Chern–Simons. So, let me roughly describe what this is. The fields is going to be $(0,1)$-forms valued in a Lie algebra, so $\mathcal{A}^{0,1}(X, \mathfrak{gl}_N)$. So

$$HCS[A] = \int_X Tr(\frac{1}{2}A \wedge dA + \frac{1}{3}A^3) \wedge \Omega_X.$$

If you look at the classical system of equations of motion, you'll find $\frac{\delta HCS}{\delta A} = 0$ gives you $\bar{\partial} A + A^2 = 0$, which is another Maurer–Cartan equation, in this case giving a holomorphic vector bundle.

I eventually want to quantize. I can do my gauge fixing conveniently with the Abelian, BRST-BV formalism. I enlarge my space of fields to a bigger space, my fields are $\mathcal{A}^{0,*}(X, \mathfrak{gl}_N)[1]$, so I have fields in $\mathcal{A}^{0,1}$, and $\mathcal{A}^{0,0}$ are my ghosts. You have another two copies, and $\mathcal{A}^{0,2}$ are my antifields and $\mathcal{A}^{0,3}$ my antighosts. The whole thing, we can do quantization, this is the perfect example so I can see the ghosts, I'll come back to this.

[A small point, the antighosts are not the antifields of the ghosts usually.]

You're right, I'll say it like this for symmetry.

So we should think of the dual of open strings as closed strings, and we'll think in this way and try to deform holomorphic Chern–Simons. Let's call $\mathcal{A}^{0,*}(X, \mathfrak{gl}_N)[1]$ by $\mathcal{E}$, my space of fields. We have the classical master equation $\{HCS, HCS\} = 0$. And actually, this equation is nothing but saying that this comes from a differential graded Lie algebra. The deformation theory for this holomorphic Chern–Simons, this deformation is controlled by the following complex, the BRST complex. Let me write some notation. I'll say $\mathcal{O}_{loc}(\mathcal{E})$ is the space of local functionals on $\mathcal{E}$ and local means like, these functionals are written as $\int_X \mathcal{L}$, integrals of a [unintelligible]density, very special distributions on the space of fields. Bracketing with the Chern–Simons functional gives a differential on $\mathcal{O}_{loc}(\mathcal{E})$ and this gives the deformations.

Usually if you work with an arbitrary Lie algebra, this will be complicated. There's a nice simplification where everything simplifies, the so-called *large N limit*, which is also related to many notions from physics. For our purposes this is simple. Our differential $\{HCS, \quad\}$ represents a Chevalley–Eilenberg differential, so computing the homology is just computing the Lie algebra cohomology. There's a classical theorem that says that Lie algebra cohomology at large N, a theorem of Loday–Quillen–Tsygan says that this is the same as the cyclic complex. This will be the key thing we're going to use. If you work it out very carefully in large N. If you work it out very carefully you get something like this.

Let me use more notation,

$$\mathcal{O}_{loc}(\underbrace{CC.}_{\text{cyclic complex}}(\mathcal{A}^{0,*})[1])$$

this is related to the local multi-trace operator.

So $CC.(\mathcal{A}^{0,*})$ can be represented by $t^{-1}PV[t^{-1}]$ with the natural differential $Q = \bar{\partial} + t\partial$. So we know this is kind of dual to the space of fields for the Kodaira–Spencer bracket. You integrate polyvector fields on Calabi–Yaus.

If you try to look at tree-level amplitudes for this theory, you'll see Givental's formalism. [missed example]. At the superpotential you'll see Seidel's primitive form [?]. The punchline is that the Kodaira–Spencer gravity for us will give rise to a kind of universal object in deforming this holomorphic Chern–Simons at large N. This is the linear dual, which is a single trace operator.

This comes from the abstract calculation. You can write down a very precise formula for how this is coupled, let me write down some formula to give you a sense.

So I start with holomorphic Chern–Simons, the following data. I try to couple $\mathcal{A}^{0,*}(X, \mathfrak{gl}_N)[1] \oplus PV(X)[[t]]$. Write an element A or μ, so let me write $HCS(A) +$

$I(A,\mu)$ for my deformation. So $I(A,\mu)$ can be written as some kind of integrations on configuration space

$$\int_{C^{m,n}}$$

where $C^{m,n}$ is the configuration space of points on the disk, m points on the boundary and n points in the interior. This kind of formula appears in Kontsevich's deformation quantization. This is a cyclic extension of his graph formula, and is basically due to Willwacher and Calaque, trying to prove cyclic formality. Let me give you a sense by trying to write down a formula.

For example, here's how it looks like. Let's consider the case when I have a polyvector μ from Kodaira–Spencer gravity, and we can write this in local coordinates as $\mu^{i_1\cdots i_m}\partial i_1 \wedge \cdots \partial i_m$. Then $A \in \mathcal{A}^{0,*}(x,\mathfrak{gl}_N)[1]$, and you try to write down a first order deformation [picture]. The coupling works, you apply the μ as differential operators to the boundary. This corresponds to a local Lagrangian density in the following form

$$c\int Tr(\mu + A \wedge (\partial A)^m) \wedge \Omega_X.$$

So for example, if you like local coordinates, you could write this like

$$c\sum \int (\mu^{i_1\cdots i_m}(A\partial_{i_1}A\cdots\partial_{i_m}A) \wedge \Omega_X.$$

This is Kontsevich's graph formula, modified by Willwacher, the coupling constant is an integral over this configuration space.

In general it will be very similar, the coupling is realized by graph integrals, where you have interior points and graph data, and you can apply polyvectors in different ways. [picture].

So the main point at this point is quantization. This is the classical story, you can treat this as a disk amplitude. The interior are closed strings, the boundaries are open strings. You want to work out the quantization. For example, you can use Feynman diagrams but you have to worry about gauge anomalies. Let me start with S_0, the action functional, which is $HCS(A)$, coupled to this gravity by 1st order. Basically we want to use the initial data and deformation theory to reconstruct the quantization. Now the space of fields has two componentns $\mathcal{A}^{0,*}(X,\mathfrak{gl}_N)[1] \oplus PV[[t]]$, now in particular, we have a Poisson bracket of degree 1. Only working to first order, this also satisfies the master equaiton $\{S_0, S_0\} = 0$, which follows from this being an L_∞-morphism.

I want to quantize this one in the BV-formalism, and the basic idea is that I start from solutions of the classical master equation S and try to find something S that satisfies the quantum master equation. Roughly speaking this looks like

$$(\hbar\Delta)e^{\frac{S}{\hbar}} = 0.$$

You can work with this algebra, but for us there's a more serious problem, because, let me write it here.

The problem is that this is an ∞-dimensional space. The space of fields is a very big space. It turns out that Δ is actually singular. The kernel is represented by δ-functions, related to having ∞-many degrees of freedom. So first we need to make sense of this equation.

We can renormalize the theory as a solution. The idea is, this is a remarkable thing from physics. Analyzing this requires using terms to cancel singularities.

We hope our gauge theory is preserved at the quantum level. This may not be solveable in this equation. So you make some obstructions for solving this equation, and sometimes this is called gauge anomaly. If you can find the solution, then the key point is that you can compute the coordinate functions for some observables, then

$$\langle O\rangle = \int_{\mathcal{L}} O\mathcal{E}^{S/\hbar}.$$

Once you have a quantization you have these evalutations. [missed a little].

What's happening in our system? If you work with holomorphic Chern Simons itself you find anomalies. [missed a little].

I have a bunch of data, open and closed data. The closed data has polyvector fields. I have $t^{-1}PV[-1] \overset{\partial}{\to} \partial PV$. and there's a boundary that makes the whole thing unpleasant.

[missed a little]. The upshot is that, well, you can combine everthing and if you do so the complex is simplified to $PV((t)), Q = \bar{\partial} + tD$. The upshot is the following. The anomalies all cancel. If you try to work with one loop, there's anomaly at the annulus level. Let me use this to represent holomorphic Chern–Simons, if you look at 1-loop things, this represents [missed]. If you renormalize and couple you get something nonzero [pictures]. Remarkably ther's another sector, which gives this diagram [picture]. You apply a closed form BV, which basically represents something like a closed BV, and you get $\{I, I\}_c = 0$.

Then you use a deformation and eventually it becomes closed strings. That's the point. Now, the theorem is the following.

Theorem 7.1. *We work with a local piece of the Calabi–Yau, so just $\mathbb{C}^d$, for d odd. There is a canonical quantization from this coupled system, starting with S_0, satisfying symmetries that I don't want to mention. The anomalies cancel out, and the connecting map, things cancel. This is a large N statement with $\mathfrak{gl}(N|N)$ to have trace 0.*

So for example, you can use Feynman diagrams, to build up these functions for a nontrivial topology. You can have some input from open strings. I should stop here.

8. July 8: Mikhail Kapranov: Combinatorial approach fo Fukaya categories of surfaces II

[I have some announcements to make before we start the lecture. Despite the weather we will still go on the trip. We will meet at the lobby at the POSCO international center at 1:50. We'll leave at 2:00 sharp. Tell Soojin if you are interested. Please take your belongings as this hall will be closed at 12:30. There will be a group photo session after the last lecture. If you'd like to join the banquet tomorrow please tell Soojin if you are interested before lunch.] Something I'm missing from Monday is enhancements of triangulated categories.

In particular, we need this to have functorial cones and some other things. We were talking Monday about classical triangulated categories. Maybe I'll say one more thing here. In a triangulated category, $Hom_{TC}(A, B)$ is typically the set of equivalence classes of something, and what you're really interested in is this something, and that's the enhancement.

Here there have been several different points of view. One I'll focus on and give more detail about.

(1) The first approach is Grothendieck derivators. I will not discuss this. It's more in the classical spirit. It will be more convenient for me to use a different approach.
(2) Pretriangulated dg categories. I'll expand on this in a moment, this was begun by Bondal and myself and enhanced by Tabuada–Töen
(3) The stable ∞-categories of Lurie
(4) (often used in the Fukaya category literature) A_∞-enhancements.

In the relations among all these things. Between the pretriangulated dg-categories and stable ∞-categories there is basically an equivalence given by the dg nerve, introduced by Lurie and by Behrend–Getzler. Pretriangulated implies that the ∞-category is stable, that was a poster talk in this conference. There is also an A_∞-nerve with a similar property related to stability (due to Faonte). I'll focus on the pretriangulated dg category case in more detail.

Consider dg categories, where $Hom_{\mathcal{C}}(A,B)$ is a cochain complex over some field $\mathbf{k}$. In this case, the equivalence classes is the homology of this complex, $H^0\mathcal{C}$ is a usual category with the same objects as $\mathcal{C}$ and taking $H^0Hom(A,B)$ as morphisms. In particular, the homotopy category of an additive category $\mathcal{A}$ is the same as $H^0C^*(\mathcal{A})$.

When we have a dg category $\mathcal{C}$ then as in Kenji Fukaya's talk, there is a Yoneda embedding $\mathcal{C} \to Mod_{\mathcal{C}}$ which is $Fun^o(\mathcal{C}, C^*(Vect_k))$. We can factor this to $Pre-Tr(\mathcal{C})$ which is the closure of $\mathcal{C}$ under $\oplus$, Σ, and cones. This is such that $H^0Pre-Tr\mathcal{C}$ is always triangulated. You could ask for closure under direct sums, there are deep reasons to do this sometimes, but let me not do this or talk about this to save time.

We call $\mathcal{C}$ pre-triangulated if $i_{\mathcal{C}} : \mathcal{C} \to Pre-Tr\mathcal{C}$ is a *quasi-equivalence* if it induces quasi-isomorphisms on Hom complexes (always true for $i_{\mathcal{C}}$) and $H^0(i_{\mathcal{C}})$ is essentially surjective. These properties imply that $H^0(\mathcal{C})$ is triangulated, and then $\mathcal{C}$ is called an enhancement of $\mathcal{V} = H^0(\mathcal{C})$. It's hard to do anything without such an enhancement.

To every $\mathcal{C}$ we can associate the derived category of modules over $\mathcal{C}$, $DMod_{\mathcal{C}}$, which is $Mod\mathcal{C}[qi^{-1}]$, the derived category of modules, and I'll also use *Morita equivalence*, which is a functor inducing an equivalence of the derived category of modules.

Having said this I can move to the Waldhausen S-construction. So let me call the second part of my lecture

8.1. **The Waldausen S-construction.** This is for surface Postnikov systems and if time permits or on Friday, as well, Fukaya categories.

I think the S stands for Graeme Segal. I rushed the previous part, paying lip service to formalities. But now let me slow down a little bit, I don't want to rush this part.

Let me recall the Grothendieck group of an Abelian category. Let $\mathcal{A}$ be an Abelian category. Then $K_0(\mathcal{A})$, the *Grothendieck group*, is generated by symbols $d(A)$ for $A \in \mathcal{A}$, subject to the relations $d(A_1) + d(A_2) = d(A_{12})$ when there is a short exact sequence

$$0 \to A_1 \to A_{12} \to A_2 \to 0.$$

You can think of this as being physical. You have some sort of interaction, three particles collide and five come out, so you say that the sum of three particles is equal to the sum of the five particles, and the elementary particles are the generators of the group—if you have them.

[Now I understand physics.]

[Let me say, that from a physical perspective this is totally wrong.]

Yes, yes, yes. Nonetheless. Let me pass to higher K-theory, syzygies among these relations. Suppose for example that we have $det(A)$, which are objects of some tensor category $(\mathcal{P}, \otimes)$ and isomorphisms $det(A_1) \otimes det(A_2) \to det(A_2)$, there is a question about how these isomorphisms behave, for $A_1 \subset A_2 \subset A_3$, the question is we can go in two different ways.

$$\begin{tikzcd} det(A_1) \otimes det(A_2/A_1) \otimes det(A_3/A_2) \arrow[r] \arrow[d] & det(A_2) \otimes det(A_3/A_2) \arrow[d] \\ det(A_1) \otimes det(A_3/A_1) \arrow[r] & det(A_3) \end{tikzcd}$$

and you want to know if this commutes. So this leads to a simplicial category $S.(\mathcal{A})$ that comes naively from this sort of picture. The $\cdot$ means that we have $S_n\mathcal{A}$ for each n. So $S_n^{naive}(\mathcal{A})$ is the category of filtrations $A_1 \subset \cdots \subset A_n$ and their isomorphisms. These should have face operators ∂_i for $i = 0, \ldots n$, for $i \neq 0$, ∂_i just forgets A_i. Then ∂_0 takes the quotient by A_1. If you do this, then strictly speaking, there should be some identities, and they don't hold. $\partial_0\partial_1$ and $\partial_0\partial_0$ are isomorphic but not equal and that's annoying. The S-construction of Waldhausen rigidifies this. This may seem like a small detail but it's important for things to move smoothly. The correct refinement due to Segal and Waldhausen is this, $S_n(\mathcal{A})$ is the category of diagrams where the quotients are given as part of the structure.

$$\begin{tikzcd} A_{00} \arrow[r] & A_{01} \arrow[r] \arrow[d] & A_{02} \arrow[r] \arrow[d] & \cdots \arrow[r] & A_{0n} \arrow[d] \\ & A_{11} \arrow[r] & A_{12} \arrow[r] \arrow[d] & \cdots \arrow[r] & A_{1n} \arrow[d] \\ & & \ddots & \cdots & \vdots \arrow[d] \\ & & & & A_{nn} \end{tikzcd}$$

such that every horizontal arrow is mono, $A_{ii} = 0$, all vertical arrows are epi, and for all $i < j < k$ we have $0 \to A_{ij} \to A_{ik} \to A_{jk}$ is a short exact sequence. So A_{ij} is like A_j/A_i in the naive version.

Then everything is okay. Then $K_i(\mathcal{A})$ is the same as $\pi_{i+1}|S.(\mathcal{A})|$. You need a simplicial object to do geometric realization here.

Now I want to discuss the shape of this diagram. It's possible to write it somewhat differently. You can write its geometric shape. Put A_{ij} into the vertices of a certain polytope $\Delta(1, n)$, which is the convex hull of midpoints of edges of the simplex Δ^n. The vertices are $\frac{e_i+e_j}{2}$, and you take the convex hull. In particular, $\Delta(1,2)$ is a triangle. $\Delta(1,3)$ is an octahedron. We had this picture before. In general, you have this in higher dimensions. This polytope is related to the Grassmannian of two-dimensional subspaces in $\mathcal{C}^{n+1}$. here you have the action of $U(1)^{n+1}$. The image of the moment map is exactly this polytope. This is related by the Plücker embedding in $\mathbb{P}(\bigwedge^2 \mathbb{C}^{n+1})$. This has some sort of symplectic flavor. There are symplectic forms here. On a very formal level, we get something that suggests symplectic geometry. One can prove the following fact about this.

Theorem 8.1. *$S.(\mathcal{A})$ has the 2-Segal (triangulation-invariance) property. For every n and any triangulation of the $(n+1)$-gon, there is a map, we can view a triangulation as a simplicial complex. Every triangle is a simplex so this matches up to isomorphism on edges. You have $Map(\mathcal{T}, S.\mathcal{A})$ the space of $\mathcal{T}$-membranes. You can lift a triangluation to the simplex. Every triangle here gives me a triangle in the simplex. So $\mathcal{T}$ can also be embedded into Δ^n. There is a restriction map $Map(\Delta^n, S.(\mathcal{A}))$ which is $S_n(\mathcal{A})$. [pictures]. This map from $Map(\Delta^n, S.(\mathcal{A})) = S_n(\mathcal{A})$ to $Map(\mathcal{T}, S.(\mathcal{A}))$ is an equivalence of categories. In particular, $Map(\mathcal{T}, S\cdots(\mathcal{A}))$ is independent of $\mathcal{T}$. Passing between two triangulations is something like associativity and this is related to the associativity of Hall algebras. There was a famous paper about Hall-hole haloes, of Denef. This was for the Hall effect.*

Now I want to get to a more general setting. This was in the more elementary setting of Abelian categories. Now we can do this for pre-triangulated categories. What do we want? We should speak not about short exact sequences but about exact triangles. So this is the second level. The third level should be an exact octohedron. The nth level should be an "exact hypersimplex." So formally, we convert $\mathcal{C}$ into a stable ∞-category, call it $\mathcal{C}^\infty$. Then define $S_n(\mathcal{C})$ to be the category of diagrams in C^∞, except we don't talk about mono or epi maps, we can talk about Cartesian squares. One feature of this kind of category is that Cartesian and coCartesian squares are the same. So all squares (for instance

$$\begin{tikzcd} A_{ij} \arrow[r] \arrow[d] & A_{ik} \arrow[d] \\ 0 \arrow[r] & A_{jk} \end{tikzcd}$$

for exact triangles) are (co)Cartesian.

Theorem 8.2. *This is 2-Segal.*

This by the way is how you define K-theory for enhanced triangulated categories. Now we want something else.

(1) First, we want to have $S_n\mathcal{C}$ not just a category or homotopy type but a dg category. We want a category of exact triangles or exact octahedra.
(2) More important, if $\mathcal{C}$ is 2-periodic, we want a cyclic symmetry, we want $S_n\mathcal{C}$ to have $\mathbb{Z}/n+1$ symmetry. So for $n = 2$ this is $\mathbb{Z}/3$-rotation invariance of exact triangles.

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, "+1"] & \end{tikzcd}$$

goes by τ to

$$\begin{tikzcd} C \arrow[rr] & & A[1] \arrow[dl] \\ & B[1] \arrow[ul] & \end{tikzcd}$$

with $\tau^3 = \Sigma^2$. For this, we redefine $S_n\mathcal{A}$ as $Map_{dgCat}(S_n, \mathcal{A})$ for an appropriate system S_n of dg categories. We should have S_n be co-simplicial, a category $\Delta \to dgCat$ and also have a $\mathbb{Z}/n+1$ action on S_n. So then we have an extension to Connes' cyclic category

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d] & dgCat \\ \Lambda \arrow[ur] & \end{tikzcd}$$

9. Herman Verlinde: Conformal Bootstrap, Hyperbolic Quantum Geometry and Holography II

In today's lecture I'll discuss the first word of the title, the conformal bootstrap. I'll try to emphasize the geometric parts of what the conformal bootstrap is. There are many interests in conformal field theory, and the bootstrap is a way of solving conformal field theory. This has to do with things like, well, if you discuss the boiling point of water, this is the three dimensional Ising model, which is a conformal field theory. Various constructions have not been done analytically but using this powerful idea. Usually this is defined using a Lagrangian, but with the bootstrap we bypass that notion, and the associativity of the operator product, the consistency, gives you such a strong constraint that the structures you can write down are very limited.

A very quick summary of what you could call a definition of the conformal field theory. In two dimensions you could look at a cylinder, and think about radial time evolution and think that the states of the theory are Hilbert states like $|\psi\rangle$ and if I put a state that evolves in time in, I can put $\mathcal{O}_t$ at the origin and time evolution becomes radial evolution away from the origin. Here I'm making use of conformal invariance. I use coordinates z and $\bar{z}$. Then it only uses the complex structure of the space and locally I can redefine my coordinates, and if z and $\bar{z}$ go to new coordinates z' and $\bar{z}'$ I should get something looking similar, so [something about $DiffS^1$] and this thing has a Lie algebra known as the Virasoro algebra. The generators, I could label them by infinitesimal transformations, which are vector fields I can write as formal expansions $V(z) = \sum_{-\infty}^{\infty} v_n z^{-n+1}$, and then $[L_V, L_W] = L_{[V,W]} + \frac{c}{12}\langle v, w\rangle$, where $\langle v, w\rangle = Res(v\partial^3 w)$. I have $\overline{Diff(S^1)}$ as well corresponding to $\bar{z}$, and so the symmetry breaks into what we call *left movers* and *right movers*. In the Hilbert space it turns out you can write it as a direct sum over labels

$$H_{CFT} = \bigoplus_{(a,\bar{a})} V_a \otimes \bar{V}_{\bar{a}}$$

where these are lowest / highest weight representations of the Virasoro algebra, which means like $L_V = \sum_n v_n L_{-n}$, these are called the Virasoro generators. It's a bit easier to say that V_a is the span of all things $L_{-n_1} \cdots L_{-n_s}|h_a\rangle$ where $L_0|h_a\rangle = h_a|h_a\rangle$, where this n_a is the conformal weight and this state is annihilated by L_n for all positive n. I have to be able to give a collection of h_a's. There's the famous BPZ paper, they showed that for $c < 1$, there is a discrete set of values of h that you can allow. The reason that's a self-consistent set is because you can formulate the bootstrap for that. Unitarity already gives very strong constraints. There are all kind of special things happening for $c < 1$ and as I said these are called rational CFTs and there are a finite number of h_as. In fact c is a positive number, a discrete set of allowed values between 0 and 1 or some subset of $c > 1$, in my point of view, for gravity, the regime $c \gg 1$ is the interesting regime. Doing quantum geometry yesterday, you have to think of c as $\frac{1}{\hbar}$ where $\hbar$ is the quantumness of the geometry. Classical is the regime where $c \gg 1$. The most quantum regime $c < 1$ has lots of special features. There are other soluble CFTs which all have additional symmetries on top of the CFT symmetries. But what I want to look at is what people call irrational CFTs where c can be bigger than one but I make no assumptions about additional symmetries. Then indeed this sum here, if you have additional symmetries, it's more natural to expand these modules to give additional symmetry generators. Here I'm assuming I only know about Virasoro symmetries. These things have a certain eigenvalue for L_0, the conformal weights. When I act, well, $L_0 L_{-n}|h\rangle = (n+h)L_{-n}|h\rangle$

It turns out that $h_a - \bar{h}_a$ turns out to be an integer, since rotation by 2π gets me back to the same state. This gives a restriction about the left and right movers. In first approximation I know nothing further about the allowed values of h_a. I want to write down equations about which choices of h_a are consistent. This turns out to be a rather strong restriction. If the h are random numbers, you could expect that the spectrum of a and $\bar{a}$ is the same spectrum, but they could appear in different combinations. With no other symmetries, then a and $\bar{a}$ will likely be equal.

Anyway, that's our task, and this is the structure of the Hilbert space. The nice thing about conformal field theories is that you can use nontrivial topology. You start out with something like [a pair of pants] where all three directions have their own time. I'm looking here at this object, I can do the CFT on the partition sum, it associates to this [unintelligible]an element of $\mathcal{H}_{CFT} \otimes \mathcal{H}_{CFT} \otimes \mathcal{H}_{CFT}$, and if I had one in and two out, then I should dualize one of the Hilbert spaces. Then I want to associate to this a particular element of the tensor product. I should be able to act by a $Diff(S^1)$ element on one side and pull it to the other side. Let me introduce the useful notation that will show up as we go along. What is L_V? We want to extract an operator to this. We'll write it as a contour integral

$$L_V = \oint \frac{dz}{2\pi i} V(z)T(z)$$

and this only depends on z, not on $\bar{z}$. I'll talk more about the stress energy tensor in a minute, but let me say that the element I associate to the pair of pants $|\psi\rangle$ should satisfy $(L_{v_1} + L_{v_2} + L_{v_3})|\psi\rangle = 0$ if L_{v_i} are the boundary values of a holomorphic vector field on the surface Σ_3. I can do this more generally, with a surface with more holes in it and more Hilbert spaces and I can go on like that.

The key step, and this is where the bootstrap comes in, is to go to four holes [picture] and I can obtain that by cutting it open into two segments with three holes and I can ask about my Σ_3 and Σ_3' which I can glue to Σ_4, and I glue together appropriately

$$|\psi_{\Sigma_4}\rangle = |\psi_{\Sigma_3}\rangle * |\psi_{\Sigma_3'}\rangle$$

and I can get the same geometry by gluing in a different way, cutting in a different way. [missed some about a question about formality of the variable z.] So maybe, I didn't quite finish this story, I should call this, if these boundary components are a, b, c, d, then I could call these Σ_{ab} and Σ_{cd}, and then they are glued along e and I could also take Σ_{ad} and Σ_{bc} glued along f. The way to think about this is the following. These two things have to be equal, it's a way to divide up the geometry and I'm doing this by hand. The condition that these are equal, this is already the bootstrap equation. It turns very nontrivial when I combine it with the condition that the Hilbert space has this structure. Now I can factorize the tensor product and project the object restricted to each $V_a \otimes V_{\bar{a}}$. Maybe I should make the drawing again. Instead of making this an element of Hilbert spaces I make it an element of the tensor product of the modules $|\Psi\rangle_{\langle - \rangle} \in V_a \otimes V_b \otimes V_c \otimes V_d$. This was obtained by the gluing procedure between the two specified states with a specified thing in the middle.

Maybe I want to interrupt this idea a little bit but let me say that this leads to conformal bootstrapping because I can decompose these states into a product of left and right movers. I can abbreviate this figure by its skeleton. I indicate it as a double line diagram [picture] and I want to keep track of the labelling. This thing here will be called a *conformal block*. It's a Virasoro conformal block because that's the only condition I've imposed.

I've finished my lightning definition of what a conformal field theory is. In the remaining minutes I'd like to write down the bootstrapping equation and mention the claim that conformal blocks are naturally the wavefunctions or states obtained by quantizing the Teichmüller space. The problem I discussed yesterday, these wavefunctions are identified with the conformal blocks.

Let me make a couple comments about how this works. So $T(z)$ is a quadratic differential, and the space of quadratic differentials is the [unintelligible]. If I now indeed make this other step, I project this thing now, and so rather than drawing it like [picture], it becomes more natural to write it as a punctured surface, and it's possible to place them at $(0,0)$, $(1,1)$, (∞,∞), and $(x,\bar{x})$. Then $\langle T(z)\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}_d\rangle$, the definitions I made of the primary state tell me that this is

$$\hat{T}(z)\langle \mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}_d\rangle$$

then

$$\hat{T}(z) = \sum_i \left(\frac{h_i}{(z-z_i)^2}\frac{1}{z-z_i}\frac{\partial}{\partial z_i}\right)$$

the conformal Ward identity, where $z_i = \{0,1,\infty,x\}$. Now I can view this as the quantization of a corresponding classical problem

$$T(z) = \sum_i \left(\frac{h_i}{(z-z_i)^2} + \frac{c_i}{z-z_i}\right)$$

and I claim that $T(z)$ parameterizes the space and there are now actually four parameters from c_i and z_i, but I view the c_i and z_i as coordinates on a phase space, and the constraints express the $SL(2,\mathbb{C})$ invariance of the conformal [unintelligible]functions, $\sum_i c_i = 0$, $\sum_i (h_i + z_i c_i) = 0$, and $\sum (z_i h_i + z_i^2 c_i) = 0$. In addition, there's the claim that if you view this as a parameterization of Teichmüller space then the Weil–Petersen form is of the form $\sum dz_i \wedge dc_i$ so at the level of Poisson brackets, I have $\{z_i, c_i\}_{WP} = 1$. If I go to the quantum theory I replace c_i with $\frac{\partial}{\partial z_i}$. Let me make one further comment here, I omitted one parameter, I have an $\hbar$ that I multiply by $T(z)$ and by each h_i and so $c_i = \hbar\frac{\partial}{\partial z_i}$, and this makes a little more precise the relation between $\hbar$ and $\frac{1}{c}$. This is an indication about how the problem of quantizing relates the conformal blocks to this Teichmüller space.

One last point, I've indicated the labels on my picture, how do I obtain these more geometrically, let me see if I can do this, I can also say that $T(z)$ is connected to the name "opers" and an oper is basically associated with the equation

$$\left(\frac{\partial^2}{\partial z^2} + b^2 T(z)\right)\psi_{(1,2)}(z) = 0$$

which means that $(L_{-1}^2 + b^2 L_{-2})|\psi_{(1,2)} = 0$, we call this a *null state*. Now we can write down the equation and differentiate the previous equation and write

$$\frac{\partial^2}{\partial z^2}\langle\psi^{(2)}_{(1,2)}\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}_d\rangle = b^2\hat{T}(z)\langle\psi^{(2)}_{(1,2)}\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}_d\rangle$$

and I can try to parallel translate, I can look at

$$\left(\frac{\partial}{\partial z} + \begin{pmatrix} 0 & b^2 T \\ 1 & 0 \end{pmatrix}\right)\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

and now I can view this as a kind of flatness equation. Then I want to start defining a conformal block, which tells me I should fix the h and the conformal channel in the middle, and the way I do this is by doing the parallel transport wrapping around the cycle in the middle, using my brother's operator, the Verlinde operator, you take two of these operators, writing the identity as a product of two of these guys, and you move one of them around and let them annihilate, and this is basically related to the $tr(g_1 g_2)$, and L_α is precisely the operation you pick up if you transport the null operator around the cycle. If I require that this thing have some fixed geodesic length as an eigenvalue, then the state that has that eigenvalue is the conformal block.

I've argued two things, that the Hilbert space of states can be interpreted as conformal blocks, and the last thing I want to write down is the expression of the equality of the two ways of cutting. If you use the fact that the Hilbert space factorizes into left and right movers, then I can sum over e, and the sum over $e, \bar{e}$ of my pictures, their absolute value squared, is $\langle\mathcal{O}_a\mathcal{O}_b\mathcal{O}_c\mathcal{O}_d\rangle$ and this should be by duality the same as the sum over $f, \bar{f}$ of the absolute value squared. My spectrum should be such that if I sum over all intermediate values in one picture then I should get the same on the other kind of picture. So there's a transformation, the Fuchsian matrix, that relates these two, $F_{ef}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ which should be unitary. Then e is the eigenstate of ℓ_α and the other side is the eigenstate of ℓ_β. Tomorrow I'll explain about how to do this for [unintelligible].

10. July 9: Kenji Fukaya: Floer homology for 3-manifolds with boundary III

I am supposed to talk about analysis, but before I want to talk about one thing I couldn't get to last time, which is gluing formulas. So we consider $\partial M_1 = \Sigma = -\partial M_2$, with E_1 and E_2 which pull back to $E|_\Sigma$. Then we have the immersions $R(M_1), R(M_2) \to R(\Sigma)$. Then as I explained, we can construct b_{M_i} in $CF(R(M_i))\otimes\Lambda_+$, and this satisfies the Maurer–Cartan equation $\sum m_k(b_{M_i},\dots,b_{M_i}) = 0$. If we write M for the union then we have

Theorem 10.1.

$$HF(M;E) \cong HF((R(M_1), b_{M_1}), (R(M_2), b_{M_2}))$$

Let me talk some about this before moving to the analysis.

So consider $W \subset \mathbb{C}$, roughly the set I'll draw here [picture]. This is a neighborhood of the union of the imaginary and real line. We consider $W \times \Sigma$ with the metric $\chi(s)^2 g_\Sigma \oplus ds^2 \oplus dt^2$ where $\chi(s) = e^{\frac{1}{s}}$ for $s < 0$ and 0 for $s > 0$. So this is singular when $s > 0$.

Then we glue, let me recall, take $M_1 \times \mathbb{R}$ and its boundary is $\mathbb{R} \times \Sigma$ and I want to glue it here [picture]. If I name the boundary components $\partial_i \times \Sigma$ then I'm gluing $M_1 \times \mathbb{R}$ to ∂_4 and $M_2 \times \mathbb{R}$ to ∂_3, the negative real parts of the boundary. You have a 4-manifold X with a metric g which is degenerate on one side. Then X has two boundary components, $\partial_1 \times \Sigma$ and $\partial_2 \times \Sigma$. X also has four ends, there's a part tha looks like $M \times (-\infty, 0]$. Another looks like $\Sigma \times (0,1) \times (0,\infty)$. A third end looks like $M_1 \times (-\infty, 0]$ and the fourth end looks like $M_2 \times [0, \infty)$. So this X has two boundary components and four ends.

So now we consider the moduli of anti-self dual connections. We have $CF(M,E) \to CF(R(M_1), R(M_2))$, and the generators are flat connections on M, a. On the right side the generators are $c \in R(M_1) \cap R(M_2)$. So I want to find this chain map and show that it is a chain homotopy equivalence.

So I consider the anti-self-dual equations (as explained yesterday at the degeneracies) $F_{\mathcal{A}} + *_g F_{\mathcal{A}} = 0$. So I need boundary conditions like $\mathcal{A}|_{\partial_2 \times \Sigma}(s,t)$ lies in $R(M_2)$, and similarly for ∂_1 and M_1. Now we need some more conditions, now $\mathcal{A}$ converges to a, a flat connection on M at the negative real end. At the positive real end, I have this strip, a family of flat connections on this strip. This end should naturally be c, the intersection of $R(M_1)$ and $R(M_2)$. The two more ends, you see something like $M_1 \times \mathbb{R}$ with these equations. I'd say that there $\mathcal{A}$ converges to $\alpha \in [R(M_2)]$. [pictures]. This should look something like $R(M_1) \times_{R(\Sigma)} R(M_2)$. On the other side you have $\mathcal{A}$ converges to β which is in $[R(M_2)]$.

This is complicated. You can think about the picture of four Lagrangian intersections [picture]. In this space the three-manifold plays the role of the Lagrangian submanifold. This is a kind of boundary condition, very different from previous ones.

So what I need to do, the boundary conditions, we need to require that the energy of these equations is some fixed number. Then we take the sum of the count of this moduli space times T weighted by the energy

$$\sum \#\mathcal{M}(\cdots) T^E [h] = \varphi_0(a).$$

So I fix some flat connection and some intersection point, we count the order of the moduli space to get the coefficient and then sum up.

To prove that this is a chain map, we need to consider the case when this moduli space is one-dimensional and look at the boundary.

You may have a disk bubble at the boundary, or at the four ends, something could escape in the four ends. If you want to cancel. Of course, in general there is a disk bubble. You use bounding chains in general to deal with disk bubbles and we have the bounding cochains in this case too. I could write an equation but it will take too much time.

Then I have the four ends. The first end is $\varphi \circ \partial$, where this is gauge theory Floer theory boundary. The second boundary is $\partial \circ \varphi$, and this is the Lagrangian Floer theory boundary. The other two are zero; why are they zero? We required that $\sum n_k([R_H], b, \ldots b) = 0$ which means that $[R_M]$ is a cycle. So these two are zero, and the moduli space of dimension one, its boundary is just $\partial\varphi + \varphi\partial$.

So this is a chain map. I want to prove that this chain map induces an isomorphism in homology. Consider energy 0 solutions. You observe that actually the flat connections correspond one to one to connections. Energy 0 is the identity, and if you put the higher energies it's a chain map. So then this becomes a chain equivalence.

The rest of the time I want to talk about the analysis, which looks a little exotic.

So I want to say something about this equation. The equation I want to discuss is this (this is in a recent preprint; the next thing is half in a paper in GAFA and half not written). So here are equations

(1)
$$\left(\frac{\partial A}{\partial t} - d_A\Psi\right) - \frac{*}{2}\left(\frac{\partial A}{\partial s} - d_A \circ I\right) = 0$$

(2)
$$\chi(s)^2\left(\frac{\partial \Psi}{\partial s} - \frac{\partial \Phi}{\partial t} + [\Phi, \Psi]\right) * \frac{*}{2}F_A[illegible] = 0$$

Theorem 10.2. *If we have $\mathcal{A} = A + \Phi ds + \Psi dt$ and these two conditions, $\mathcal{A}$ is defined on the punctured disk crossed with Σ and energy is everywhere finite, then [unintelligible].*

So first of all I want to recall the following things. E is an $SO(3)$-bundle, then $E_{\mathbb{C}}$ is an $O(2)$-bundle and $\wedge^{dt}E_{\mathcal{C}} \cong \mathcal{C}$, that's the same stuff. So $\mathcal{A} = Aut(E)$ and $\mathcal{A}_{\mathbb{C}} = \mathrm{Aut}_{\mathbb{C}}(E)$. The first acts on connections on E. Then from $\mathcal{A}$ an $SO(3)$-connection, we can write $\mathcal{A}_{\mathbb{C}}$ as a $1,0$ part and $0,1$ part, and $\mathcal{A}$ corresponds to $\mathcal{A}_{\mathcal{C}}^{0,1}$. The complex gauge group actually acts on connections on E.

So $\bar{\partial} + \mathcal{A}_{\mathbb{C}}^{0,1} = \bar{\partial}_{\mathcal{A}_{\mathbb{C}}}$, then this goes to $g^{-1}\bar{\partial}_{\mathcal{A}_{\mathbb{C}}}$. There's a famous story. We do this for $X = \Sigma$, and all connections on Σ, we get a moment map

$$Conn\Sigma \to \Gamma(\Sigma, \wedge^{0,2} \otimes \mathfrak{so}(3))$$

$A \mapsto F_A$, a curvature, which is a moment map of this gauge group. In the finite dimensional case, we have stable connections modulo the complexified gauge group this is $\mu^{-1}(u)$ modulo $g_{\mathbb{R}}$. At the beginning, this implies that for A a connection on Σ, with $|F_a| < \epsilon$, then there exists a complex gauge transformation whcih takes A to $g_{\mathbb{C}}^{*}A_0$ and $F_A = 0$. The flat connections all being irreducible implies that they are stable. Then [missed]. You can assume that $g_{\mathbb{C}} = exp(\sigma)$ and $\bar{\sigma}^t = \sigma$. You can focus on the purely imaginary transformations.

The next thing, let X be a 4-manifold, say $\Sigma \times D^2$, and consider the complexified gauge transformations, the claim is that equation 1 is invariant by $g_{\mathbb{C}}$. The second equation is not. This first equation is $\bar{\partial}_{A_{\mathcal{C}}^{0,1}}\bar{\partial}_{A_{\mathcal{C}}^{0,1}} = 0$. The first equation says this is [missed].

So our energy is assumed to be finite, but we may assume that $E(\mathcal{A})$ is very small. Then F_A is small everywhere. You can estimate using ellipticity. You have this two-parameter family of connections. These connections are complex gauge equivalent to flat connections. Then this implies there is some complex gauge transformation $g_{\mathbb{C}}$ so that $A(s,t) = g_{\mathbb{C}}(s,t)_* A_0(s,t)$ where A_0 is flat.

Now I can remark that I consider this one, can just take [illegible] which is a connection on $(D^2 \backslash 0) \times \Sigma$. You started with a family of connections flat when s is positive, but then by the transformation they are always flat.

We have s, t mapping to $\varphi(s,t) = [A_0(s,t)]$. So this gives a map $D^2 \backslash 0 \to R(\Sigma)$ which is holomorphic. So this extends to $D^2 \to R(\Sigma)$. Now we extend this, but this is not the end of the story. I want to do the following thing. Now we extend A_0 to everywhere. What we know is that this $\mathcal{A} = g_{\mathbb{C}}^* \mathcal{A}_0$ and $g_{\mathbb{C}} = exp(\sigma)$ with $\bar{\sigma}^{-1} = \sigma$. You need to check some real gauge transformatinos for this last condition. The real part you can apply [missed]. I wnat to show that $g_{\mathbb{C}}^* \mathcal{A}_0$ is smooth. Then we will obtain the theorem.

Lemma 10.1.
$$|\sigma|_{C^k} \le C_k (\log \chi)^{2k} \chi^2$$
where $\chi(s) = e^{\frac{1}{s}}$ *for* $s < 0$.

I don't want to prove this, but let me talk a bit about it. [comments].

This one I want to explain. The proof just uses, we have to use the second equation. I want to see what the second equation means. So $F_{exp(\sigma)A_0}$ is something like $*d_{A_0}\sigma$ plus second order. This is just a calculation. Just check a bit and you have this. Then the second equation is
$$d_{A_0}\sigma + c|\sigma|^2 + \chi(s)^2[\cdots] = 0.$$
And d_{A_0} is invertible, so you have second order plus exponentially small things. That's a proof but you have to write down the estimate carefully.

The last thing I want to say is about Fredholm theory, which is not written, So we have $*F_{\mathcal{A}} + F_{\mathcal{A}} = 0$. You want to cook up a solution space which is nonlinear. You should find a Fredholm complex that controls this equation. You have the Cauchy–Riemann equation on this side and antiself-dual equations on this side and the boundary makes it so this can't be Fredholm. So overlap it a little bit. On one side you have one and the other side the other and you have a little overlap. You have the AHS complex for $s \le 0$ and the Cauchy–Riemann complex for $s \ge -\epsilon$. You have elements a and b and you have something on the overlap which is basically the axiom for the Fredholm complex. You can still do something on the overlap. The claim is that you get a Fredholm complex. For finite dimensionality of the index of the operator, you consider a sequence, and you want to find a bounded sequence that converges. Even with ellipticity, you can't expect convergence, but if you go to $s \ge -\frac{\epsilon}{2}$. So first take a convergent subsequence that converges there. Then you change by a boundary to make it converge in the desired place. So you have an element that converges near the boundary, and then you can change it to converge everywhere. I'm sorry I can't say more.

11. Ludmil Katzarkov: Sheaf of categories and applications

Let me thank the organizers for giving me the opportunity to speak here, it's always nice to be in Pohang, maybe in drier weather. The sheaf of categories is the main example, but this is part of a more general phenomenon which should go under the name *Kähler metrics on categories*, I'll start with

(1) a model and motivations, and then define
(2) Kähler metrics on categories, and then consider
(3) examples.

The second part is a joint project with Kontsevich, [unintelligible], and Pandit, and the third part is published in a different form with Simpson, Nole, and Pandit.

Let me remind you that this is coming from the moduli space of Higgs bundles. I have a Riemann surface C with $g(C) > 1$. The pair (E, Φ) is called a *Higgs bundle* when E is a rank two bundle on C and Φ is an endomorphism in $H^*(C, End \otimes K_C)$, with coefficients in the canonical class of this curve. The Higgs bundles form a moduli space M_{Higgs}, the dimension of this moduli space is $6g-6$, and so to a pair like that, you can correspond a so-called linearization, a spectral covering, a subscheme of the cotangent bundle of C, $\tilde{C} \subset T^*C$, and $\tilde{C}$ is given by the equation $dt(\lambda - det\Phi) = 0$, where λ is the topological section in this cotangent bundle, so $\tilde{C}$ is a two-sheeted covering of C ramified as we can easily see from the definition in $4g-4$ points. So now that's what was observed by Hitchin in 1986 and then later on this theory was developed to any group and any algebraic variety X, and eventually [unintelligible]extended this to any projective variety. The result that started with Hitchin, Hitchin observed that there's an analytic isomorphism, real analytic, between the moduli space of Higgs bundles and representations of the fundamental group of C in $SL(2,\mathbb{C})$, and as we can easily compute this is also $6g-6$ dimensional, as every element corresponds to a matrix, and you can do this in three dimensions of ways, and you have the action of $SL(2,\mathbb{C})$, and that indeed gives a $6g-6$-dimensional space of representations. This observation led to many applications of this moduli space of Higgs bundles. They were considered as a non-Abelian Hodge structure. They have nice properties like functoriality and so-called "strictness."

I'll briefly talk about this strictness, which we'll want, and I'll try to define this in the most painless way. This moduli space of Higgs bundles has a C^*-action on it. This C^* action extends on the compactification of this moduli space and the fixed points of this action defined by λ acts on (E,Φ) by taking it to $(E,\lambda\Phi)$, and the fixed points of this action are called complex variations of Hodge structure. These are Higgs bundles that split in a peculiar way. The complex variations of Hodge structure let you put some variation of classical Hodge theory on this non-Abelian Hodge theory. Let me mention one consequence. You could have the following geometric situation. Say you have an algebraic surface S and you have a configuration of curves on it, C_i for $1 \le i \le \ell$, let's say for simplicity they are all $\mathbb{P}^1$ (actually it doesn't matter) and assume now you have a representation ϱ of $\pi_1(S)$ on a reductive group over $\mathbb{C}$ (in particular this could be $SL(2,\mathbb{C})$) and then the rich structure you have on the moduli space of Higgs bundles lets you prove the following. Assume that ϱ restricted to C_i is trivial for all i, then $\rho(\pi_1(T)) \to G_C$ is trivial as well (here T is the union of the curves). So that is certainly wrong in the case that this is not an algebraic surface. You use the whole machinery of the moduli space of Higgs bundles to prove this statement. It has some immediate consequences. In particular, it leads to the proof of the following theorem

Theorem 11.1. *Let X be a smooth projective variety such that $\pi_1(X)$ is contained in some $G_{\mathbb{C}}$, a complex linear group, say $GL(n,\mathbb{C})$. Then $\widetilde{X}$, the universal covering of X, is holomorphically convex.*

This is the strongest uniformization theorem after Riemann, who proved

Theorem 11.2. *If the dimension of X is 1 then $\widetilde{X}$ is one of $\mathbb{P}^1$, cC, or D, so either Stein or compact.*

The idea of holomorphic convexity is a combination of holomorphic and Stein, a space is holomorphically convex if contracting all compact submanifolds gives something Stein. Let me give an abstract definition of holomorphic convexity.

Definition 11.1. X, a complex manifold, is called *holmorphically convex* if and only if for every infinite sequence of poitns $x_1, \dots, x_n$ without a limit point, there exists a holomorphic function f, unbounded on this sequence.

Clearly something that is compact is trivially holomorphically convex. What does this statement about holomorphic convexity have to do with the statement I mentioned from the theory of Higgs bundles and the moduli space of representations? This goes back to [unintelligible](not the one of that name mentioned in Kenji's talk but a different one) who noticed the following:

Assume that we have a situation as the one described before, I have an algebraic surface and a configuration of curves T on it, made of C_i, and the image of $\pi_1(C_i)$ in $\pi_1(S)$ is trivial but the image of $\pi_1(T)$ in $\pi_1(S)$ is ∞. Assume we have such a situation. Then what does this mean? It means that in the universal covering $\widetilde{S}$, the configuration of curves opened up, since the whole thing has infinite fundamental group, and I can find an infinite sequence of points on this chain which by the maximum principle will have that every holomorphic function is constant. So I cannot find an unbounded function on this sequence. So if I exhibit this phenomenon, the universal covering will not be holomorphically convex.

Non-Abelian Hodge theory does not allow that, as one can see, which is the major part of the proof of this theorem.

Unfortunately, there is a restriction, which tells me I have to have a Higgs bundle, so this doesn't work for groups not contained in reductive or linear groups. So we need a different kind of Hodge theory. We have a nice action, a nice extension to the compactification, and a little bit more. Another way to think about the Higgs bundles is that they correspond to maps $U : \tilde{X} \to G/K$, or if you like $\varrho : \pi_1(X) \to G$, I get such a harmonic map, and if I have a family of such representations, I get $\widetilde{U} : \tilde{X}$ to some building, [unintelligible]. These are basic ideas from non-Abelian Hodge theory. What we'd like is a new theory with this kind of strictness, connected to this type of categories.

I'll move to the second part of my talk, which is Kähler metrics on categories. This will eventually change many times, but that's a little bit my point of view of these things. Eventually there will be I guess a more uniform way to look at this and I guess it won't be my way. Let me present what I think about this. I'll certainly try to imitate the Higgs bundle story here. So let me start with an A_∞ category over $\mathcal{C}$, a non-Archimedean field $\mathbf{k}$. So let me introduce some data,

- D0 a category $\mathcal{C}^0$ which is A_∞ and a functor $\mathcal{C}^0 \otimes_{O_\mathbf{k}} \mathbf{k} \to \mathcal{C}$, where $O_\mathbf{k}$ is the integers of $\mathbf{k}$. You can think of this as a fibered catgeory where the fiber over E consists of $Met(E)$, so-called *metrized objects*, which are objects with a Hermitian norm, which I'll define. So $Met(E)$ is pairs $(\tilde{E}, h)$ where $h : \tilde{E} \otimes_{O_\mathbf{k}} \mathbf{k} \to E$ is a morphism that identifies these two in the category upstairs. This plays the role of my family of Higgs bundles. We should also have a stability condition $Z : \mathbf{k}^{Cont}(\mathcal{C}) \to \mathbb{C}$, so let me put here, continuous, which means $\mathbf{k}^{0,alg}$, well I can just say $\mathbf{k}^0(\mathcal{C})$, and I fix a stability condition, which is going to play the role of the Kähler class. The Kähler metrics are the class of this stability condition.
- D1 I also have a flow on $Met(E)$ which is given by the action of $\mathbb{R}$, which I can think of as $\log \mathbb{R}^*_{>0}$, I think of $\mathbb{R}$ as morphisms of E. The idea is, I have this flow and if I have this moduli space of Kähler metrics and the flow

converges, then just like I had the complex variation of Hodge structures, I'll get nothing else but the stability condition for the category.

D2 I have a function $Mass$ from $Ob(\mathcal{C}^0) \to \mathbb{R}_{\geq 0}$

D3 I have two other functions, *negative and positive amplitude*, Amp_- and Amp_+ from $Ob(\mathcal{C}^0) \to \mathbb{R}$, and

D4 I have a function $S^{\mathbb{C}}$ on the objects into the complex numbers, the *complexified Kähler potential*, there's a little bit of unclarity what is the general situation for this function so I won't discuss it now.

This data should satisfy the following six axioms. We have the three actions, $\mathbb{Z}$ the (*shift*), $\mathbb{R}_{\geq 0}$ (the flow) and $\mathbb{R}$ (the scaling)

A1 $\mathbb{Z} \ltimes (\mathbb{R}_{\geq 0} \times \mathbb{R})$ acting on $Ob(\mathcal{C})$ should satisfy that $[1] \circ \text{rescale} \circ [-1] = 1$ and $[1] \circ \text{flow} \circ [-1] = \pi\text{rescale} + \text{flow}$ and $[\text{rescale}, \text{flow}] = 1$

A2 The shift, on $Mass$, on Amp_+, and on Amp_-, this should preserve the mass, acts on the amplitudes by adding π, and preserves the potential. Similarly, the flow makes the mass get smaller, Amp_- gets bigger and Amp_+ gets smaller, and the equlities I wrote before tell us how the rescaling behaves in this.

A3 Additivity — the $\oplus$ in this category $\mathcal{C}^0$ commutes with this group $\mathbb{Z} \ltimes (\mathbb{R}_{\geq 0} \times \mathbb{R})$ and then $Z(\quad \oplus \quad) = Z + Z$ and $S^{\mathbb{C}}(\quad \oplus \quad) = S^{\mathbb{C}} \oplus S^{\mathbb{C}}$, this becomes a bit like tropical geometry as $Amp_-(\quad \oplus \quad) = \min(Amp_-, Amp_-)$ and $Amp_+(\quad \oplus \quad) = \max(Amp_+, Amp_+)$

A4 Existence of the limit, this Simpson type of compactification; for every gauge, the limit $e^{\text{flow } t}(E,h)$ is in the Berkovich $F \log$ compactification of $Met(E)$, so I'll say a couple of words a little bit later,

A5 The function $im(e^{i\theta} S^{\mathbb{C}}(E,h))$ is bounded below by a subharmonic function, here $\theta - \pi/2 \leq Amp_- \leq Amp_+ \leq \theta + \pi/2$, and

A6 The mass of (E,h) is at least $|Z(E)|$ for all E, h, and so with the flow. . .

Now the definition will be that this data 0 through 4 satisfying these axioms will be called the moduli space of metrized objects, and if we allow E to move this will be the moduli space of Kähler metrics on $\mathcal{C}$. So now let me give an example, which is $D^b(pt)$, and in this case, let's say that E is some complex $V_1 \to V_2 \to \cdots \to V_n$, then $Met(E,h)$ is an inductive limit over $\mathbf{k}$ of $G_i/K_1 \times \cdots \times G_i/K_{i+k})$, and you need to take an open domain here, that is nothing else but $d|\ | < |\ |^{i-1,0}$. Now in this case, the mass function is just the sum of the dimension of the homologies $\sum h^i$, the amplitudes Amp_- and Amp_+ are the lengths of the complex (times π; this is a little bit fishy), and the potential $S^{\mathbb{C}}$, let's say that I have two objects, $S^{\mathbb{C}}(E_1, E_2, \alpha) = \tau(Cone(E_1 \xrightarrow{\alpha} E_2))$. So now the compactification of this Met in this case are the stable objects in this category with [unintelligible]filtration, by analogy with the moduli space of Higgs bundles.

The example I wanted to talk about is connected with Kapranov's work which is the sheaf of categories, $SL(2)$ Higgs bundles and the corresponding map is the compactification of Teichmüller space that Verlinde was talking about but I guess I am grossly over time.

12. June 10: Mikhail Kapranov: Combinatorial approach fo Fukaya categories of surfaces III

Recall that last time we had $\mathcal{C}$ a pre-triangulated dg category, and we constructed $S.\mathcal{C}$, the Waldhausen construction, and we'd like to represent $S.\mathcal{C}$ as $Map_{dgCat}(S_n, \mathcal{C})$ with S_n a co-cyclic (Connes) object, 2-periodic in $dgCat$.

12.1. **Constructing S_n via matrix factorizations.** If A is a commutative ring or W is in the center if it's noncommutative, we consider complexes M^0 and M^1 with d between the two, with $d^2 = W$. The category of these things is $MF(W)$. So Hom between two of these is an actual complex, this is a 2-periodic dg category, like connections with scalar curvature.

This has a graded version also. Suppose that we have an Abelian group L, suppose A is graded, $A = \bigoplus_{\lambda \in L} A_\lambda$, and $|W| = 0$, then we can talk about graded matrix factorizations, $MF^L(A, W)$, where everything preserves grading.

The simplest example has $A = k[z]$ and $W = z^{n+1}$, the A_n-singularity, this has degree $n+1$ and we take $L = \mathbb{Z}/n+1$ and $deg(z) = 1$. Then we can define $S_n = MF^{\mathbb{Z}/n+1}(k[z], z^{n+1})$. Then the cyclic structure is manifestly clear. This is a dg category and $\mathbb{Z}/n+1$ acts on the nose. The theorem is that

Theorem 12.1. *(S_n) form a co-cyclic object in dgCat and $S_n\mathcal{C} \cong Map(S_n, \mathcal{C})$ for any two-periodic $\mathcal{C}$.*

From this we can continue the discussion of surface Postnikov systems a little bit more rigorously. We start with (S, M), a marked surface, with S hyperbolic. Suppose $\mathcal{T}$ is a triangulation with vertices in M, or any polygonal decomposition. We can combine triangles into a polygon. In every triangle, the arcs are not oriented but the surface is oriented. So there is a cyclic order on the vertices. This means we can associate to the triangle canonically the associated S_T for the set of vertices T. Then we can define the category of Postnikov systems of type $\mathcal{T}$ as $Post_{\mathcal{T}}(\mathcal{C}) = Map_{dg}(\mathcal{T}, S.(\mathcal{C}))$. We look at collections of triangles in this thing, [unintelligible], and the corollary is the following theorem

Theorem 12.2. *Up to Morita equivalence (and all higher coherences among such equivalences) the category $Post_{\mathcal{T}}(\mathcal{C})$ depends only on the surface and the set of marked points but not on the triangulation. We use the fact that the partially ordered set (building) of all polyhedral decompositions is contractible.*

You have triangluations as vertices, flips as edges, some pentagon relations for 2-cells, and so on. [picture].

So we get canonically a differential graded category $Post_{(S,M)}(\mathcal{C})$ or better yet $\mathbb{F}(S, M, \mathcal{C})$ since it's a version of the Fukaya category. Now let me say a little more about that connection to the Fukaya category.

I recall the very basic setup, if (X, ω) is a symplectic manifold, then one result is the construction of a triangulated category $\mathbb{F}(X)$ (a priori A_∞) whose objects are pairs $(\Lambda, \mathcal{L})$ where Λ is a Lagrangian manifold and $\mathcal{L}$ a $U(1)$-local system, so that very naively, $Hom(\Lambda, \mathcal{L}), (\Lambda', \mathcal{L}')$ is

$$\bigoplus_{x \in \Lambda \cap \Lambda'} Hom(\mathcal{L}_x, \mathcal{L}'_x)$$

when these are transverse. The differential and A_∞ structure comes by counting pseudoholomorphic disks. Now this is of course a sophisticated analytic construction. People were trying to consider situations where this was more algebraic and can be approached in a more straightforward way. So in particular, there was a proposal of Kontsevich to localize the Fukaya category for some particular symplectic manifolds, on some skeleton, some kind of "Stein" symplectic manifold (meaning it can be represented as the neighborhood of a skeleton.)

For such X, $\mathbb{F}(X)$ can be localized on (possibly singular) Lagrangian skeleton $K \subset X$. I'll write two points and then discuss a cetain point of view on them.

What does it mean to localize? One should be able to construct a sheaf R of dg or A_∞ categories R_K on K so that $\mathbb{F}(K) \cong H^0(K, R_K)$.

For K a graph on a surface, the stalk of R_K at x with valence $n+1$ should be (possibly 2-periodic), well, $\mathcal{D}^{(2)} Rep\ A_n$, and in here we have the Coxeter functor which satisfies $C^{n+1} = \mathrm{id}$. Recall that a Waldhausen diagram A_{ij} is determined up to isomorphism by its first row $A_1 \to \cdots \to A_n$, since $A_{ij} = Cone(A_{0i} \to A_{0j}$. Now I want to make a remark on this approach in general.

We can think of the Fukaya category as a categorification of homology, not just forming an Abelian group but a category. It's clear that

(1) $\mathbb{F}(X)$ categorifies a part of the middle dimensional homology or cohomology, with support.
(2) Kontsevich's proprosed R_K categorifies $\underline{H}^n_K(\mathbb{Z}/X)$. [pictures].

In all known examples, $H^{\neq n}_k(\mathbb{Z}_x) = 0$. This is a categorification of the spectral sequence for cohomology with support. Defining $\mathbb{F}(X)$ as $H^0(R_K)$ is an analogue to $H^n_k(X, \mathbb{Z})$:

$$H^n_k(X, \mathbb{Z}) = H^0(K, \underline{H}^n_K(\underline{\mathbb{Z}}_X)).$$

Another point also proposed by Kontsevich is to look at coefficients for the Fukaya category. If you're talking about homology, you might say we're only interested in the integers, but when you try to understand complicated manifolds in terms of simpler ones, there's just no way—it's very useful to do this. Even if you're interested in the regular Fukaya category of (X, ω_X), under $\pi : X \to Y$, it gets some coefficients. Here π is some kind of map compatible with Poisson brackets.

Our proposal is that you should look for coefficients, not the analogue just of sheaves, but of perverse sheaves. That's what I want to explain in the remaining time. Let me first of all recall the definitions and then say why perversity is important (because it has something to do with cohomology with support).

So let me recall the situation. Suppose X is a complex manifold, and that it has a complex analytic stratification $S = (X_\alpha)$. So one can imagine the manifold is smooth, the strata are smooth but not closed and the closures may be singular. Suppose $\mathbf{k}$ is a base field. Then we have the category $Perf(X, S)$, an Abelian subcategory in S-constructible complexes of vector spaces $\mathcal{D}^b_{S-const}(X)$, so in particular this includes local systems on X, like usual sheaves.

I'll basically recall perversity. $(Perv-)$ means that $\underline{H}^i(\mathcal{F})$ is supported on codimension at least i. Then $Perv^+$ is dual. This means that H^0 is everwhere, H^1 only on divisors, and so on. So if $\mathbf{k}$ is the field of complex nubers, then $Perv(X, S)$ is $\mathcal{D}_X - mod^{hol,reg}_{S-smooth}$.

This has remarkable properties, such as the *purity property*. Suppose X and S are given and $K \subset X$ is a totally real submanifold. This means like $\mathbb{R}^n$ inside $\mathbb{C}^n$. We know that for a Kahler manifold, [unintelligible], so anyway, K is totally real such that the intersection of K iwth a stratum is totally real in the stratum. Then for all $\mathcal{F} \in Perv(X, S)$ we have $\underline{\mathbb{H}}^i_k(\mathcal{F}) = 0$ for $i \neq n$. This holds for the constant sheaf but not ordinary sheaves. So perverse sheaves are natural in problems with skeleta. For us this was the convincing reason that we should look at perverse sheafs.

This means that the functor from perverse sheaves $Perv(X,S)$ to $Sh(K)$ which takes $\mathcal{F}$ to $R_K(\mathcal{F}) = \underline{\mathbb{H}}_k^n$.

We can find an analogue of Maxim's proposal if we [unintelligible].

We want to understand this category in terms of representations of a quiver. We want to define the category as the category of representations of the quiver Q_-, so we should have vector specs and maps, if we want an equivalence. For every sheaf we get a vector space. We take a skeleton, we put V_i the stalks of $R_K(\mathcal{F})$, and now let me finish with one example, perverse sheaves on the disk with a possible singular point at 0. The stratification consists of 0 and its complement. So then $Perv(\mathcal{D},0)$ consists of ϕ and ψ and $a:\phi\to\psi$ and $b:\psi\to\phi$, subject to the relations $1_\psi - ab$ is invrertible.

So ϕ is the stalk of the sheaf L at $K=0$ and the other at K. In this case we have a categorical analogue, which is the concept of a spherical functor. This was introduced by Anno and Lagvinenko. We have two pre-triangulated cotegories and a map $S:\mathcal{D}_0\to\mathcal{D}_1$. There is a right adjoint $\mathcal{D}^*$. There is a map $Id_{\mathcal{D}_0}\to S^*S$ and we can take the cone on this, and we get T_0. Similarly, we can take the cone on $\{SS^*\to Id_{\mathcal{D}_1}\}$. Now S is called *spherical* if T_i are both eqeuivalence.

So we can look at perverse schobers, the conjectural categorical analogue of perverse sheaves. We have the agricultural terminology from French, now in German there is nothing like this. So Schober is a German word for a stack. If you say stack in this context it would be confusing. Some people wanted to use German agricultural terms, so Hirzebruch called sheafs Garbes and presheafs Garbendatum. So Schober means stack, like a stack of hay. So I'm out of time, thank you very much.

13. Vladimir Hinich: Enriched infinity-categories

[missed the beginning]

We all like usual categories and we know that categories in homological algebra are enriched over something. I want to present what seems to me a very obvious way to define such objects and maybe also to work with them in the infinity category world. I have probably first to say something about what infinity categories are or how one can think about them. It's not completely common knowledge. There are various more or less commonly thought of as equivalent ways to define infinity categories. I do not want to stick to a certain model, but instead want to give a vocabulary of what one can do with them. I'll mention certain models to give examples or stress certain points.

It's useful to have in mind the picture in Kapranov's second lecture when he said a few words about dg categories. I'm talking about ∞-categories, and some people call it $(\infty,1)$-categories for those who don't know in what meaning I use this term.

Let me list the features of this thing.

(1) an ∞-category $\mathcal{C}$ has objects and for a pair of objects x and y we have a space of maps $Map(x,y)$, defined up to homotopy. I'm not giving a precise definition, so I'm allowed to say an obscure sentence like this. Sometimes space means topological space and sometimes Kan simplicial set, depending on what you like.

(2) There is a composition $Map(y,z)\times Map(x,y) \to Map(x,z)$ defined uniquely up to homotopy. Sometimes you have an explicit composition and sometimes you have the data that lets you recover this up to homotopy. This is associative up to homotopy.

The first example is a conventional category where the set of maps is a discrete space.

In particular, for each ∞-category $\mathcal{C}$ you can define a conventional category $ho(\mathcal{C})$ which has the same objects and connected components of as the set of maps;

$$Hom_{ho}(x,y) = \pi_0(Map(x,y)).$$

We have an obvious map $\pi : \mathcal{C} \to ho(\mathcal{C})$ which is the identity on objects and projection to components on maps.

Definition 13.1. A map α is an equivalence if $\pi(a)$ is invertible. What is usual isomorphism is replaced by equivalence. For isomorphism you should have strictly associative multiplication.

(3) An *equivalence* of ∞-categories, well, I won't define this but a map $f : \mathcal{C} \to \mathcal{D}$ is an equivalence (Dwyer–Kan equivalence) if
   (a) The maps $Map(x,y) \to Map(fx,fy)$ are equivalences of spaces (perhaps weak homotopy equivalences depending on the definition of spaces).
   (b) The map $ho(\mathcal{C}) \to ho(\mathcal{D})$ is an equivalence of categories.

This is the world we live in. It's not clear the objects of an ∞-category do not form a set but a space. Let me mention immediately one concrete model which is however very inconvenient, but easy to understand. That is, categories enriched over Kan simplicial sets or topological spaces, maybe even easier.

What is the difference between what I said and the model? In this the composition is associative on the nose. Here everything is strict and a structure describable in this way is rich enough to cover the things that interest us, but morphisms of categories are too restrictive. We might put a model category structure on these guys. You should replace $\mathcal{C}$ by something cofibrant and then everything would be okay.

This is a model where you don't see immediately how to define a *space* of objects.

An ∞-category $\mathcal{C}$ is called an ∞-groupoid if all arrows are equivalences and I would, so any ∞-category $\mathcal{C}$ contains a maximal ∞-subgroupoid which I call $K(C)$ and of course this is the same as if you start with a conventional category, the maximal subgroupoid. This is like the moduli space of this category. I'd like to persuade you that ∞-groupoids are basically the same as spaces. At this moment it's basically a slogan. In this concrete model, what I can do is just present a functor that becomes an equivalence on homotopy categories.

You have a functor from simplicial groupoids to Kan simplicial sets given by the nerve, there are different versions but they are basically equivalent. This is the maximal subspace of an ∞-category and I'd like to look at this as follows.

Let me introduce a number of important categories. There is an ∞-category $\mathcal{S}$ of spaces, an ∞-category Cat of ∞-categories, and an adjoint pair of functors

$$\mathcal{S} \underset{}{\overset{K}{\leftrightarrows}} Cat$$

Another important feature is the following. If I have two ∞-categories, I can form $Fun(\mathcal{C},\mathcal{D})$. You'd like a model where everything is cofibrant. Since most things are not cofibrant, this model is not very nice.

For regular categories we know that functors form a category. This means ∞-categories should be a form of two-category but we have no language to explain this.

What we know about ∞-categories is enough to construct limits and colimits. An object x in $\mathbb{C}$ is initial if $Map(x,y)$ is contractible for all y. This is the way we are talking about uniqueness in this world. A similar notion gives terminal objects, and then after some preliminary work we can explain what it means to have limits and colimits.

Now (and probably the last thing that comes in mind) once we have a model category M we can associate to it a certain ∞-category $\mathcal{N}(M)$, in this language the Dwyer–Kan localization of M with respect to isomorphisms, $\mathcal{L}(M,W)$. I understood this idea only recent although this is long known for intelligent people. The model category structure is a choice of coordinates for those who like coordinate, but we really care about the ∞-category.

The model structure is a good tool to construct something. As a choice of coordinates sometimes it's useful.

For the ∞-categories obtained from model categories, the limits and colimits correspond to homotopy limits and colimits in the model category, so this is really the correct notion. Of course if you apply this to regular categories you get the regular definition, but that's less interesting.

I can't move to my suggestion because I first have to explain how to talk about monoidal categories. I will use the term category instead of ∞-category, and if I want to mention a conventional category I will call it a *conventional category.* This is not a trivial question and the easiest thing is to take Segal's definition, an associative algebra object in categories. A monoidal category is a functor $\Delta^{op} \to Cat$ such that two conditions are satisfied:

- $M_0 = *$
- the Segal condition that $M_n \to M_1 \times \cdots \times M_1$ which comes from embedding the intervals $\{i, i+1\}$ into $\{0,\dots,n\}$ is an equivalence.

This is basically Segal's definition for the algebraic structure on the loop space. This is an ∞-version of an associative algebra. I'd like to leav this definition on the blackboard because I'll come back to it later.

What I want to explain is a part of this, I want to have a notion of associative algebra in a monoidal category. This is also not a very obvious notion. A monoidal category is an associative algebra in the category of categories, but I used a trick in the monoidal structure of the category of categories, it's Cartesian. So it's not very honest. This is correct, though, and easy to give.

There is a simple conventional monoidal category that I call *Ass* which for those who know this world is sort of a PRO for associative algebras. I can say what it is precisely but this is not important. Even in the conventional world you can use monoidal functors from this one to your monoidal category to give algebras in that monoidal category. So that's my definition here.

So this is my notion of algebra. I'm ready to say the formulation of the problem that I'm trying to suggest.

Given a monoidal ∞-category M, what is an M-enriched one? When M is not just monoidal but Cartesian, there is a well-known definition from a Segal type construction. In this case the constructions will be basically equivalent.

I need that M has colimits, so let me define Cat^L, which is categories with colimits and functors preserving colimits. If I have two ∞-categories, then I have a product. Then this is monoidal. I can define $X \otimes Y$ like a tensor product of vector spaces. You can talk about universal objects representing bilinear maps.

We define $X \times Y \to Z$ to be bilinear if it preserves colimits in each argument, and let $X \otimes Y$ represent bilinear maps.

I require M to be an algebra with respect to this structure, so my monoidal structure preserves colimits in each argument.

The idea (I'm ready now) is extremely simple. There's an idea even easier than the notion of category. This is the notion of a quiver. If you look at the dictionary, this is a container for arrows. This doesn't have a composition. Let X be in $\mathcal{S}$ and M in $Alg(Cat^L)$. Then $Quiv_X(M)$ is the category of functors $Fun(X^{op} \times X, M)$. To each choice of sourc and target you get an object in M. This works perfectly for ordinary categories.

Definition 13.2 (Pre-definition)**.** An M-enriched category with objects X is just an algebra object in $Quiv_X(M)$.

Two remarks.

(1) The most nontrivial part is to define a monoidal structure on quivers.
(2) I don't like that I had to fix the space of objects. This can easily be avoided. You consider families of these guys over spaces, and a relative monoidal structure where you can multiply only in the same fiber. I won't spend time on this, this is a formal thing.

The first remark is very important.

Let me say what I am going to do to solve the first problem. A toy version, which is very nice. Everybody knows there is a notion of matrix in linear algebra. If you have an associative ring R and a finite set X, there is a notion of a ring of matrices whose columns and rows are numbered by X with values in R. How will you define this? If you don't care about multiplication, you'll define this as $Map(X \times X \to R)$. It's a table of numbers. Let me write down $Map(X^{op} \times X \to R)$ where *op* means nothing here.

The student asks how to multiply these. If you like formulas you will write down a formula and spend thousands of blackboards proving it's associative. In my case you have an infinite number of coherences to check. Here you have another way. You say, I will construct a free module $Map(X, R)$, and I'll define my ring as $End_R(Map(X, R))$. This is what we will be doing here. Let me stress once more. It's hard to define an associative algebra structure in the ∞ world. There are exceptions. Once you define an A_∞ structure on a category, you write down m_k for all k and prove that they are compatible. Usually you try to use a universal property. One example I've mentioned you probably haven't paid attention to. If you have a Cartesian structure, this gives you a unique (up to some certain usual things) algebra structure. A similar example was suggested by Lurie. The endomorphisms of an object M are defined by a universal property, the universal object that admits an action of $X \times M \to M$. This doesn't require associativity.

But if you have a universal object then it will automatically give you a composition. There's parts of this that are hard to check but luckily Lurie already did it.

To define a multiplicative structure on the category of quivers, I'll present it as the category of endomorphisms of another category.

The claim is that $Quiv_X(M)$ is $Fun^L_{LMod(M)}(Fun(X,M))$ (here $LMod(M)$ means left tensor modules over M).

Let me just add that $Fun(X,M)$ are all functors, this is the same as $Fun^L(P(X),M)$, where $P(X)$ are simplicial presheaves, because presheaves are constructed universally from the colimit property for X. This is, when X is small, $M \otimes P(X^{op})$, sort of a dual object.

So this gives you a monoidal structure.

Let me show some very simple case. I'm not sure you'll give me time to explain the compatibility with the stronger notion. Let's look at $M = \mathcal{S}$. What do we expect to get if our monoidal category is the category of spaces? We expect to get the same definition. This will also be an exercise in monoidal structures, not so obvious how to do it. This is what I intend to present.

We have the category $\mathcal{C}$. There is a natural thing, connected to it, a functor $Y : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$. This should be our answer. This should be somehow presented as an associative algebra object in the appropriate category. What does it mean to have quivers over $\mathcal{C}$ of $\mathcal{S}$. I gave a definition when $\mathcal{C}$ was a space, now it's a category. So this is $Fun^L(P(\mathcal{C}), P(\mathcal{C}))$. If you look carefully at the definitions, you see that this is of course the same as $Fun(C^{op} \times \mathcal{C}, \mathcal{S})$, and the isomorphism takes the standard Yoneda Y to the identity in $Fun^L(P(\mathcal{C}), P(\mathcal{C}))$. We have as a collection of objects $\mathcal{C}$ which should be a space, but this is a nice start.

Once you have an algebra in this category, it restricts to an algebra in any subspace of $\mathcal{C}$, and it's natural to take the maximal subspace $Quiv_{K(\mathcal{C})}(\mathcal{S})$. This is lax so it carries algebras to algebras.

I intended to say something more but maybe it's less important. This is equivalent to a definition with complete Segal spaces. There's an issue about completeness. Of course it should just be added, otherwise, yeah. You should also add completeness or a condition or localize if you have two different ways to take care of things. You get Segal spaces and not complete Segal spaces.

14. Aleksei Bondal: Categorical Interpretation of flops

Thank you very much for inviting me to this exciting conference. I've already learned a lot. My talk is about joint work with A. Bodzcuta. First I should recall some ideology from a joint paper with Orlov in 1995, which is the homological interpretation, a homological minimal model program. First I should recall what is the minimal model program, denoted MMP, and then homological put an H in front, HMMP. This is about birational geometry. For X a surface, with $E = \mathbb{P}^1 \subset X$, with self intersection number -1, then there is a map from $X \to X'$ which contracts E to a point. On X', you can find another -1 curve, contract it, and so on. You can continue this process and get to a minimal model. It might happen that you find another curve and contract it, you might come to another minimal model that is not isomorphic.

For example, consider a blowup of $\mathbb{P}^2$ in two points. When you blow up a point, $\mathbb{P}^1$ appears as in this picture, so you can go down to $\mathbb{P}^2$. But if downstairs you look at a line through these two points, then you can pull it back upstairs and that also has intersection number -1 and you can contract just this one line and the result is $\mathbb{P}^1 \times \mathbb{P}^1$. People tried to generalize to higher dimension, and realized that smooth varieties are not enough, you should consider singular varieties, and the MMP is about how to generalize this story.

I won't describe this whole theory which is really huge, so I'll just say a few words. You might consider some morphisms of various types, you could have divisorial contraction $f : X \to Y$ as you had here. You have in X a divisor D, a curve being a divisor means it's codimension 1 and it might be contracted to something C that is not a divisor, and the dimension of C, well f is a birational morphism, the dimension of X and Y are equal, say n, and $dim\ D = n - 1$, but the dimension of C is less than $n - 1$. It could be fibered or it could be the whole divisor contracted to a point.

The other situation that we found is so-called flips. This is also a birational morphism $X \to Y$, maybe I'll describe this in more precise terms later. This is not a morphism, it's just a birational map, there points of indeterminacy of the map, an example of a birational map is the map in the opposite direction $X' \to X$, it's a birational morphism outside the contracted point. That's why I call it a map, not a morphism. Nevertheless, you should somehow compare canonical classes in X and Y. One of the conjectures from this paper, the idea is that somehow the canonical class governs the behaviour of the derived category. The flip is the following. You can blow up X to $f : \tilde{X} \to X$ and everything is birational, and you have a composition to $\tilde{X} \xrightarrow{g} Y$ which is everywhere defined. A birational map can be presented as a birational morphism from a blowup. Maybe now I'll assume for simplicity that everything is smooth, which is not necessary in general. So you can look at $f^*(\omega_X) \hookrightarrow \omega_{\tilde{X}}$ and $g^*(\omega_Y) \hookrightarrow \omega_{\tilde{X}}$. This is called a flip if one of these is contained in the other, but we call it a *flop* if these two pullbacks coincide.

Conjectures from this paper of '91, we call in this paper (well, another one), we called these generalized flips and generalized flops. The point is that you consider a contraction of X to something singular Z and you have another variety X^+ and you have a birational morphism $X \to X^+$, and you have somehow for a birational morphism of this type, you can present it in this way. For instance in dimension 3, Kollar proved, a birational morphism between Calabi–Yaus is a generalized flop, and in this case every generalized flop is a composition of ordinary flops.

The MMP goes by using divisorial contraction and flips to get a minimal model, and there are more than one minimal model but they are related by flops. You can see that these two models in my example in two dimensions have different canonical classes. The flops have the same canonical classes. That makes the high dimensional setting more complicated and more interesting.

Now about HMMP, the idea, first, is that you should consider the derived category $\mathcal{D}(X)$, by which I mean the bounded derived category of coherent sheaves on X. This minimization should be viewed as being about the minimization of this derived category. In what sense? There are various conjectures. One of them is this.

Conjecture 14.1. If you have a generalized flip, then the derived category of Y is fully faithfully embedded in the derived category of X.

If you have a fully faithful embedding of one category into the other $\mathcal{A} \hookrightarrow \mathcal{B}$, you have a decomposition $\mathcal{B} = \langle \mathcal{A}, {}^\perp \mathcal{A} \rangle$. You need this inclusion to have an adjoint. You should have for each X in $\mathcal{B}$ a decomposition $A \to X \to A'$ with $A \in \mathcal{A}$ and A' in ${}^{\perp}\mathcal{A}$ which is something so that $Hom(\mathcal{A}, \perp \mathcal{A})$ is empty.

Conjecture 14.2. Flops induce derived equivalences.

Today I will discuss flops only, and under fairly strong conditions.

Maybe I should describe an example of a flop.

Consider a 3-dimensional smooth variety X and a curve E in it isomorphic to $\mathbb{P}^1$, but now $N_{X/E}$ is rank two, assume it's $\mathcal{O}_E(-1) \oplus \mathcal{O}_E(-1)$. This is an assumption. This will really be a flop, not a derived flop. You have a contraction of X to something singular, Y, by contracting E to a point. You can blow up and you have $\tilde{X}$ with a divisor $\mathcal{D} \cong \mathbb{P}^1 \times \mathbb{P}^1$ which projects to E. Another nice fact, you have this map that contracts $\mathcal{P}^1 \times \mathcal{P}^1 \to \mathcal{P}^1$, and you could also contract in the other direction, the other $\mathcal{P}^1$, so to X^+. So this X^+, naively, you have $\mathbb{P}^1$ and its complement. You remove $\mathcal{P}^1$ and glue it back in a different way. Locally, the singularity, the E and E^+ map to the point, and around this there is a three dimensional [unintelligible].

It might happen that the result of the flop, it still exists as a complex analytic variety but might not be algebraic. But if Y is the spectrum of a local complete ring, then it's okay in the vicinity of the singularity.

The situation where I have one curve and I want to contract it in the direction of the flow, so over the singular point if you have just one component curve you can check that it should just be $\mathbb{P}^1$. The normal bundle $N_{X/E}$ (assuming it's a smooth projective variety) is either $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(2)$ or $\mathcal{O}(1) \oplus \mathcal{O}(-3)$. We worked out the first two cases and the third case, which is the most complicated, was worked out by Tom Bridgeland, who said that if you have X over Y, X is smooth, and Y has a terminal singularity (I won't explain this) and is Gorenstein (the singularities are minor) then the flop exists, and X and X' are isomorphic but the maps are different.

The point is that he didn't just show it, he constructed X^+ as the moduli space of an object in X. In the derived category of X there is a T-structure. But there is another such structure which will be important for my talk today. In fact, he constructed a family depending on an integer p. For my talk, $p = 0$ or -1.

$$ {}^{-1}Per(X/Y) = \{E \in D^b(X) | R^1 f_* \mathcal{H}^0(E) = 0; \quad f_*\mathcal{H}^{-1}(E) = 0; \quad \mathcal{H}^i E = 0, i \neq 0, -1; \quad Hom(\mathcal{H}^0(E), \mathcal{A}_f) = 0. $$

Here $\mathcal{A}_f$ is the null-category of f, which is the objects E in $Coh(X)$ such that $Rf_*E = 0$. This is important for flops; I'll explain why later.

The basic idea of the T-structure is that we have the derived categories of X and Y and the pushforward

$$\mathcal{D}(X) \xrightarrow{Rf_*} \mathcal{D}(Y)$$

and $\mathcal{A}_f$ is a subcategory of $\mathcal{C}_f$ which is a subcategory in $\mathcal{D}(X)$, the kernel of this map to $\mathcal{D}(Y)$.

You can think of this as a situation in topology, when you glue T-structures, you should glue with some accuracy because if you take the adjoint functor it's no longer the bounded derived category and $f^!$ is unbounded in the other direction.

This parameter p is how you shift one of these categories, the T-structure in one of the categories. Neither of these T-structures is the standard T-structure in coherent sheaves.

This was Bridgeland. The next step for us was done by Michel Van den Bergh. He interpreted these two T-structures as the category of modules over some algebras downstairs. The point is, assume now that Y is affine, consider a kind of local situation, and one can prove that the derived category of X is equivalent to the derived category of perverse sheaves for -1 and for 0,

$$D^b(X) \cong D^b({}^{-1}Per(X/Y)) \cong D^b({}^0Per(X/Y)).$$

He claimed that there exists a so-called projective generator $\mathcal{M}$ in ${}^{-1}Per(X/Y)$ and $\mathcal{N} \subset^0 Per(X/Y)$, which are projective objects which generate their categories; for any object A in the category there is a nontrivial morphism from the generator into A.

Then these are actually vector bundles over X, and you can consider the local endomorphism algebra and push forward to get $A_f = f_* End_X \mathcal{M}$ and $B_f = f_* End_{\mathcal{N}}$. The claim is that A_f-modules is equivalent to the category of ${}^{-1}Per(X/Y)$ and similarly B_f-modules for ${}^0Per(X/Y)$.

Now our first result, these facst imply together, I forgot to say, he showed that if you consider $D(X^+)$, these categories are just exchanged. Under this equivalence, for X and X^+, these coincide, that is,

$${}^{-1}Per(X/Y) \cong^0 Per(X'/Y), \quad {}^0Per(X/Y) \cong^{-1} Per(X'/Y)$$

which happens because $A_f \cong B_{f^+}$ and $B_f \cong A_{f^+}$.

The functor goes, we have this picture

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "p"'] \arrow[dr, "p^+"] & \\ X \arrow[rr, dotted] & & X^+ \end{tikzcd}$$

and you get a map $F = Rp^+_* Lp^*$.

[missed a comment about Bridegland]

It was Chen who first generalized to the case when X and X^+ are Gorenstein terminal (Bridgeland had only Y Gorenstein, X and X^+ were smooth) but he also showed that the functor $\mathcal{D}(X) \to \mathcal{D}(\mathcal{M} = X^+)$ was just F.

So the flop functor in the opposite direction is $Rp_* Lp^{+*}$, and $F^+F \neq \mathrm{id}$. One point of my talk is to show that this is spherical with respect to [unintelligible], and it can actually be shown to be in two ways up to shift.

I said that Van den Bergh does not actually quite use the flop functor. It doesn't give you a nontrivial automorphism of the derived category.

Maybe I should say a few words about the setup of Van den Bergh. We put the condition that Y has canonical hypersurface singularities of multiplicity 2, which means basically that you can locally around every point describe your variety Y in $\mathcal{Y}^{n+1}$ where f starts from x_1^2 plus something that doesn't contain x_1, this is a multiplicity 2 singularity, they all look like this. If you look at the exceptional locus $Ex(f)$ of f in X, the places where this is not one to one, then the codimension in X of $Ex(f)$ is at least 2 and the relative dimension of f is 1. So you could do $Y : x_1 \to -x_1$ and $x_i \to x_i$, this is an involution, you can pull back along this involution, the thing is mapped into Y in a different way.

This is the setup in which we work. The first thing in which we did was a description of Van den Bergh's and the flop functor. We can identify the derived category of X and these perveres objects. Now I assume Y is affine to simplify the story. Then $\mathcal{P}_{0(1)}$ is the category of projective objects with respect to the T-structure in ${}^{0(-1)}Per(X/Y)$. Then $Hot(\mathcal{P}_{0(-1)})$, this is equivalent to $\mathcal{D}(X)$. and

$Hot(\mathcal{P}_1) \to Hot(\mathcal{P}_0^+)$. We termwise push forward and pull back, so that $VdB = (f^{+*}f_*(\quad))^{\vee\vee}$ and this is the Van den Bergh functor. That's not a big deal. The point which is a much harder statement is that something similar is true for the flop functor.

Theorem 14.1. *(B,B.)*

$$f^{+*}f_*(\quad)$$

is the flop functor.

The proof of this fact is based on a lemma, if M is an object in $\mathcal{P}_{-1}$, you can push forward and pull it back, Lf^*f_*M, a priori these could go on forever, but it turns out that L^1 is trivial. L^2 is nontrivial, I call it $P = L^2f^*f_*M$, because it lies in $\mathcal{A}_f$.

Then

Theorem 14.2. *P is a projective generator in $\mathcal{A}_f$, and $\mathcal{D}^b(\mathcal{A}_f) \xrightarrow{\psi} \mathcal{D}(X)$ is a spherical functor. From this it follows that $\mathcal{A}_f$ is A_p-modules End P.*

The spherical twist $T_X = (F^+f)^{-1})$ and the cotwist C_ψ is the identity functor twisted by 4.

Maybe I'll stop.